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On the functional equation $F(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]) = 0$

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In the present paper we show that the functional equation

$$F\big(x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big) = 0, \tag{1}$$

where $\varphi(x)$ denotes the required function and $F(x, y_0, \ldots, y_n)$ and $f_1(x), \ldots, f_n(x)$ are known functions, possesses infinitely many continuous solutions under suitable assumptions regarding the functions F and f_i.

Analogous properties of the equation

$$F\big(x, \varphi(x), \varphi[f(x)]\big) = 0$$

were proved in our previous paper [2], as also partially by T. Kitamura [1]. The present paper is a direct generalization of those results.

Given an invertible function $f(x)$, we denote by $f^k(x)$ its k-th iteration, i. e. we put

$$f^0(x) = x, \qquad \begin{matrix} f^{k+1}(x) = f\big(f^k(x)\big), \\ f^{k-1}(x) = f^{-1}\big(f^k(x)\big), \end{matrix} \qquad k = 0, \pm 1, \pm 2, \ldots$$

One can prove the following lemmas:

Lemma I. *Suppose that the function $f(x)$ is continuous and strictly increasing in an interval $\langle a, b\rangle$. In order that $f(\langle a, b\rangle) = \langle a, b\rangle$ it is necessary and sufficient that a and b be roots of the equation*

$$f(x) = x. \tag{2}$$

Lemma II. *Let the function $f(x)$ be continuous and strictly increasing in an interval $\langle a, b\rangle$. If a and b $(a < b)$ are two consecutive roots of equation (2) and if $f(x) > x$ in (a, b), then, for every $x \in (a, b)$, the sequences $\{f^n(x)\}$ and $\{f^{-n}(x)\}$ are monotone and*

$$\lim_{n\to\infty} f^n(x) = b, \qquad \lim_{n\to\infty} f^{-n}(x) = a.$$

The proofs of these lemmas are to be found in [3].

LEMMA III. *Let functions $g(x)$ and $h(x)$ be continuous and strictly increasing in an interval $\langle a, b\rangle$. If $g(a) = h(a) = a$, $g(b) = h(b) = b$ and $g(x) > h(x) > x$ in (a, b), then the function $k(x) \overset{\text{df}}{=} g[h^{-1}(x)]$ is also continuous and strictly increasing in the interval $\langle a, b\rangle$, and $k(a) = a$, $k(b) = b$, $k(x) - x > 0$ in (a, b).*

Proof. The function $h^{-1}(x)$ is, like $h(x)$, continuous and strictly increasing in $\langle a, b\rangle$, $h^{-1}(a) = a$, $h^{-1}(b) = b$. Then the function $k(x)$ is continuous and strictly increasing in $\langle a, b\rangle$ as a superposition of continuous and increasing functions; moreover we have

$$k(a) = g[h^{-1}(a)] = g(a) = a, \qquad k(b) = g[h^{-1}(b)] = g(b) = b,$$

and, for $x \in (a, b)$,

$$k(x) - x = g[h^{-1}(x)] - x > h[h^{-1}(x)] - x = 0,$$

which was to be proved.

In the sequel we shall assume that:

(i) The functions $f_i(x)$ $(i = 1, \dots, n)$ are defined, continuous and strictly increasing in an interval $\langle a, b\rangle$, $f_i(a) = a$, $f_i(b) = b$ $(i = 1, \dots, n)$, and

$$x < f_1(x) \leqslant f_i(x) \leqslant f_{n-1}(x) < f_n(x) \quad \text{for} \quad x \in (a, b),\ i = 2, \dots, n-2.$$

(ii) The function $F(x, y_0, y_1, \dots, y_n)$ is defined and continuous in a set Ω of $(n+2)$-dimensional space of the variables $(x, y_0, y_1, \dots, y_n)$.

(iii) The set $\mathcal{F}$ of the points $P = (x, y_0, y_1, \dots, y_n)$ such that $P \in \Omega$ and $F(P) = 0$ is not empty, and the projections Ω_n and Ω_0 of the set $\mathcal{F}$ on the subspaces of the variables $(x, y_0, y_1, \dots, y_{n-1})$ and $(x, y_1, \dots, y_n)$ respectively are the cubes

$$a < x < b, \qquad c < y_i < d, \qquad i = 0, \dots, n-1,$$

and

$$a < x < b, \qquad c < y_j < d, \qquad j = 1, \dots, n$$

respectively.

(iv) The equation $F(x, y_0, y_1, \dots, y_n) = 0$ is solvable in Ω with respect to y_n and y_0:

$$y_n = G(x, y_0, \dots, y_{n-1}), \qquad y_0 = H(x, y_1, \dots, y_n).$$

LEMMA IV. *Under suppositions (ii)-(iv) the functions G and H are defined and continuous in Ω_n and Ω_0 respectively, and satisfy there the inequalities*

$$c < G < d, \qquad c < H < d.$$

Proof. Let $(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1})$ be an arbitrary point of the set Ω_n. Thus there exists a number $\bar{y}_n$ such that the point $(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1}, \bar{y}_n)$ belongs to $\mathcal{F}$. On account of (iv) the relation $F(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1}, \bar{y}_n) = 0$ is equivalent to the relation $\bar{y}_n = G(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1})$. And since the projection of the point $(\bar{x}, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_{n-1}, \bar{y}_n)$ on the subspace of the variables $(x, y_1, \dots, y_{n-1}, y_n)$ belongs to Ω_0, we have in particular $c < \bar{y}_n < d$. Thus the function $G(x, y_0, \dots, y_{n-1})$ is defined on Ω_n and satisfies there the inequalities $c < G < d$. The continuity of the function G follows from the continuity of the function F in Ω.

The proof for the function H is quite analogous.

THEOREM. *Under suppositions (i)-(iv) equation (1) possesses infinitely many solutions that are continuous in the open interval (a, b).*

Proof. Let us take an arbitrary $x_0 \in (a, b)$ and let us put

$$x_1 = f_n(x_0), \qquad \begin{aligned} x_i &= f_n[f_{n-1}^{-1}(x_{i-1})], \qquad & i > 1, \\ x_{-i} &= f_1^{-i}(x_0), & i > 0. \end{aligned} \tag{3}$$

Writing $k(x) \overset{\text{df}}{=} f_n[f_{n-1}^{-1}(x)]$ we have

$$x_i = k(x_{i-1}) = k^{i-1}(x_1), \qquad i = 2, 3, \dots$$

On account of (i) and lemma III the functions $k(x)$ and $f_1(x)$ fulfil the hypotheses of lemma II. Thus the sequences x_i and x_{-i} are monotone and

$$\lim_{i\to\infty} x_i = b, \qquad \lim_{i\to\infty} x_{-i} = a.$$

Consequently

$$(a, b) = \bigcup_{-\infty}^{\infty} \langle x_i, x_{i+1}). \tag{4}$$

Let $\bar{\varphi}(x)$ be an arbitrary function, defined and continuous in the interval $\langle x_0, x_1)$ so that the following conditions are satisfied:

$$c < \bar{\varphi}(x) < d \quad \text{for} \quad x \in \langle x_0, x_1), \tag{5}$$

$$\lim_{x\to x_1-} \bar{\varphi}(x) = G\big(x_0, \bar{\varphi}(x_0), \bar{\varphi}[f_1(x_0)], \dots, \bar{\varphi}[f_{n-1}(x_0)]\big). \tag{6}$$

The above conditions are not contradictory, for, according to (i), $f_i(x_0) \in \langle x_0, x_1)$ $(i = 1, \dots, n-1)$, whence there exist values $\bar{\varphi}[f_i(x_0)] \in (c, d)$, $i = 1, \dots, n-1$, and thus the function G is defined at the point $P_0\big(x_0, \bar{\varphi}(x_0), \bar{\varphi}[f_1(x_0)], \dots, \bar{\varphi}[f_{n-1}(x_0)]\big)$, and satisfies the inequalities $c < G(P_0) < d$.

Now let us put

$$(7)\qquad \varphi(x) \overset{\mathrm{df}}{=} \begin{cases} \bar{\varphi}(x) & \text{for } x \in \langle x_0, x_1), \\ G\big(f_n^{-1}(x), \varphi[f_n^{-1}(x)], \varphi[f_1(f_n^{-1}(x))], \ldots, \varphi[f_{n-1}(f_n^{-1}(x))]\big) & \\ & \text{for } x \in \langle x_i, x_{i+1}),\ i = 1, 2, \ldots, \\ H\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big) & \text{for } x \in \langle x_{-i}, x_{-i+1}),\ i = 1, 2, \ldots \end{cases}$$

We shall show that the function $\varphi(x)$ is defined and continuous in the whole interval (a, b), and that it satisfies equation (1) and the inequalities

$$(8)\qquad c < \varphi(x) < d, \qquad x \in (a, b).$$

According to relation (4) it is enough to show that the function $\varphi(x)$ is defined, continuous, and satisfies inequalities (8) in each of the intervals $\langle x_0, x_i)$ and $\langle x_{-i+1}, x_1)$, $i = 1, 2, \ldots$

For the intervals $\langle x_0, x_i)$ the proof will be by induction.

For $i = 1$ the theorem is evident.

Let us suppose now that the function $\varphi(x)$ is defined, continuous, and satisfies condition (8) in an interval $\langle x_0, x_p)$, $p \geqslant 1$. According to (7) we have for $x \in \langle x_p, x_{p+1})$

$$(9)\qquad \varphi(x) = G\big(f_n^{-1}(x), \varphi[f_n^{-1}(x)], \varphi[f_1(f_n^{-1}(x))], \ldots, \varphi[f_{n-1}(f_n^{-1}(x))]\big).$$

On account of the monotonity of the function $f_n^{-1}(x)$, the monotonity of the sequence x_i, and by relations (3) we have for $x \in \langle x_p, x_{p+1})$

$$(10)\qquad f_n^{-1}(x) \geqslant f_n^{-1}(x_p) > f_n^{-1}(x_1) = x_0$$

and

$$f_n^{-1}(x) \leqslant f_n^{-1}(x_{p+1}) = f_{n-1}^{-1}(x_p).$$

Since $f_{n-1}^{-1}(x) < x$, we have $f_n^{-1}(x) < x_p$. Then, for $x \in \langle x_p, x_{p+1})$

$$(11)\qquad f_n^{-1}(x) \in \langle x_0, x_p).$$

We have also by (10)

$$f_i[f_n^{-1}(x)] > f_n^{-1}(x) > x_0, \qquad \text{for} \quad x \in \langle x_p, x_{p+1}),\ i = 1, \ldots, n-1,$$

and by (3)

$$f_i[f_n^{-1}(x)] < f_i[f_n^{-1}(x_{p+1})] \leqslant f_{n-1}[f_n^{-1}(x_{p+1})] = x_p$$
$$\text{for} \quad x \in \langle x_p, x_{p+1}),\ i = 1, \ldots, n-1.$$

Then also

$$f_i[f_n^{-1}(x)] \in \langle x_0, x_p) \qquad \text{for} \quad x \in \langle x_p, x_{p+1}),\ i = 1, \ldots, n-1.$$

Hence, by (11), on account of our inductive hypothesis, for $x \in \langle x_p, x_{p+1})$ the values

$$\varphi[f_n^{-1}(x)] \quad \text{and} \quad \varphi[f_i(f_n^{-1}(x))], \qquad i = 1, \ldots, n-1,$$

are defined and we have the inequalities

$$c < \varphi[f_n^{-1}(x)] < d \quad \text{and} \quad c < \varphi[f_i(f_n^{-1}(x))] < d, \qquad i = 1, \ldots, n-1.$$

Then, on account of lemma IV and relation (9), the function $\varphi(x)$ is defined, continuous, and satisfies relations (8) in the interval $\langle x_p, x_{p+1})$. Now we are to show that the function $\varphi(x)$ is continuous for $x = x_p$. We have

$$\varphi(x_p) = G\big(f_n^{-1}(x_p), \varphi[f_n^{-1}(x_p)], \varphi[f_1(f_n^{-1}(x_p))], \ldots, \varphi[f_{n-1}(f_n^{-1}(x_p))]\big),$$

$$\lim_{x \to x_p+} \varphi(x) = \varphi(x_p)$$

on account of the continuity of the function $\varphi(x)$ in the interval $\langle x_p, x_{p+1})$.

For $p > 1$

$$\lim_{x \to x_p-} \varphi(x) = \lim_{x \to x_p-} G\big(f_n^{-1}(x), \varphi[f_n^{-1}(x)], \varphi[f_1(f_n^{-1}(x))], \ldots, \varphi[f_{n-1}(f_n^{-1}(x))]\big)$$
$$= G\big(f_n^{-1}(x_p), \varphi[f_n^{-1}(x_p)], \varphi[f_1(f_n^{-1}(x_p))], \ldots, \varphi[f_{n-1}(f_n^{-1}(x_p))]\big) = \varphi(x_p)$$

on account of the continuity of the function $\varphi(x)$ for $x \in \langle x_0, x_p)$ and the continuity of the function G.

For $p = 1$

$$\lim_{x \to x_1-} \varphi(x) = \varphi(x_1)$$

on account of relation (6). The proof for the intervals $\langle x_{-i+1}, x_1)$ is similar.

That the function $\varphi(x)$ satisfies equation (1) follows immediately from relations (7).

Taking as $\bar{\varphi}(x)$ all functions continuous in the interval $\langle x_0, x_1)$ and fulfilling conditions (5) and (6) we obtain all solutions of equation (1) that are continuous in the interval (a, b) and pass through Ω.

Remark I. If we take as $\bar{\varphi}(x)$ all functions defined in $\langle x_0, x_1)$ and fulfilling condition (5), then formulae (7) will define all solutions of equation (1) (passing through Ω) in the interval (a, b). The hypothesis of the continuity of the function F will then be superfluous.

Remark II. Hypothesis (iv) can be replaced by the following hypotheses:

(iv*) The set $\mathcal{F}$ is a simply connected piece of hypersurface.

(iv**) The derivatives F_{y_n} and F_{y_0} exist and are continuous in Ω, $F_{y_n} \neq 0$ and $F_{y_0} \neq 0$ in Ω.

Remark III. If in hypothesis (iii) only the set Ω_n (or only Ω_0) occurs, and in hypothesis (iv) we postulate only the possibility of solving the equation $F = 0$ with respect to the variable y_n (or only with respect to y_0), then the solution of equation (1) can not exist in the whole interval (a, b). Anyhow, as is obvious from the proof, we can then infer the existence of continuous solutions of equation (1) in the interval $\langle a+\varepsilon, b)$ (or in the interval $(a, b-\varepsilon\rangle)$, where ε is an arbitrary positive number. In this case we must choose $x_0 \leqslant a+\varepsilon$ (or $x_0 \geqslant f_n^{-1}(b-\varepsilon)$) in the proof of the theorem.

References

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A simple proof of a certain result of Z. Opial

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1. Suppose that $x(t)$ is of class C^1 for $t \in \langle 0, h\rangle$ $(h > 0)$, and that the following condition holds:

$$x(0) = x(h) = 0. \tag{1}$$

Under the assumptions given above Z. Opial [2] has proved the inequality

$$\int_0^h |x(t)x'(t)|\,dt \leqslant \tfrac{1}{4}h \int_0^h x'^2(t)\,dt. \tag{2}$$

The purpose of this note is to present a simple proof of this result of Opial.

In order to do so we shall use the following consequence of a well-known inequality of Buniakowski (see for example [1], p. 146):

$$\Big(\int_a^b |u(t)|\,dt\Big)^2 \leqslant (b-a)\int_a^b u^2(t)\,dt. \tag{3}$$

Let us observe that Opial, in his proof of (2), also used (3).

2. Denote by $y(t) = \int_0^t |x'(t)|\,dt$ and by $z(t) = \int_t^h |x'(t)|\,dt$. We have the following obvious relations:

$$y'(t) = |x'(t)| = -z'(t), \tag{4}$$

and

$$|x(t)| \leqslant y(t), \qquad |x(t)| \leqslant z(t), \qquad 0 \leqslant t \leqslant h. \tag{5}$$