

**Multiple positive solutions to singular
boundary value problems for superlinear
second order FDEs**

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Abstract. We study the existence of positive solutions to the singular boundary value problem for a second-order FDE

$$\begin{cases} u'' + q(t)f(t, u(w(t))) = 0, & \text{for almost all } 0 < t < 1, \\ u(t) = \xi(t), & a \leq t \leq 0, \\ u(t) = \eta(t), & 1 \leq t \leq b, \end{cases}$$

where $q(t)$ may be singular at $t = 0$ and $t = 1$, $f(t, u)$ may be superlinear at $u = \infty$ and singular at $u = 0$.

1. Introduction and main results. Boundary value problems (abbr. as BVP) associated with singular second order differential equations have a long history and many different methods and techniques have been used and developed in order to obtain various qualitative properties of their solutions. For details, see, for instance, papers [1–6, 10–13, 20, 22–32] and the references therein. However, there are only a few works on singular boundary value problems for superlinear ODEs (see [1, 2, 23]). As far as the author knows, works on the existence of multiple positive solutions to singular boundary value problems for superlinear ODEs are quite rare.

In the recent years, together with the development of theory of the functional differential equations (abbr. as FDE), more authors have paid attention to BVPs nonsingular second order FDEs such as

$$[p(t)x'(t)]' = f(t, x_t, x(t)),$$

or

$$x''(t) = f(t, x(t), x(\sigma(t)), x'(t), x'(\tau(t)))$$

(for examples, see [5, 17–19, 21, 33, 43]).

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To the best of the author's knowledge, there has not been much work done about positive solutions for singular boundary value problems with deviating arguments [1, 7, 8, 14], although they are of importance in applications. As pointed out in [7], the background for these singular BVPs for FDEs lies in many areas of physics, applied mathematics and variational problems of control theory.

In this paper, we investigate the existence of positive solutions for a singular BVP for second-order FDE of the form

$$(1.1) \quad \begin{cases} u'' + q(t)f(t, u(w(t))) = 0, & \text{for almost all } 0 < t < 1, \\ u(t) = \xi(t), & a \leq t \leq 0, \\ u(t) = \eta(t), & 1 \leq t \leq b. \end{cases}$$

Throughout we assume that

(P_1) w is a continuous function defined on $[0, 1]$ and satisfies

$$p_1 = \inf\{w(t) : 0 \leq t \leq 1\} < 1, \quad q_1 = \sup\{w(t) : 0 \leq t \leq 1\} > 0.$$

Hence the set $E := \{t \in [0, 1] : 0 \leq w(t) \leq 1\}$ is compact and $\text{meas } E > 0$. Moreover, $\text{meas}\{t \in [0, 1] : w(t) = 0 \text{ or } w(t) = 1\} = 0$.

Moreover, we assume

(P_2) ξ and η are continuous functions defined on $[a, 0]$ and $[1, b]$, respectively, where $a := \min\{0, p_1\}$ and $b := \max\{1, q_1\}$; moreover, $\xi(t) > 0$ on $[a, 0)$ and $\eta(t) > 0$ on $(1, b]$, $\xi(0) = \eta(1) = 0$.

In [1], R. P. Agarwal and Donal O'Regan considered the singular boundary value problems

$$(1.2) \quad \begin{cases} u'' + q(t)f(t, u) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases}$$

and

$$(1.3) \quad \begin{cases} u'' + q(t)f(t, u(t-r)) = 0, & t \in (0, 1) \setminus \{r\}, \\ u(t) = \xi(t), & -r \leq t \leq 0, \\ u(1) = 0, & 0 < r < 1, \end{cases}$$

where $q \in C(0, 1)$ with $q > 0$ on $(0, 1)$ and $\int_0^1 q(s) ds < \infty$, and $f : [0, 1] \times (0, \infty) \rightarrow [0, \infty)$ is continuous. Moreover, $\xi(t) > 0$ on $[-r, 0)$ and $\xi(0) = 0$. For example, they considered the singular boundary value problem

$$(1.4) \quad \begin{cases} u'' + \sigma(u^{-a_1} + u^{b_1} + 1) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, & \sigma \geq 0 \text{ a constant,} \end{cases}$$

with $0 \leq a_1 < 1$ and $b_1 \geq 0$, and showed that it has at least one positive solution under a restriction on σ . We notice that they allowed $b_1 > 1$ (the superlinear problem).

Singular boundary value problems for delay differential equations were first examined by Erbe and Kong [7, 8]; they studied the BVP (1.1) for more general boundary data. In [8], it is shown that if

- (A₁) $q(t)f(t, u) : (0, 1) \times (0, \infty) \rightarrow (0, \infty)$ is continuous and integrable on $[0, 1]$ for each fixed $u \in (0, \infty)$;
- (A₂) $f(t, u)$ is decreasing in u for each fixed t ;
- (A₃) $\lim_{u \rightarrow 0^+} f(t, u) = \infty$ uniformly on compact subsets of $(0, 1)$;
- (A₄) $\lim_{u \rightarrow \infty} f(t, u) = 0$ uniformly on compact subsets of $(0, 1)$;
- (A₅) for all $\theta > 0$,

$$0 < \int_0^1 q(t)f(t, \theta g_1(w(t))) dt < \infty,$$

where

$$g_1(t) := \begin{cases} t - a, & a \leq t \leq 1/2, \\ b - t, & 1/2 \leq t \leq b, \end{cases}$$

then (1.1) has at least one positive solution, as an application of a fixed point theorem for mappings that are decreasing with respect to a cone in a Banach space.

As pointed out in [1], assumption (A₄) is very restrictive. In [14], the author and Wang replace (A₄) with the very general condition that f is sublinear at ∞ . Also assumption (A₃) is removed. Moreover, they observe that the dominating function g_1 in (A₅) is not fit for (1.1).

Motivated by the results mentioned above, in this paper we replace (A₄) with a still more general condition (i.e., f may be superlinear at ∞). Also assumption (A₃) is removed and the dominating function g_1 in (A₅) is the same as in [14]. We also relax condition (A₂). For example, it is of interest to discuss the BVP (1.1) with

$$f(t, u) = \sigma(u^{-a_1} + u^{b_1}), \quad \sigma > 0 \text{ a constant,}$$

with $a_1 > 0$ and $b_1 \geq 0$. For $b_1 > 1$ (the superlinear problem), we find that (1.1) has at least two positive solutions under a restriction on σ .

It is also worth remarking here that the boundary data of (1.1) can be replaced by more general boundary data [7, 8, 14] and the existence of multiple positive solutions could again be discussed.

We now introduce the definition of a solution to (1.1).

A function u is said to be a *solution* to (1.1) if (i) u is continuous and nonnegative on $[a, b]$, (ii) $u(t) = \xi(t)$ on $[a, 0]$ and $u(t) = \eta(t)$ on $[1, b]$, (iii) $u'(t)$ exists and is locally absolutely continuous in $(0, 1)$, (iv) $u''(t) = -q(t)f(t, u(w(t)))$ for almost all $t \in (0, 1)$. Furthermore, a solution u is said to be *positive* if $u(t) > 0$ in $(0, 1)$.

The main purpose of this paper is to establish the existence of multiple positive solutions to (1.1). Motivated by the example $f(t, u) = \sigma(u^{-a_1} + u^{b_1})$,

$\sigma > 0$ with $a_1 > 0$ and $b_1 \geq 0$, we establish the following general existence results.

THEOREM 1. *Assume that $f(t, u) = F(t, u) + Q(t, u)$, and*

(H₁) *$q \in C(0, 1)$ is nonnegative and there exist $\alpha, \beta \in [0, 1)$ such that*

$$0 < \int_E s(1-s)q(s) ds \leq \int_0^1 s^\alpha(1-s)^\beta q(s) ds < \infty,$$

where E is determined by (P₂);

(H₂) *$F : [0, 1] \times (0, \infty) \rightarrow [0, \infty)$ is continuous and nonincreasing in $u > 0$;*

(H₃) *for each fixed $u > 0$,*

$$\int_E t(1-t)q(t)F(t, u) dt > 0;$$

(H₄) *for each fixed $\theta > 0$,*

$$\begin{aligned} 0 < \int_0^1 s(1-s)q(s)F(s, u_0(w(s)) + \theta g(w(s))) ds \\ \leq \int_0^1 s^\alpha(1-s)^\beta q(s)F(s, u_0(w(s)) + \theta g(w(s))) ds < \infty, \end{aligned}$$

where $g, u_0 : [a, b] \rightarrow [0, \infty)$ are continuous functions defined by

$$g(t) := \begin{cases} 0, & a \leq t \leq 0, \\ t(1-t), & 0 \leq t \leq 1, \\ 0, & 1 \leq x \leq b, \end{cases} \quad u_0(t) := \begin{cases} \xi(t), & a \leq t \leq 0, \\ 0, & 0 \leq t \leq 1, \\ \eta(t), & 1 \leq t \leq b; \end{cases}$$

(H₅) *$Q : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing in $u > 0$;*

(H₆) *we have*

$$\sup_{p \in (0, \infty)} \frac{p}{\int_0^1 s(1-s)q(s)[F(s, u_0(w(s)) + pg(w(s))) + Q(s, u_0(w(s)) + p)] ds} > 1.$$

Then (1.1) has at least one positive solution.

THEOREM 2. *Assume that (H₁)–(H₅) of Theorem 1 hold, and*

(H₆^{*}) *there is a $p > 0$ such that*

$$\frac{p}{\int_0^1 s(1-s)q(s)[F(s, u_0(w(s)) + pg(w(s))) + Q(s, u_0(w(s)) + p)] ds} > 1;$$

(H₇^{*}) *$\lim_{u \rightarrow \infty} Q(t, u)/u = \infty$ uniformly on compact subsets of $(0, 1)$.*

Then (1.1) has at least two positive solutions.

REMARK 1.1. u_0 is a nonnegative solution to (1.1) with $f \equiv 0$.

REMARK 1.2. If $\lim_{u \rightarrow \infty} Q(t, u)/u = 0$ uniformly on $[0, 1]$, then (H_6) of Theorem 1 holds.

Clearly, our hypotheses allow but do not require $q(t)f(t, u)$ to be singular at $u = 0$, $t = 0$, and $t = 1$.

In this paper, we obtain multiple positive solutions to (1.1) by arguments involving only positivity properties of the Green function and a fixed point theorem in cones. In Section 2, the proof of Theorems 1 and 2 is discussed. Next, some corollaries of Theorems 1 and 2 are given in Section 3. Finally, in Section 4, an example is given to explain the main results.

The proof of Theorems 1 and 2 will be based on an application of the following fixed point theorem due to Krasnosel'skiĭ [16].

THEOREM 3. *Let X be a Banach space, and let $K \subset X$ be a cone in X . Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let*

$$\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$$

be a completely continuous operator such that either

- (i) $\|\Phi y\| \leq \|y\|$ for $y \in K \cap \partial\Omega_1$ and $\|\Phi y\| \geq \|y\|$ for $y \in K \cap \partial\Omega_2$; or
- (ii) $\|\Phi y\| \geq \|y\|$ for $y \in K \cap \partial\Omega_1$ and $\|\Phi y\| \leq \|y\|$ for $y \in K \cap \partial\Omega_2$.

Then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

2. Proof of main results. It follows from (H_6) (or (H_6^*)) that there is a $p > 0$ such that

$$(2.1) \quad \frac{p}{\int_0^1 s(1-s)q(s)[F(s, u_0(w(s)) + pg(w(s))) + Q(s, u_0(w(s)) + p)] ds} > 1.$$

Also, it follows from $(H_1), (H_3)$ and (H_4) that there exists a $\delta \in (0, 1/4)$ such that

$$(2.2) \quad 0 < \int_{E_\delta} s(1-s)q(s) ds < \infty,$$

and

$$(2.3) \quad \begin{aligned} 0 < \int_{E_\delta} s(1-s)q(s)F(s, pg(w(s))) ds \\ \leq \int_{E_\delta} s(1-s)q(s)F(s, p\delta^2) ds < \infty, \end{aligned}$$

where $E_\delta = \{t \in [0, 1] : \delta \leq w(t) \leq 1 - \delta\} \subset E$.

Let $r > 0$ be such that

$$(2.4) \quad r < \min \left\{ p\delta^2, \int_{E_\delta} G(1/2, s)q(s)F(s, p\delta^2) ds \right\},$$

where $G(t, s)$ is the Green function of the BVP $-u'' = 0, u(0) = u(1) = 0$, which is explicitly given by

$$G(t, s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Suppose that u_0 is a solution of (1.1) with $f \equiv 0$ (see (H_4)).

Let $f^*(t, y) = F^*(t, y) + Q(t, y + u_0(w(t)))$, where

$$F^*(t, y) := \begin{cases} F(t, u_0(w(t)) + rg(w(t))) & \text{if } y \leq rg(w(t)), \\ F(t, u_0(w(t)) + y) & \text{if } y \geq rg(w(t)). \end{cases}$$

We now consider the modified boundary value problem

$$(2.5) \quad \begin{cases} y'' + q(t)f^*(t, y(w(t))) = 0, & \text{for almost all } 0 < t < 1, \\ y(t) = 0, & a \leq t \leq 0, \\ y(t) = 0, & 1 \leq t \leq b. \end{cases}$$

If y is a solution of (2.5), then

$$(2.6) \quad y(t) = \begin{cases} 0, & a \leq t \leq 0, \\ \int_0^1 G(t, s)q(s)f^*(s, y(w(s))) ds, & 0 \leq t \leq 1, \\ 0, & 1 \leq t \leq b. \end{cases}$$

Let K be the cone in the Banach space $X = C[a, b]$ defined by

$$K = \{y \in C[a, b] : y(t) \geq g(t)\|y\| \text{ for all } t\},$$

where $\|y\| := \sup\{|y(t)| : a \leq t \leq b\}$.

For $y \in K$, denote by $(\Phi y)(t)$ the right hand side of (2.6). From the definition of $G(t, s)$, we obtain

$$t(1-t)s(1-s) \leq G(t, s) \leq G(s, s) = s(1-s) \quad \text{on } [0, 1] \times [0, 1].$$

Hence, for any $y \in K$, we have

$$\begin{aligned} 0 \leq (\Phi y)(t) &\leq \int_0^1 s(1-s)q(s)f^*(s, y(w(s))) ds, \quad 0 \leq t \leq 1, \\ (\Phi y)(t) &\geq t(1-t) \int_0^1 s(1-s)q(s)f^*(s, y(w(s))) ds \\ &\geq t(1-t)\|\Phi y\|_{[0,1]}, \quad 0 \leq t \leq 1. \end{aligned}$$

where $\|y\|_{[0,1]} = \sup\{|y(t)| : 0 \leq t \leq 1\}$. Since $\|\Phi y\|_{[0,1]} = \|\Phi y\|$, this shows that $\Phi(K) \subset K$ and each fixed point of Φ is a solution to (2.5).

Moreover, we have the following lemma which will be proved at the end of this section.

LEMMA 2.1. $\Phi : K \rightarrow K$ is completely continuous.

We will only give the proof of Theorem 2 since the proof of Theorem 1 can be done in a similar way.

Proof of Theorem 2. Let $y \in K$ with $\|y\| = r$; then $y(t) \geq rg(t)$. It follows from (2.2)–(2.4) and (H_2) that we have (noting that $u_0(w(t)) \equiv 0$ on E)

$$\begin{aligned} (\Phi y)(1/2) &= \int_0^1 G(1/2, s)q(s)f^*(s, y(w(s))) ds \\ &\geq \int_0^1 G(1/2, s)q(s)F(s, u_0(w(s)) + r) ds \\ &\geq \int_{E_\delta} G(1/2, s)q(s)F(s, r) ds \\ &\geq \int_{E_\delta} G(1/2, s)q(s)F(s, p\delta^2) ds \\ &> r = \|y\|, \end{aligned}$$

which implies that

$$(2.7) \quad \|\Phi y\| > \|y\| \quad \forall y \in K \cap \partial\Omega_1,$$

where $\Omega_1 := \{y \in E : \|y\| < r\}$.

Let $y \in K$ with $\|y\| = p$; then $y(t) \geq pg(t) \geq rg(t)$. It follows from (2.1)–(2.4) and (H_1) – (H_6^*) that

$$\begin{aligned} (\Phi y)(t) &= \int_0^1 G(t, s)q(s)f^*(s, y(w(s))) ds \\ &\leq \int_0^1 s(1-s)q(s)f(s, u_0(w(s)) + y(w(s))) ds \\ &\leq \int_0^1 s(1-s)q(s)[F(s, u_0(w(s)) + pg(w(s))) + Q(s, u_0(w(s)) + p)] ds \\ &< p = \|y\|, \end{aligned}$$

which implies that

$$(2.8) \quad \|\Phi y\| < \|y\| \quad \forall y \in K \cap \partial\Omega_2,$$

where $\Omega_2 := \{y \in E : \|y\| < p\}$.

Therefore, from the second part of Theorem 3, we conclude that Φ has a fixed point y in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. It follows from (2.7) and (2.8) that $\|y\| \neq r$ and $\|y\| \neq p$. Thus $0 < r < \|y\| < p$. This shows that y is a positive solution to (2.5).

Since $\lim_{y \rightarrow \infty} Q(s, y)/y = \infty$ on E , there exists an $R > p$ such that

$$(2.9) \quad Q(s, y) \geq My \quad \text{whenever } y > \delta^2 R, \quad s \in E_\delta,$$

where the constant $M > 0$ is chosen so that

$$(2.10) \quad \delta^2 M \int_{E_\delta} G(1/2, s)q(s) ds > 1.$$

Let $\Omega_3 := \{y \in E : \|y\| < R\}$. Since $y \in K$ with $\|y\| = R$ implies that

$$(2.11) \quad y(w(t)) \geq \delta^2 R \quad \text{on } E_\delta,$$

it follows from (2.8)–(2.11) that (noting that $u_0(w(t)) \equiv 0$ on E)

$$\begin{aligned} (\Phi y)(1/2) &= \int_0^1 G(1/2, s)q(s)f^*(s, y(w(s))) ds \\ &\geq \int_{E_\delta} G(1/2, s)q(s)Q(s, y(w(s))) ds \\ &\geq \delta^2 MR \int_{E_\delta} G(1/2, s)q(s) ds > R = \|y\|. \end{aligned}$$

This shows that

$$(2.12) \quad \|\Phi y\| > \|y\| \quad \forall y \in K \cap \partial\Omega_3.$$

Therefore, from (2.8), (2.12) and the first part of Theorem 3, we conclude that Φ has a fixed point y in $K \cap (\bar{\Omega}_3 \setminus \Omega_2)$. It follows from (2.8) and (2.12) that $\|y\| \neq p$ and $\|y\| \neq R$. Thus $0 < p < \|y\| < R$. This shows that y is another positive solution to (2.5).

Consequently, there exist two positive solutions to (2.5): y_1 in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ and y_2 in $K \cap (\bar{\Omega}_3 \setminus \Omega_2)$. Since $0 < r < \|y_1\| < p < \|y_2\|$, and

$$y_i(w(t)) \geq rg(w(t)), \quad i = 1, 2,$$

we have (for $i = 1, 2$)

$$\begin{cases} y_i'' + q(t)f(t, u_0(w(t)) + y_i(w(t))) = 0, & \text{for almost all } 0 < t < 1, \\ y_i(t) = 0, & a \leq t \leq 0, \\ y_i(t) = 0, & 1 \leq t \leq b. \end{cases}$$

This shows that $u_1(t) := u_0(t) + y_1(t)$ and $u_2(t) := u_0(t) + y_2(t)$ are also two positive solutions to (1.1). The proof is therefore complete.

Proof of Lemma 2.1. Let D be a bounded subset of K and $M > 0$ a constant such that $\|y\| \leq M$ for $y \in D$. Then

$$\begin{aligned}
 (2.13) \quad \|\Phi y\| &\leq \int_0^1 s(1-s)q(s)f^*(s, y(w(s))) ds \\
 &\leq \int_0^1 s(1-s)q(s)F(s, u_0(w(s)) + rg(w(s))) ds \\
 &\quad + \int_0^1 s(1-s)q(s)Q(s, u_0(w(s)) + M) ds,
 \end{aligned}$$

which implies the boundedness of $\Phi(D)$.

From (2.6), it is easy to obtain

$$\begin{aligned}
 (2.14) \quad (\Phi y)'(t) &= - \int_0^t sq(s)f^*(s, y(w(s))) ds \\
 &\quad + \int_t^1 (1-s)q(s)f^*(s, y(w(s))) ds, \quad 0 < t < 1.
 \end{aligned}$$

For any $y \in D$ and $0 < t < 1$, it follows from (2.14) that

$$\begin{aligned}
 (2.15) \quad |t^\alpha(1-t)^\beta(\Phi y)'(t)| &\leq \int_0^t s(1-s)^\beta q(s)f^*(s, y(w(s))) ds \\
 &\quad + \int_t^1 s^\alpha(1-s)q(s)f^*(s, y(w(s))) ds \\
 &\leq \int_0^1 s^\alpha(1-s)^\beta q(s)f^*(s, y(w(s))) ds \\
 &\leq \int_0^1 s^\alpha(1-s)^\beta q(s)F(s, u_0(w(s)) + rg(w(s))) ds \\
 &\quad + \int_0^1 s^\alpha(1-s)^\beta q(s)Q(s, u_0(w(s)) + M) ds \\
 &=: C.
 \end{aligned}$$

Then for $0 \leq s, t \leq 1$ we get

$$\begin{aligned}
 |(\Phi y)(t) - (\Phi y)(s)| &\leq \left| \int_s^t (\Phi y)'(x) dx \right| \leq C \left| \int_s^t x^{-\alpha}(1-x)^{-\beta} dx \right| \\
 &\leq C'|t-s|^{1-\gamma}
 \end{aligned}$$

where $\gamma = \max(\alpha, \beta) < 1$ and C' is a positive constant independent of $y \in D$ and $t, s \in [0, 1]$. Since $y(t) = 0$ on $[a, 0] \cup [b, 1]$, $\Phi(D)$ is equicontinuous.

We are now going to prove that the mapping Φ is continuous on D .

Let $\{y_m\}_{m=0}^\infty \subset D$ converge to y_0 uniformly on $[a, b]$ as $m \rightarrow \infty$. From (H_1) – (H_5) we know that

$$\begin{aligned} 0 &\leq q(t)f^*(t, y_m(w(t))) \\ &\leq q(t)F(t, u_0(w(t)) + rg(w(t))) + q(t)Q(t, u_0(w(t)) + M), \end{aligned}$$

and hence

$$0 \leq G(t, s)q(s)f^*(s, y_m(w(s))) \leq s(1 - s)q(s)H(s) \quad \text{on } (0, 1),$$

where $H(t) := F(t, rg(w(t))) + \sup\{Q(t, u_0(w(t)) + y) : 0 \leq y \leq M\}$ and $t(1 - t)q(t)H(t)$ is an integrable function defined on $[0, 1]$. Consequently, we apply the dominated convergence theorem to get

$$\begin{aligned} \lim_{m \rightarrow \infty} (\Phi y_m)(t) &= \lim_{m \rightarrow \infty} \int_0^1 G(t, s)q(s)f^*(s, y_m(w(s))) ds \\ &= \int_0^1 G(t, s)q(s)f^*(s, y_0(w(s))) ds \\ &= (\Phi y_0)(t), \quad t \in [0, 1], \end{aligned}$$

which completes the proof of Lemma 2.1.

3. Some corollaries. If $w(t) = t$, then (1.1) is a singular boundary value problem of the form

$$(3.1) \quad \begin{cases} y'' + q(t)f(t, y) = 0, & 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases}$$

COROLLARY 1. Assume that $f(t, y) = F(t, y) + Q(t, y)$, and

(B₁) $q \in C(0, 1)$ is nonnegative and there exist $\alpha, \beta \in [0, 1)$ such that

$$0 < \int_0^1 s(1 - s)q(s) ds \leq \int_0^1 s^\alpha(1 - s)^\beta q(s) ds < \infty;$$

(B₂) $F : [0, 1] \times (0, \infty) \rightarrow [0, \infty)$ is continuous;

(B₃) $F(t, y)$ is nonincreasing in $y > 0$, for each fixed t ;

(B₄) for each fixed $\theta > 0$,

$$\begin{aligned} 0 &< \int_0^1 s(1 - s)q(s)F(s, \theta s(1 - s)) ds \\ &\leq \int_0^1 s^\alpha(1 - s)^\beta q(s)F(s, \theta s(1 - s)) ds < \infty; \end{aligned}$$

(B₅) $Q : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing in $y > 0$;

(B₆) $\sup_{p \in (0, \infty)} \frac{p}{\int_0^1 s(1 - s)q(s)[F(s, ps(1 - s)) + Q(s, p)] ds} > 1$.

Then (3.1) has at least one positive solution.

COROLLARY 2. Assume that (B_1) – (B_5) of Corollary 1 hold, and

(B_6^*) there is a $p > 0$ such that

$$\frac{p}{\int_0^1 s(1-s)q(s)[F(s, ps(1-s)) + Q(s, p)] ds} > 1;$$

(B_7^*) $\lim_{y \rightarrow \infty} Q(t, y)/y = \infty$ uniformly on compact subsets of $(0, 1)$.

Then (3.1) has at least two positive solutions.

If $w(t) = t - r$, $a = -r$, then (1.1) is a singular boundary value problem of the form

$$(3.2) \quad \begin{cases} u'' + q(t)f(t, u(t-r)) = 0, & t \in (0, 1) \setminus \{r\}, \\ u(t) = \xi(t), & -r \leq t \leq 0, \\ u(1) = 0, & 0 < r < 1, \end{cases}$$

where ξ is a continuous function defined on $[-r, 0]$, $\xi(t) > 0$ on $[-r, 0)$ and $\xi(0) = 0$.

By a slight modification of the proofs of Theorems 1 and 2, we can get the following results.

COROLLARY 3. Assume that $f(t, u) = F(t, u) + Q(t, u)$, and

(K_1) $q \in C(0, 1)$ is nonnegative and there exist $\alpha, \beta \in [0, 1)$ such that

$$0 < \int_r^1 s(1-s)q(s) ds \leq \int_0^1 s^\alpha(1-s)^\beta q(s) ds < \infty;$$

(K_2) $F : [0, 1] \times (0, \infty) \rightarrow [0, \infty)$ is continuous and nonincreasing in $u > 0$;

(K_3) for each fixed $u > 0$, $\int_r^1 (1-t)q(t)F(t, u) dt > 0$.

(K_4) for each fixed $\theta > 0$,

$$\begin{aligned} 0 &< \int_0^r sq(s)F(s, \xi(s-r)) ds + \int_r^1 (1-s)q(s)F(s, \theta(s-r)(1+r-s)) ds \\ &\leq \int_0^r s^\alpha q(s)F(s, \xi(s-r)) ds \\ &\quad + \int_r^1 (1-s)^\beta q(s)F(s, \theta(s-r)(1+r-s)) ds < \infty; \end{aligned}$$

(K_5) $Q : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing in $u > 0$;

(K_6) $\sup_{p \in (0, \infty)} \frac{p}{c + \int_r^1 s(1-s)q(s)[F(s, p(s-r)(1-s+r)) + Q(s, p)] ds} > 1,$
 where $c = \int_0^r s(1-s)q(s)f(s, \xi(s-r)) ds$.

Then (3.2) has at least one positive solution.

COROLLARY 4. Assume that (K_1) – (K_5) of Corollary 3 hold, and

(K_6^*) there is a $p > 0$ such that

$$\frac{p}{c + \int_r^1 s(1-s)q(s)[F(s, p(s-r)(1-s+r)) + Q(s, p)] ds} > 1,$$

where $c = \int_0^r s(1-s)q(s)f(s, \xi(s-r)) ds$;

(K_7^*) $\lim_{u \rightarrow \infty} Q(t, u)/u = \infty$ uniformly on compact subsets of $(0, 1)$.

Then (3.2) has at least two positive solutions.

4. An example. Consider the singular boundary value problem

$$(4.1) \quad \begin{cases} y'' + \sigma(y^{-a_1} + y^{b_1}) = 0, & 0 < t < 1, \\ y(0) = y(1) = 0, & \sigma > 0, \text{ a constant,} \end{cases}$$

with $0 < a_1 < 2$ and $b_1 \geq 0$. Put

$$F(t, y) = \sigma y^{-a_1}, \quad Q(t, y) = \sigma y^{b_1}, \quad q(t) = 1, \quad \max\{a_1 - 1, 0\} < \alpha = \beta < 1.$$

Obviously, (B_1) – (B_5) of Corollaries 1 and 2 are satisfied.

Put

$$(4.2) \quad T(x) := \frac{6x^{\alpha_1+1}}{c_1 + x^{\alpha_1+b_1}}, \quad \text{where } c_1 = 6 \int_0^1 [s(1-s)]^{1-a_1} ds.$$

Corollary 1 shows that (4.1) has at least one positive solution if

$$(4.3) \quad \sigma < \sup_{x \in (0, \infty)} T(x).$$

When $b_1 \leq 1$, (4.1) has at least one positive solution if

- (i) $0 \leq b_1 < 1$, or
- (ii) $b_1 = 1, \sigma < [\int_0^1 s(1-s) ds]^{-1} = 6$.

When $b_1 > 1$ ((B_7^*) of Corollary 2 holds), we can see that

$$T(p) = \sup_{x \in (0, \infty)} T(x), \quad p = \left[\frac{c_1(a_1 + 1)}{b_1 - 1} \right]^{\frac{1}{a_1+b_1}}.$$

By Corollary 2, (4.1) has two positive solutions if $b_1 > 1$ and

$$(4.4) \quad \sigma < T(p) = \frac{6}{\left[\frac{c_1(a_1 + 1)}{b_1 - 1} \right]^{\frac{b_1-1}{a_1+b_1}} + c_1 \left[\frac{c_1(a_1 + 1)}{b_1 - 1} \right]^{-\frac{a_1+1}{a_1+b_1}}}$$

since, if σ satisfies (4.4), we have

$$\frac{p}{\int_0^1 s(1-s)q(s)[F(s, ps(1-s)) + Q(s, p)] ds} > 1.$$

Thus (B_6^*) of Corollary 2 is satisfied.

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