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Kobayashi–Royden vs. Hahn pseudometric in $\mathbb{C}^2$

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Abstract. For a domain $D \subset \mathbb{C}$ the Kobayashi–Royden $\varkappa$ and Hahn h pseudometrics are equal iff D is simply connected. Overholt showed that for $D \subset \mathbb{C}^n$, $n \geq 3$, we have $h_D \equiv \varkappa_D$. Let $D_1, D_2 \subset \mathbb{C}$. The aim of this paper is to show that $h_{D_1 \times D_2} \equiv \varkappa_{D_1 \times D_2}$ iff at least one of D_1, D_2 is simply connected or biholomorphic to $\mathbb{C} \setminus \{0\}$. In particular, there are domains $D \subset \mathbb{C}^2$ for which $h_D \not\equiv \varkappa_D$.

1. Introduction. For a domain $D \subset \mathbb{C}^n$, the Kobayashi–Royden pseudometric $\varkappa_D$ and the Hahn pseudometric h_D are defined by the formulas

$$\varkappa_D(z;X) := \inf\{|\alpha| : \exists_{f \in \mathcal{O}(E,D)}\ f(0) = z,\ \alpha f'(0) = X\},$$
$$h_D(z;X) := \inf\{|\alpha| : \exists_{f \in \mathcal{O}(E,D)}\ f(0) = z,\ \alpha f'(0) = X,\ f \text{ is injective}\},$$
$$z \in D,\ X \in \mathbb{C}^n,$$

where E denotes the unit disc (cf. [Roy], [Hah], [Jar-Pfl]). Obviously $\varkappa_D \leq h_D$. It is known that both pseudometrics are invariant under biholomorphic mappings, i.e., if $f : D \to \widetilde{D}$ is biholomorphic, then

$$h_D(z;X) = h_{\widetilde{D}}(f(z); f'(z)(X)),$$
$$\varkappa_D(z;X) = \varkappa_{\widetilde{D}}(f(z); f'(z)(X)), \qquad z \in D,\ X \in \mathbb{C}^n.$$

It is also known that for a domain $D \subset \mathbb{C}$ we have: $h_D \equiv \varkappa_D$ iff D is simply connected. In particular $h_D \not\equiv \varkappa_D$ for $D = \mathbb{C}_* := \mathbb{C} \setminus \{0\}$. It has turned out that $h_D \equiv \varkappa_D$ for any domain $D \subset \mathbb{C}^n$, $n \geq 3$ ([Ove]). The case $n = 2$ was investigated for instance in [Hah], [Ves], [Vig], [Cho], but neither a proof nor a counterexample for the equality was found (existing "counterexamples" were based on incorrect product properties of the Hahn pseudometric).

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2. The main result

Theorem 1. *Let $D_1, D_2 \subset \mathbb{C}$ be domains. Then*:

1. *If at least one of D_1, D_2 is simply connected*, *then $h_{D_1\times D_2} \equiv \varkappa_{D_1\times D_2}$.*
2. *If at least one of D_1, D_2 is biholomorphic to $\mathbb{C}_*$, then $h_{D_1\times D_2} \equiv \varkappa_{D_1\times D_2}$.*
3. *Otherwise $h_{D_1\times D_2} \not\equiv \varkappa_{D_1\times D_2}$.*

Let $p_j : D_j^* \to D_j$ be a holomorphic universal covering of D_j ($D_j^* \in \{\mathbb{C}, E\}$), $j = 1, 2$. Recall that if D_j is simply connected, then $h_{D_j} \equiv \varkappa_{D_j}$. If D_j is not simply connected and D_j is not biholomorphic to $\mathbb{C}_*$, then, by the uniformization theorem, $D_j^* = E$ and p_j is not injective.

Hence, Theorem 1 is an immediate consequence of the following three propositions (we keep the above notation).

Proposition 2. *If $h_{D_1} \equiv \varkappa_{D_1}$, then $h_{D_1\times D_2} \equiv \varkappa_{D_1\times D_2}$ for any domain $D_2 \subset \mathbb{C}$.*

Proposition 3. *If D_1 is biholomorphic to $\mathbb{C}_*$, then $h_{D_1\times D_2} \equiv \varkappa_{D_1\times D_2}$ for any domain $D_2 \subset \mathbb{C}$.*

Proposition 4. *If $D_j^* = E$ and p_j is not injective*, $j = 1, 2$, *then $h_{D_1\times D_2} \not\equiv \varkappa_{D_1\times D_2}$.*

Observe the following property that will be helpful in proving the propositions.

Remark 5. For any domain $D \subset \mathbb{C}^n$ we have $h_D \equiv \varkappa_D$ iff for any $f \in \mathcal{O}(E, D)$ with $f'(0) \neq 0$, and $\vartheta \in (0,1)$, there exists an injective $g \in \mathcal{O}(E, D)$ such that $g(0) = f(0)$ and $g'(0) = \vartheta f'(0)$.

Proof of Proposition 2. Let $f = (f_1, f_2) \in \mathcal{O}(E, D_1 \times D_2)$ and let $\vartheta \in (0, 1)$.

First, consider the case where $f_1'(0) \neq 0$. By Remark 5, there exists an injective function $g_1 \in \mathcal{O}(E, D_1)$ such that $g_1(0) = f_1(0)$ and $g_1'(0) = \vartheta f_1'(0)$. Put $g(z) := (g_1(z), f_2(\vartheta z))$.

Obviously $g \in \mathcal{O}(E, D_1 \times D_2)$ and g is injective. Moreover, $g(0) = f(0)$ and $g'(0) = (g_1'(0), f_2'(0)\vartheta) = (\vartheta f_1'(0), \vartheta f_2'(0)) = \vartheta f'(0)$.

Suppose now that $f_1'(0) = 0$. Take $0 < d < \operatorname{dist}(f_1(0), \partial D_1)$ and put

$$h(z) := \frac{f_2(\vartheta z) - f_2(0)}{f_2'(0)}, \qquad M := \max\{|h(z)| : z \in \overline{E}\},$$
$$g_1(z) := f_1(0) + \frac{d}{M+1}(h(z) - \vartheta z),$$
$$g(z) := (g_1(z), f_2(\vartheta z)), \qquad z \in E.$$

Obviously $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$. Since $|g_1(z) - f_1(0)| < d$, we get $g_1(z) \in B(f_1(0), d) \subset D_1$, $z \in E$. Hence $g \in \mathcal{O}(E, D_1 \times D_2)$. Take $z_1, z_2 \in E$ such that $g(z_1) = g(z_2)$. Then $h(z_1) = h(z_2)$, and consequently $z_1 = z_2$.

Finally,

$$g(0) = (g_1(0), f_2(0)) = \left(f_1(0) + \frac{d}{M+1}h(0), f_2(0)\right) = f(0),$$

$$g'(0) = (g_1'(0), \vartheta f_2'(0)) = \left(\frac{d}{M+1}(h'(0) - \vartheta), \vartheta f_2'(0)\right) = \vartheta f'(0). \ \blacksquare$$

Proof of Proposition 3. We may assume that $D_1 = \mathbb{C}_*$ and $D_2 \neq \mathbb{C}$. Using Remark 5, let $f = (f_1, f_2) \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$ and let $\vartheta \in (0, 1)$. Applying an appropriate automorphism of $\mathbb{C}_*$, we may assume that $f_1(0) = 1$.

For the case where $f_2'(0) = 0$, we apply the above construction to the domains $\widetilde{D}_1 = f_2(0) + \operatorname{dist}(f_2(0), \partial D_2)E$, $\widetilde{D}_2 = \mathbb{C}_*$ and mappings $\widetilde{f}_1 \equiv f_2(0)$, $\widetilde{f}_2 = f_1$.

Now, consider the case where $f_2'(0) \neq 0$ and $\vartheta f_1'(0) = 1$. We put

$$g_1(z) := 1 + z, \quad g(z) := (g_1(z), f_2(\vartheta z)), \quad z \in E.$$

Obviously, $g \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$ and g is injective. We have $g(0) = (1, f_2(0)) = f(0)$ and $g'(0) = (1, \vartheta f_2'(0)) = \vartheta f'(0)$.

In all other cases, let $M := \max\{|f_2(z) : |z| \leq \vartheta\}$. Take a $k \in \mathbb{N}$ such that $|c_k| > M$, where

$$c_k := f_2(0) - k\frac{\vartheta f_2'(0)}{\vartheta f_1'(0) - 1}.$$

Put

$$h(z) := \frac{f_2(\vartheta z) - c_k}{f_2(0) - c_k},$$

$$g_1(z) := (1 + z)h^k(z), \quad g_2(z) := f_2(\vartheta z),$$

$$g(z) := (g_1(z), g_2(z)), \quad z \in E.$$

Obviously, $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$. Since $h(z) \neq 0$, we have $g_1(z) \neq 0$, $z \in E$. Hence $g \in \mathcal{O}(E, \mathbb{C}_* \times D_2)$. Take $z_1, z_2 \in E$ such that $g(z_1) = g(z_2)$. Then $h(z_1) = h(z_2)$, and consequently $z_1 = z_2$.

Finally $g(0) = (h^k(0), f_2(0)) = f(0)$ and

$$\begin{aligned} g'(0) &= (g_1'(0), \vartheta f_2'(0)) = (h^k(0) + kh^{k-1}(0)h'(0), \vartheta f_2'(0)) \\ &= \left(1 + k\frac{\vartheta f_2'(0)}{f_2(0) - c_k}, \vartheta f_2'(0)\right) = (1 + \vartheta f_1'(0) - 1, \vartheta f_2'(0)) = \vartheta f'(0). \ \blacksquare \end{aligned}$$

Proof of Proposition 4. It suffices to show that there exist $\varphi_1, \varphi_2 \in \operatorname{Aut}(E)$ and a point $q = (q_1, q_2) \in E^2$, $q_1 \neq q_2$, such that $p_j(\varphi_j(q_1)) = p_j(\varphi_j(q_2))$, $j = 1, 2$, and $\det[(p_j \circ \varphi_j)'(q_k)]_{j,k=1,2} \neq 0$.

Indeed, put $\widetilde{p}_j := p_j \circ \varphi_j$, $j = 1, 2$, and suppose that $h_{D_1 \times D_2} \equiv \varkappa_{D_1 \times D_2}$. Put $a := (\widetilde{p}_1(0), \widetilde{p}_2(0))$ and $X := (\widetilde{p}_1'(0), \widetilde{p}_2'(0)) \in (\mathbb{C}_*)^2$. Take an arbitrary $f \in \mathcal{O}(E, D_j)$ with $f(0) = a_j$. Let $\widetilde{f}$ be the lifting of f with respect to $\widetilde{p}_j$ such that $\widetilde{f}(0) = 0$. Since $|\widetilde{f}'(0)| \le 1$, we get $|f'(0)| \le |X_j|$. Consequently $\varkappa_{D_j}(a_j; X_j) = 1$, $j = 1, 2$. In particular, $\varkappa_{D_1 \times D_2}(a; X) = \max\{\varkappa_{D_1}(a_1; X_1), \varkappa_{D_2}(a_2; X_2)\} = 1$.

Let $(0, 1) \ni \alpha_n \nearrow 1$. Fix an $n \in \mathbb{N}$. Since $\varkappa_{D_1 \times D_2}(a; X) = 1$, there exists $f_n \in \mathcal{O}(E, D_1 \times D_2)$ such that $f_n(0) = a$ and $f_n'(0) = \alpha_n X$. By Remark 5, there exists an injective holomorphic mapping $g_n = (g_{n,1}, g_{n,2}) : E \to D_1 \times D_2$ such that $g_n(0) = a$ and $g_n'(0) = \alpha_n^2 X$. Let $\widetilde{g}_{n,j}$ be the lifting with respect to $\widetilde{p}_j$ of $g_{n,j}$ with $\widetilde{g}_{n,j}(0) = 0$, $j = 1, 2$.

By the Montel theorem, we may assume that the sequence $(\widetilde{g}_{n,j})_{n=1}^{\infty}$ is locally uniformly convergent, $\widetilde{g}_{0,j} := \lim_{n\to\infty} \widetilde{g}_{n,j}$. We have $\widetilde{g}_{0,j}'(0) = 1$, $\widetilde{g}_{0,j} : E \to E$. By the Schwarz lemma we have $\widetilde{g}_{0,j} = \mathrm{id}_E$, $j = 1, 2$.

Let $h_{0,j}(z_1, z_2) := \widetilde{p}_j(z_1) - \widetilde{p}_j(z_2)$, $(z_1, z_2) \in E^2$, and

$$V_j = V(h_{0,j}) = \{(z_1, z_2) \in E^2 : h_{0,j}(z_1, z_2) = 0\}, \qquad j = 1, 2.$$

Since

$$\det \left[\frac{\partial h_{0,j}}{\partial z_k}(q)\right]_{j,k=1,2} = -\det[\widetilde{p}_j'(q_k)]_{j,k=1,2} \neq 0,$$

V_1 and V_2 intersect transversally at q. Let $U \subset\subset \{(z_1, z_2) \in E^2 : z_1 \neq z_2\}$ be a neighborhood of q such that $V_1 \cap V_2 \cap \overline{U} = \{q\}$. For $n \in \mathbb{N}$, $j = 1, 2$, define

$$h_{n,j}(z_1, z_2) := g_{n,j}(z_1) - g_{n,j}(z_2), \qquad (z_1, z_2) \in E^2.$$

Observe that the sequence $(h_{n,j})_{n=1}^{\infty}$ converges uniformly on $\overline{U}$ to $h_{0,j}$, $j = 1, 2$. In particular (cf. [Two-Win]), we have $V(h_{n,1}) \cap V(h_{n,2}) \cap \overline{U} = \{z \in \overline{U} : h_{n,1}(z) = h_{n,2}(z) = 0\} \neq \emptyset$ for some $n \in \mathbb{N}$—contradiction.

We now move to the construction of φ_1, φ_2 and q. Let $\psi_j \in \mathrm{Aut}(E)$ be a nonidentity lifting of p_j with respect to p_j ($p_j \circ \psi_j \equiv p_j$, $\psi_j \not\equiv \mathrm{id}$), $j = 1, 2$. Observe that ψ_j has no fixed points (a lifting is uniquely determined by its value at one point), $j = 1, 2$.

To simplify notation, let

$$h_a(z) := \frac{z - a}{1 - \overline{a}z}, \qquad a, z \in E.$$

One can easily check that

$$\sup_{z \in E} m(z, \psi_j(z)) = 1, \quad j = 1, 2,$$

where

$$m(z, w) := |h_w(z)| = \left|\frac{z - w}{1 - z\overline{w}}\right|$$

is the Möbius distance. Hence there exist $\varepsilon \in (0,1)$ and $z_1, z_2 \in E$ with $m(z_1, \psi_1(z_1)) = m(z_2, \psi_2(z_2)) = 1 - \varepsilon$. Let $d \in (0,1)$, $h_1, h_2 \in \operatorname{Aut}(E)$ be such that $h_j(-d) = z_j$, $h_j(d) = \psi_j(z_j)$, $j = 1, 2$.

If $(p_j \circ h_j)'(-d) \neq \pm(p_j \circ h_j)'(d)$ for some j (we may assume $j = 1$), then at least one of the determinants

$$\det \begin{bmatrix} (p_1 \circ h_1)'(-d) & (p_1 \circ h_1)'(d) \\ (p_2 \circ h_2)'(-d) & (p_2 \circ h_2)'(d) \end{bmatrix},$$

$$\det \begin{bmatrix} (p_1 \circ h_1 \circ (-\operatorname{id}))'(-d) & (p_1 \circ h_1 \circ (-\operatorname{id}))'(d) \\ (p_2 \circ h_2)'(-d) & (p_2 \circ h_2)'(d) \end{bmatrix},$$

is nonzero.

Otherwise, let

$$\widetilde{\psi}_j = h_j^{-1} \circ \psi_j \circ h_j \quad \text{and} \quad \widetilde{p}_j = p_j \circ h_j, \quad j = 1, 2.$$

Observe that $\widetilde{\psi}_j(-d) = d$ and $(\widetilde{\psi}_j'(-d))^2 = 1$, $j = 1, 2$. Thus, each $\widetilde{\psi}_j$ is either $-\operatorname{id}$ or h_c, where $c = -2d/(1 + d^2)$. The case $\widetilde{\psi}_j = -\operatorname{id}$ is impossible since $\widetilde{\psi}_j$ has no fixed points. By replacing p_j by $\widetilde{p}_j$ and ψ_j by $\widetilde{\psi}_j$, $j = 1, 2$, the proof reduces to the case where $\psi_1 = \psi_2 = h_c =: \psi$ for some $-1 < c < 0$.

We claim that there exists a point $a \in E$ such that if an automorphism $\varphi = \varphi_a \in \operatorname{Aut}(E)$ satisfies $\varphi(a) = \psi(a)$ and $\varphi(\psi(a)) = a$, then $\varphi'(a) \neq \pm\psi'(a)$. Suppose for a moment that such an a has been found. Notice that $\varphi \circ \varphi = \operatorname{id}$ and hence $\varphi'(\psi(a)) = 1/\varphi'(a)$. Put $\varphi_1 := \operatorname{id}$, $\varphi_2 := \varphi$, $q := (a, \psi(a))$. We have

$$\begin{aligned}
&\det \begin{bmatrix} (p_1 \circ \varphi_1)'(a) & (p_1 \circ \varphi_1)'(\psi(a)) \\ (p_2 \circ \varphi_2)'(a) & (p_2 \circ \varphi_2)'(\psi(a)) \end{bmatrix} \\
&\qquad = \det \begin{bmatrix} p_1'(a) & p_1'(\psi(a)) \\ p_2'(\varphi(a))\varphi'(a) & p_2'(\varphi(\psi(a))\varphi'(\psi(a)) \end{bmatrix} \\
&\qquad = \det \begin{bmatrix} (p_1 \circ \psi)'(a) & p_1'(\psi(a)) \\ p_2'(\psi(a))\varphi'(a) & (p_2 \circ \psi)'(a)\frac{1}{\varphi'(a)} \end{bmatrix} \\
&\qquad = \det \begin{bmatrix} p_1'(\psi(a))\psi'(a) & p_1'(\psi(a)) \\ p_2'(\psi(a))\varphi'(a) & p_2'(\psi(a))\psi'(a)\frac{1}{\varphi'(a)} \end{bmatrix} \\
&\qquad = p_1'(\psi(a))p_2'(\psi(a)) \det \begin{bmatrix} \psi'(a) & 1 \\ \varphi'(a) & \frac{\psi'(a)}{\varphi'(a)} \end{bmatrix} \neq 0,
\end{aligned}$$

which finishes the construction.

It remains to find a. First observe that the equality $\varphi_a'(a) = \psi'(a)$ is impossible since then we would have $\varphi_a = \psi$ and consequently $\psi \circ \psi = \operatorname{id}$; contradiction. We only need to find an $a \in E$ such that $\varphi_a'(a) \neq -\psi'(a)$.

One can easily check that

$$\varphi_a = h_{-a} \circ (-\,\mathrm{id}) \circ h_{h_a(\psi(a))} \circ h_a.$$

Direct calculations show that $\varphi_a'(a) = -\psi'(a) \Leftrightarrow a \in \mathbb{R}$. Thus it suffices to take any $a \in E \setminus \mathbb{R}$. ∎

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