Kobayashi–Royden vs. Hahn pseudometric in \( \mathbb{C}^2 \)

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Abstract. For a domain \( D \subset \mathbb{C} \) the Kobayashi–Royden \( \kappa \) and Hahn \( h \) pseudometrics are equal iff \( D \) is simply connected. Overholt showed that for \( D \subset \mathbb{C}^n, n \geq 3 \), we have \( h_D \equiv \kappa_D \). Let \( D_1, D_2 \subset \mathbb{C} \). The aim of this paper is to show that \( h_{D_1 \times D_2} \equiv \kappa_{D_1 \times D_2} \) iff at least one of \( D_1, D_2 \) is simply connected or biholomorphic to \( \mathbb{C} \setminus \{0\} \). In particular, there are domains \( D \subset \mathbb{C}^2 \) for which \( h_D \not\equiv \kappa_D \).

1. Introduction. For a domain \( D \subset \mathbb{C}^n \), the Kobayashi–Royden \( \kappa \) and the Hahn pseudometric \( h \) are defined by the formulas

\[
\kappa_D(z; X) := \inf \{|\alpha| : \exists f \in \mathcal{O}(E, D) \ f(0) = z, \ \alpha f'(0) = X\},
\]

\[
h_D(z; X) := \inf \{|\alpha| : \exists f \in \mathcal{O}(E, D) \ f(0) = z, \ \alpha f'(0) = X, f \text{ is injective}\},
\]

where \( E \) denotes the unit disc (cf. [Roy], [Hah], [Jar-Pfl]). Obviously \( \kappa_D \leq h_D \). It is known that both pseudometrics are invariant under biholomorphic mappings, i.e., if \( f : D \to \tilde{D} \) is biholomorphic, then

\[
h_D(z; X) = h_{\tilde{D}}(f(z); f'(z)(X)),
\]

\[
\kappa_D(z; X) = \kappa_{\tilde{D}}(f(z); f'(z)(X)), \quad z \in D, \ X \in \mathbb{C}^n.
\]

It is also known that for a domain \( D \subset \mathbb{C} \) we have \( h_D \equiv \kappa_D \) iff \( D \) is simply connected. In particular \( h_D \not\equiv \kappa_D \) for \( D = \mathbb{C}_* := \mathbb{C} \setminus \{0\} \). It has turned out that \( h_D \equiv \kappa_D \) for any domain \( D \subset \mathbb{C}^n, n \geq 3 \) ([Ove]). The case \( n = 2 \) was investigated for instance in [Hah], [Ves], [Vig], [Cho], but neither a proof nor a counterexample for the equality was found (existing “counterexamples” were based on incorrect product properties of the Hahn pseudometric).

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2. The main result

**Theorem 1.** Let $D_1, D_2 \subset \mathbb{C}$ be domains. Then:

1. If at least one of $D_1, D_2$ is simply connected, then $h_{D_1 \times D_2} \equiv \mathcal{X}_{D_1 \times D_2}$.
2. If at least one of $D_1, D_2$ is biholomorphic to $\mathbb{C}$, then $h_{D_1 \times D_2} \equiv \mathcal{X}_{D_1 \times D_2}$.
3. Otherwise $h_{D_1 \times D_2} \not\equiv \mathcal{X}_{D_1 \times D_2}$.

Let $p_j : D_j^* \to D_j$ be a holomorphic universal covering of $D_j$ ($D_j^* \subset \{\mathbb{C}, E\}$), $j = 1, 2$. Recall that if $D_j$ is simply connected, then $h_{D_j} \equiv \mathcal{X}_{D_j}$. If $D_j$ is not simply connected and $D_j$ is not biholomorphic to $\mathbb{C}$, then, by the uniformization theorem, $D_j^* = E$ and $p_j$ is not injective.

Hence, Theorem 1 is an immediate consequence of the following three propositions (we keep the above notation).

**Proposition 2.** If $h_{D_1} \equiv \mathcal{X}_{D_1}$, then $h_{D_1 \times D_2} \equiv \mathcal{X}_{D_1 \times D_2}$ for any domain $D_2 \subset \mathbb{C}$.

**Proposition 3.** If $D_1$ is biholomorphic to $\mathbb{C}$, then $h_{D_1 \times D_2} \equiv \mathcal{X}_{D_1 \times D_2}$ for any domain $D_2 \subset \mathbb{C}$.

**Proposition 4.** If $D_j^* = E$ and $p_j$ is not injective, $j = 1, 2$, then $h_{D_1 \times D_2} \not\equiv \mathcal{X}_{D_1 \times D_2}$.

Observe the following property that will be helpful in proving the propositions.

**Remark 5.** For any domain $D \subset \mathbb{C}^n$ we have $h_D \equiv \mathcal{X}_D$ iff for any $f \in \mathcal{O}(E, D)$ with $f'(0) \not\equiv 0$, and $\vartheta \in (0, 1)$, there exists an injective $g \in \mathcal{O}(E, D)$ such that $g(0) = f(0)$ and $g'(0) = \vartheta f'(0)$.

**Proof of Proposition 2.** Let $f = (f_1, f_2) \in \mathcal{O}(E, D_1 \times D_2)$ and let $\vartheta \in (0, 1)$.

First, consider the case where $f'_1(0) \not\equiv 0$. By Remark 5, there exists an injective function $g_1 \in \mathcal{O}(E, D_1)$ such that $g_1(0) = f_1(0)$ and $g'_1(0) = \vartheta f'_1(0)$. Let $g(z) := (g_1(z), f_2(\vartheta z))$.

Obviously $g \in \mathcal{O}(E, D_1 \times D_2)$ and $g$ is injective. Moreover, $g(0) = f(0)$ and $g'(0) = (g'_1(0), f'_2(0) \vartheta) = (\vartheta f'_1(0), \vartheta f'_2(0)) = \vartheta f'(0)$.

Suppose now that $f'_1(0) \equiv 0$. Take $0 < d < \text{dist}(f_1(0), \partial D_1)$ and put

\[
\begin{align*}
    h(z) &:= \frac{f_2(\vartheta z) - f_2(0)}{f'_2(0)}, & M &:= \max\{|h(z)| : z \in E\}, \\
    g_1(z) &:= f_1(0) + \frac{d}{M + 1} (h(z) - \vartheta z), \\
    g(z) &:= (g_1(z), f_2(\vartheta z)), & z \in E.
\end{align*}
\]
Obviously $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$. Since $\|g_1(z) - f_1(0)\| < d$, we get $g_1(z) \in B(f_1(0), d) \subset D_1$, $z \in E$. Hence $g \in \mathcal{O}(E, D_1 \times D_2)$. Take $z_1, z_2 \in E$ such that $g(z_1) = g(z_2)$. Then $h(z_1) = h(z_2)$, and consequently $z_1 = z_2$.

Finally, 
\[ g(0) = (g_1(0), f_2(0)) = \left( f_1(0) + \frac{d}{M+1}h(0), f_2(0) \right) = f(0), \]
\[ g'(0) = (g'_1(0), \psi f'_2(0)) = \left( \frac{d}{M+1}(h'(0) - \psi), \psi f'_2(0) \right) = \psi f'(0). \]

**Proof of Proposition 3.** We may assume that $D_1 = \mathbb{C}$, and $D_2 \neq \mathbb{C}$. Using Remark 5, let $f = (f_1, f_2) \in \mathcal{O}(E, \mathbb{C} \times D_2)$ and let $\psi \in (0, 1)$. Applying an appropriate automorphism of $\mathbb{C}$, we may assume that $f_1(0) = 1$.

For the case where $f'_2(0) = 0$, we apply the above construction to the domains $\tilde{D}_1 = f_2(0) + \text{dist}(f_2(0), \partial D_2)E$, $\tilde{D}_2 = \mathbb{C}$ and mappings $f_1 \equiv f_2(0)$, $f_2 = f_1$.

Now, consider the case where $f'_2(0) \neq 0$ and $\psi f'_2(0) = 1$. We put 
\[ g_1(z) := 1 + z, \quad g(z) := (g_1(z), f_2(\psi z)), \quad z \in E. \]

Obviously, $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$ and $g$ is injective. We have $g(0) = (1, f_2(0)) = f(0)$ and $g'(0) = (1, \psi f'_2(0)) = \psi f'(0)$.

In all other cases, let $M := \max\{|f_2(z) : |z| \leq \psi\}$. Take a $k \in \mathbb{N}$ such that $|c_k| > M$, where
\[ c_k := f_2(0) - k\frac{\partial f_2(0)}{\partial f'_2(0)} - 1. \]

Put
\[ h(z) := \frac{f_2(\psi z) - c_k}{f_2(0) - c_k}, \]
\[ g_1(z) := (1 + z)h^k(z), \quad g_2(z) := f_2(\psi z), \]
\[ g(z) := (g_1(z), g_2(z)), \quad z \in E. \]

Obviously, $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$. Since $h(z) \neq 0$, we have $g_1(z) \neq 0$, $z \in E$. Hence $g \in \mathcal{O}(E, \mathbb{C} \times D_2)$. Take $z_1, z_2 \in E$ such that $g(z_1) = g(z_2)$. Then $h(z_1) = h(z_2)$, and consequently $z_1 = z_2$.

Finally $g(0) = (h(0), f_2(0)) = f(0)$ and
\[ g'(0) = (g'_1(0), \psi f'_2(0)) = (h'(0) + kh^{k-1}(0)h'(0), \psi f'_2(0)) \]
\[ = \left( 1 + k\frac{\partial f_2(0)}{f_2(0) - c_k}, \psi f'_2(0) \right) = \left( 1 + \psi f'(0), \psi f'_2(0) \right) = \psi f'(0). \]

**Proof of Proposition 4.** It suffices to show that there exist $\varphi_1, \varphi_2 \in \text{Aut}(E)$ and a point $q = (q_1, q_2) \in E^2$, $q_1 \neq q_2$, such that $p_j(\varphi_j(q_1)) = p_j(\varphi_j(q_2))$, $j = 1, 2$, and $\det((p_j \circ \varphi_j)'(q_k))_{j,k=1,2} \neq 0$. 


Indeed, put \( \tilde{p}_j := p_j \circ \varphi_j, j = 1, 2 \), and suppose that \( h_{D_1 \times D_2} \equiv \mathcal{R}_{D_1 \times D_2} \). Put \( a := (\tilde{p}_1(0), \tilde{p}_2(0)) \) and \( X := (\tilde{p}_1(0), \tilde{p}_2(0)) \in (\mathbb{C}_s)^2 \). Take an arbitrary \( f \in \mathcal{O}(E, D_j) \) with \( f(0) = a_j \). Let \( \tilde{f} \) be the lifting of \( f \) with respect to \( \tilde{p}_j \) such that \( \tilde{f}(0) = 0 \). Since \( |f'(0)| \leq 1 \), we get \( |f'(0)| \leq |X_j| \). Consequently \( \mathcal{R}_{D_j}(a_j; X_j) = 1, j = 1, 2 \). In particular, \( \mathcal{R}_{D_1 \times D_2}(a; X) = \max\{\mathcal{R}_{D_1}(a_1; X_1), \mathcal{R}_{D_2}(a_2; X_2)\} = 1 \).

Let \( (0, 1) \ni \alpha_n \rightarrow 1 \). Fix an \( n \in \mathbb{N} \). Since \( \mathcal{R}_{D_1 \times D_2}(a; X) = 1 \), there exists \( f_n \in \mathcal{O}(E, D_1 \times D_2) \) such that \( f_n(0) = a \) and \( f_n'(0) = \alpha_n X \). By Remark 5, there exists an injective holomorphic mapping \( g_n = (g_{n,1}, g_{n,2}) : E \rightarrow D_1 \times D_2 \) such that \( g_n(0) = a \) and \( g_n'(0) = \alpha_n X \). Let \( \tilde{g}_{n,j} \) be the lifting with respect to \( \tilde{p}_j \) of \( g_{n,j} \) with \( \tilde{g}_{n,j}(0) = 0, j = 1, 2 \).

By the Montel theorem, we may assume that the sequence \((\tilde{g}_{n,j})_{n=1}^\infty\) is locally uniformly convergent, \( \tilde{g}_{0,j} := \lim_{n \rightarrow \infty} \tilde{g}_{n,j} \). We have \( \tilde{g}_{0,j}(0) = 1, \tilde{g}_{0,j} : E \rightarrow E \). By the Schwarz lemma we have \( \tilde{g}_{0,j} = \text{id}_E, j = 1, 2 \).

Let \( h_{0,j}(z_1, z_2) := \tilde{p}_j(z_1) - \tilde{p}_j(z_2), (z_1, z_2) \in E^2 \), and

\[
V_j = V(h_{0,j}) = \{(z_1, z_2) \in E^2 : h_{0,j}(z_1, z_2) = 0\}, \quad j = 1, 2.
\]

Since

\[
\det \left[ \frac{\partial h_{0,j}}{\partial z_k}(q) \right]_{j,k=1,2} = -\det[\tilde{p}_j'(q_k)]_{j,k=1,2} \neq 0,
\]

\( V_1 \) and \( V_2 \) intersect transversally at \( q \). Let \( U \subseteq \{(z_1, z_2) \in E^2 : z_1 \neq z_2\} \) be a neighborhood of \( q \) such that \( V_1 \cap V_2 \cap \overline{U} = \{q\} \). For \( n \in \mathbb{N}, j = 1, 2 \), define

\[
h_{n,j}(z_1, z_2) := g_{n,j}(z_1) - g_{n,j}(z_2), \quad (z_1, z_2) \in E^2.
\]

Observe that the sequence \((h_{n,j})_{n=1}^\infty\) converges uniformly on \( \overline{U} \) to \( h_{0,j}, j = 1, 2 \). In particular (cf. [Two-Win]), we have \( V(h_{n,1}) \cap V(h_{n,2}) \cap \overline{U} = \{z \in \overline{U} : h_{n,1}(z) = h_{n,2}(z) = 0\} \neq \emptyset \) for some \( n \in \mathbb{N} \)—contradiction.

We now move to the construction of \( \varphi_1, \varphi_2 \) and \( q \). Let \( \psi_j \in \text{Aut}(E) \) be a nonidentity lifting of \( p_j \) with respect to \( p_j \) (\( p_j \circ \psi_j \equiv p_j, \psi_j \neq \text{id} \), \( j = 1, 2 \)). Observe that \( \psi_j \) has no fixed points (a lifting is uniquely determined by its value at one point), \( j = 1, 2 \).

To simplify notation, let

\[
h_a(z) := \frac{z - a}{1 - \overline{a}z}, \quad a, z \in E.
\]

One can easily check that

\[
\sup_{z \in E} m(z, \psi_j(z)) = 1, \quad j = 1, 2,
\]

where

\[
m(z, w) := |h_w(z)| = \left| \frac{z - w}{1 - z\overline{w}} \right|
\]
is the Möbius distance. Hence there exists \( \varepsilon \in (0, 1) \) and \( z_1, z_2 \in E \) with 
\[ m(z_1, \psi_1(z_1)) = m(z_2, \psi_2(z_2)) = 1 - \varepsilon. \]
Let \( d \in (0, 1) \), \( h_1, h_2 \in \text{Aut}(E) \) be such that 
\[ h_j(-d) = z_j, h_j(d) = \psi_j(z_j), \quad j = 1, 2. \]

If \((p_j \circ h_j)'(-d) \neq \pm (p_j \circ h_j)'(d)\) for some \( j \) (we may assume \( j = 1 \)), then

at least one of the determinants
\[
\begin{vmatrix}
(p_1 \circ h_1)'(-d) & (p_1 \circ h_1)'(d) \\
(p_2 \circ h_2)'(-d) & (p_2 \circ h_2)'(d)
\end{vmatrix},
\]
is nonzero.

Otherwise, let
\[
\tilde{\psi}_j = h_j^{-1} \circ \psi_j \circ h_j \quad \text{and} \quad \tilde{p}_j = p_j \circ h_j, \quad j = 1, 2.
\]

Observe that \( \tilde{\psi}_j(-d) = d \) and \((\tilde{\psi}_j'(-d))^2 = 1, j = 1, 2. \) Thus, each \( \tilde{\psi}_j \) is
either \(-\text{id}\) or \( h_c \), where \( c = -2d/(1 + d^2) \). The case \( \tilde{\psi}_j = -\text{id} \) is impossible
since \( \tilde{\psi}_j \) has no fixed points. By replacing \( p_j \) by \( \tilde{p}_j \) and \( \psi_j \) by \( \tilde{\psi}_j \), \( j = 1, 2, \)
the proof reduces to the case where \( \psi_1 = \psi_2 = h_c := \psi \) for some \(-1 < c < 0.\)

We claim that there exists a point \( a \in E \) such that if an automorphism \( \varphi = \varphi_a \in \text{Aut}(E) \) satisfies 
\( \varphi(a) = \psi(a) \) and \( \varphi(\psi(a)) = a \), then \( \varphi'(a) \neq \pm \psi'(a) \). Suppose for a moment that such an \( a \) has been found. Notice that 
\( \varphi \circ \varphi = \text{id} \) and hence \( \varphi'(\psi(a)) = 1/\varphi'(a) \). Put \( \varphi_1 := \text{id}, \varphi_2 := \varphi, \quad q := (a, \psi(a)). \) We have
\[
\begin{vmatrix}
(p_1 \circ \varphi_1)'(a) & (p_1 \circ \varphi_1)'(\psi(a)) \\
(p_2 \circ \varphi_2)'(a) & (p_2 \circ \varphi_2)'(\psi(a))
\end{vmatrix}
\]
\[
= \det \begin{bmatrix}
p_1'(a) & p_1'(\psi(a)) \\
p_2'(\varphi(a)) & p_2'(\varphi(\psi(a))\varphi'(\psi(a)))
\end{bmatrix}
\]
\[
= \det \begin{bmatrix}
(p_1 \circ \psi)'(a) & p_1'(a) \\
p_2'(\varphi(a)) & (p_2 \circ \psi)'(a) \frac{1}{\varphi'(a)}
\end{bmatrix}
\]
\[
= \det \begin{bmatrix}
p_1'(\psi(a)) & p_1'(\psi(a)) \\
p_2'(\psi(a)) & p_2'(\psi(a))\varphi'(a) \frac{1}{\varphi'(a)}
\end{bmatrix}
\]
\[
= p_1'(\psi(a))p_2'(\psi(a)) \det \begin{bmatrix}
\psi'(a) & 1 \\
\psi'(a) & \varphi'(a)
\end{bmatrix} \neq 0,
\]
which finishes the construction.

It remains to find \( a \). First observe that the equality \( \varphi'_a(a) = \psi'(a) \) is impossible since then we would have \( \varphi_a = \psi \) and consequently \( \psi \circ \psi = \text{id}; \)
contradiction. We only need to find an \( a \in E \) such that \( \varphi'_a(a) \neq -\psi'(a) \).
One can easily check that
\[ \varphi_a = h^{-a} \circ (-\text{id}) \circ h_{a(\psi(a))} \circ h_a. \]
Direct calculations show that \[ \varphi'_a(a) = -\psi'(a) \Leftrightarrow a \in \mathbb{R}. \] Thus it suffices to take any \( a \in E \setminus \mathbb{R}. \)

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