ANNALES
POLONICI MATHEMATICI
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On the notion of gradient

I. Essentiality of regularity suppositions

by S. Gołąb (Kraków)

The gradient of a scalar field σ is well known to be defined as a vector field the components of which are partial derivatives of the function σ with respect to the independent variables x_i. For a gradient to exist the function σ must be differentiable i. e. there must exist the partial derivatives

(1) $$\frac{\partial\sigma}{\partial x_i} = v_i \qquad (i = 1, 2, \ldots, n).$$

The existence alone of the derivatives (1) is not, however, sufficient for a gradient to exist, since, should (1) represent a field of covariant vectors, the components v_i ought to be, in going over to a new system of coordinates, transformed in accordance to the rule

(2) $$\bar{v}_i = \sum_{j=1}^{n} v_j \frac{\partial x_j}{\partial \bar{x}_i} \qquad (i = 1, 2, \ldots, n),$$

if the transformation of coordinates

(3) $$\bar{x}_i = \varphi_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n)$$

is of the class C^1 (i. e. $\partial\varphi_i/\partial x_k$, $i, k = 1, \ldots, n$, exist and are continuous), and the jacobian J of the transformation (3) is different from zero

(4) $$J = \det\left(\frac{\partial\varphi_i}{\partial x_k}\right) \neq 0.$$

These last assumptions, concerning the regularity of the allowable transformations (3), are made with the view that the set of transformations should form the so-called *pseudogroup* (see e. g. [1]).

Now, for the relations to arise (2), it is necessary and sufficient that

(5) $$\bar{v}_i = \frac{\partial\sigma}{\partial\bar{x}_i} = \sum_{j=1}^{n} \frac{\partial\sigma}{\partial x_j}\cdot\frac{\partial x_j}{\partial\bar{x}_i} \qquad (i = 1, 2, \ldots, n)$$

or simply that there should occur the classic formula for the differentiation of composite functions of many variables, and for this purpose, as it is well known, differentiability alone will not do, whereas continuity of the partial derivatives (1) or differentiability in Stolz-Fréchet sense will already suffice. Differentiability in the Stolz-Fréchet sense is simultaneously, as I showed in 1937 [2], a necessary condition that there arise formula (5) for every differentiable transformation.

Hence comes the conclusion that *in order that, with an established pseudogroup of transformations of the class C^1, formula* (1) *should represent, at a definite point* (x_i) *the components of a (covariant) vector, it is necessary and sufficient for a function to possess at the point* (x_i) *a differential in the Stolz-Fréchet sense.*

The majority of manuals treating of vector analysis are confined only to mentioning the sufficient conditions (class C^1 of the function σ) or assume them tacitly without expressly giving a motivation for the assumption accepted.

Although, in books on analysis, I have come across some examples of differentiable functions for which formula (5) does not hold or which even cease to be differentiable with respect to new variables $\bar{x}_i$, yet I have not met, in manuals on vector analysis, an effective example of a differentiable scalar field for which (1) would not represent a vector field.

It is the aim of this note to give just such a simple example for $n = 2$ (as a matter, of course, the generalization for an arbitrary n does not afford any more difficulties), and along with it I have tried to choose the scalar field σ so that it be continuous (as we know the differentiability does not involve continuity for functions with the number of independent variables greater than one), and moreover, that the partial derivatives be bounded.

So let us assume

(6) $$\sigma(x, y) \overset{\text{df}}{=} \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{for} \quad x^2+y^2 > 0, \\ 0 & \text{for} \quad x = y = 0. \end{cases}$$

The function σ thus defined is sure to be regular outside the origin of the coordinate system, the only questionable thing being its regularity at the point $(0, 0)$.

Now let ε be an arbitrary positive number, and let us consider a neighbourhood of the point $(0, 0)$ satisfying the condition

$$x^2+y^2 < \varepsilon^2.$$

Then we have $\sqrt{x^2+y^2} < \varepsilon$ and consequently

$$|x| < \varepsilon, \quad |y| < \varepsilon.$$

For $x^2+y^2 > 0$ we shall finally have the inequality

$$|\sigma| = \left|\frac{xy^2}{x^2+y^2}\right| = |x| \cdot \frac{y^2}{x^2+y^2} \leqslant |x| < \varepsilon$$

and from this it follows that the function σ is continuous at the point $(0, 0)$. Since further

$$\frac{\sigma(h, 0)-\sigma(0, 0)}{h} = 0 \quad \text{and} \quad \frac{\sigma(0, h)-\sigma(0, 0)}{h} = 0,$$

we have

(7) $$\left(\frac{\partial\sigma}{\partial x}\right)_{x=y=0} = \left(\frac{\partial\sigma}{\partial y}\right)_{x=y=0} = 0.$$

Later on, we have for $x^2+y^2 > 0$

$$\frac{\partial\sigma}{\partial x} = \frac{y^2(y^2-x^2)}{(x^2+y^2)^2}, \quad \frac{\partial\sigma}{\partial y} = \frac{2x^3y}{(x^2+y^2)^2}.$$

Hence we get first

(8) $$\left|\frac{\partial\sigma}{\partial x}\right| = \frac{y^2}{(x^2+y^2)^2}\,|y^2-x^2| \leqslant \frac{y^2(x^2+y^2)}{(x^2+y^2)^2} = \frac{y^2}{x^2+y^2} \leqslant 1,$$

and subsequently we have for $|x| \leqslant |y|$

$$\left|\frac{\partial\sigma}{\partial y}\right| = \frac{2|x|^3|y|}{(x^2+y^2)^2} \leqslant \frac{2y^4}{(x^2+y^2)^2} \leqslant \frac{2y^4}{y^4} = 2,$$

whereas for $|x| > |y|$

$$\left|\frac{\partial\sigma}{\partial y}\right| \leqslant \frac{2x^4}{(x^2+y^2)^2} \leqslant \frac{2x^4}{x^4} = 2$$

or else always

(9) $$\left|\frac{\partial\sigma}{\partial y}\right| \leqslant 2;$$

thus the function σ possesses bounded partial derivatives not only in the neighbourhood of the point $(0, 0)$ but even all over the plane.

It will be shown now that the partial derivatives do not determine the vector field all over the plane. Singularity will obviously arise at the point $(0, 0)$ only. If it were contrary to our assertion, then the vector fixed at the point $(0, 0)$ would, concerning relations (7), be a null vector.

Let us now make some simple (affine) transformation of the coordinates

$$x = u, \quad y = u+v \tag{10}$$

where u, v are new variables. This transformation is allowable, since its jacobian becomes

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 \neq 0. \tag{11}$$

In the new coordinates the function σ will be expressed by a formula

$$\sigma = \frac{u(u+v)^2}{2u^2+2uv+v^2} = \varphi(u, v) \quad \text{for} \quad u^2+v^2 > 0.$$

Obviously

$$\varphi(h, 0) = h^3/2h^2 = h/2,$$

and so (since $\varphi(0, 0) = 0$ because for $u = v = 0$ we have $x = y = 0$)

$$\frac{\varphi(h, 0)-\varphi(0, 0)}{h} = \frac{1}{2}$$

or else

$$\left(\frac{\partial\varphi}{\partial u}\right)_{u=v=0} = \frac{1}{2}, \tag{12}$$

thus, in the system of coordinates (u, v), the vector fixed at the point $(0, 0)$ would have its first component different from zero, and thus it could not be a null vector. In this way the property of the vector to be a null vector would not be an invariant of the regular transformations of coordinates, and we have come to a contradiction in assuming $(\partial\sigma/\partial x, \partial\sigma/\partial y)$ to represent a vector field all over the plane.

References

[1] S. Gołąb, *Über den Begriff der Pseudogruppe von Transformationen*, Math. Ann. 116 (1939), p. 768-780.

[2] — *Sur une condition nécessaire et suffisante pour l'existence d'une différentielle totale*, Ann. Soc. Pol. Math. 16 (1937), p. 31-40.

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On the notion of gradient

II. A certain extremal property of direction of the gradient vector

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If for a given scalar field σ (having at a point p under consideration a total differential or being at that point of the class C^1, i. e. having at that point continuous partial derivatives of the first order (see [1])) we form a gradient v whose components are the partial derivatives

$$v_i = \partial\sigma/\partial x_i \quad (i = 1, \ldots, n), \tag{1}$$

then if the gradient v at the point p differs from zero, the direction of vector v defines that of the maximum increase of the field σ. By this the following is implied: If we take at the point p a fixed arbitrary radius r and its current point m, and if by s we denote the distance m from p and by $\varphi(s)$ the value of the function σ at the point m, then the right-side derivative of the function $\varphi(s)$ for $s = 0$ will be the greatest (and at the same time positive) for the radius r passing through the vector v. It is a well known fact and its proof is simple.

The aim of this note is to give some other similar property of the vector v.

About the function σ we assume that

1. it is continuous in the neighbourhood of the point p,
2. has at the point p a total differential,
3. the vector v with components

$$v_i = (\partial\sigma/\partial x_i)_p \quad (i = 1, \ldots, n)$$

is different from zero, i. e.

$$\sum_{i=1}^{n} |v_i| > 0,$$

4. there exists an infinite sequence of hypersurfaces S_ν with the following properties:

(I) S_ν is a closed $(n-1)$-dimensional surface containing p internally;