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Newton numbers and residual measures of plurisubharmonic functions

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Abstract. We study the masses charged by $(dd^c u)^n$ at isolated singularity points of plurisubharmonic functions u. This is done by means of the local indicators of plurisubharmonic functions introduced in [15]. As a consequence, bounds for the masses are obtained in terms of the directional Lelong numbers of u, and the notion of the Newton number for a holomorphic mapping is extended to arbitrary plurisubharmonic functions. We also describe the local indicator of u as the logarithmic tangent to u.

1. Introduction. The principal information on local behaviour of a subharmonic function u in the complex plane can be obtained by studying its Riesz measure μ_u. If u has a logarithmic singularity at a point x, the main term of its asymptotics near x is $\mu_u(\{x\})\log|z-x|$. For plurisubharmonic functions u in $\mathbb{C}^n$, $n>1$, the situation is not so simple. The local properties of u are controlled by the current $dd^c u$ (we use the notation $d=\partial+\overline{\partial}$, $d^c=(\partial-\overline{\partial})/(2\pi i)$) which cannot charge isolated points. The trace measure $\sigma_u = dd^c u\wedge\beta_{n-1}$ of this current is precisely the Riesz measure of u; here $\beta_p = (p!)^{-1}(\pi/2)^p(dd^c|z|^2)^p$ is the volume element of $\mathbb{C}^p$. A significant role is played by the *Lelong numbers* $\nu(u,x)$ of the function u at points x:

$$\nu(u,x)=\lim_{r\to 0}\,(\tau_{2n-2}r^{2n-2})^{-1}\sigma_u[B^{2n}(x,r)],$$

where τ_{2p} is the volume of the unit ball $B^{2p}(0,1)$ of $\mathbb{C}^p$. If $\nu(u,x)>0$ then $\nu(u,x)\log|z-x|$ gives an upper bound for $u(z)$ near x; however, the difference between these two functions can be comparable to $\log|z-x|$.

Another important object generated by the current $dd^c u$ is the Monge–Ampère measure $(dd^c u)^n$. For the definition and basic facts on the complex Monge–Ampère operator $(dd^c)^n$ and Lelong numbers, we refer the reader to the books [12], [14] and [8], and for more advanced results, to [2]. Here we

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mention that $(dd^c u)^n$ cannot be defined for all plurisubharmonic functions u, but if $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus K)$ with $K \subset\subset \Omega$, then $(dd^c u)^n$ is well defined as a positive closed current of bidimension $(0,0)$ (or, which is the same, as a positive measure) on Ω. This measure cannot charge pluripolar subsets of $\Omega \setminus K$, and it can have positive masses at points of K, e.g. $(dd^c \log|z|)^n = \delta(0)$, the Dirac measure at 0, $|z| = (\sum |z_j|^2)^{1/2}$. More generally, if $f : \Omega \to \mathbb{C}^N$, $N \ge n$, is a holomorphic mapping with isolated zeros at $x^{(k)} \in \Omega$ of multiplicities m_k, then $(dd^c \log|f|)^n|_{x^{(k)}} = m_k\,\delta(x^{(k)})$. So, the masses of $(dd^c u)^n$ at isolated singularity points of u (the residual measures of u) are of especial importance.

Let a plurisubharmonic function u belong to $L^\infty_{\mathrm{loc}}(\Omega \setminus \{x\})$; its residual mass at the point x will be denoted by $\tau(u,x)$:

$$\tau(u,x) = (dd^c u)^n|_{\{x\}}.$$

The problem under consideration is to estimate this value.

The following well known relation compares $\tau(u,x)$ with the Lelong number $\nu(u,x)$:

$$\tau(u,x) \ge [\nu(u,x)]^n. \tag{1}$$

Equality in (1) means that, roughly speaking, the function $u(z)$ behaves near x as $\nu(u,x)\log|z-x|$. In many cases however relation (1) is not optimal; e.g. for

$$u(z) = \sup\{\log|z_1|^{k_1}, \log|z_2|^{k_2}\}, \qquad k_1 > k_2, \tag{2}$$

we have $\tau(u,0) = k_1k_2 > k_2^2 = [\nu(u,0)]^2$.

As follows from the Comparison Theorem due to Demailly (see Theorem A below), the residual mass is determined by asymptotic behaviour of the function near its singularity, so one needs to find appropriate characteristics for the behaviour. To this end, a notion of local indicator was proposed in [15]. Note that $\nu(u,x)$ can be calculated as

$$\nu(u,x) = \lim_{r\to-\infty} r^{-1} \sup\{v(z) : |z-x| \le e^r\} = \lim_{r\to-\infty} r^{-1}\mathcal{M}(u,x,r),$$

where $\mathcal{M}(u,x,r)$ is the mean value of u over the sphere $|z-x| = e^r$ (see [4]). In [5], the *refined*, or *directional*, *Lelong numbers* were introduced as

$$\begin{aligned} \nu(u,x,a) &= \lim_{r\to-\infty} r^{-1} \sup\{v(z) : |z_k - x_k| \le e^{ra_k},\ 1 \le k \le n\} \\ &= \lim_{r\to-\infty} r^{-1} g(u,x,ra), \end{aligned} \tag{3}$$

where $a = (a_1,\ldots,a_n) \in \mathbb{R}^n_+$ and $g(u,x,b)$ is the mean value of u over the set $\{z : |z_k - x_k| = \exp b_k,\ 1 \le k \le n\}$. For x fixed, the collection $\{\nu(u,x,a)\}_{a\in\mathbb{R}^n_+}$ gives a more detailed information about the function u near x than $\nu(u,x)$ does, so one can expect a more precise bound for $\tau(u,x)$ in terms of the directional Lelong numbers. It was noticed already in [5] that the mean value of u over $\{z : |z_k - x_k| = |\exp y_k|,\ 1 \le k \le n\}$ is a plurisubharmonic function of $y \in \mathbb{C}^n$, Re $y_k \ll 0$, so $a \mapsto \nu(u,x,a)$ is a concave function on $\mathbb{R}^n_+$. The idea was developed in [15] where a *local indicator* $\Psi_{u,x}$ of the function u at x was constructed as a plurisubharmonic function in the unit polydisk $D = \{y \in \mathbb{C}^n : |y_k| < 1,\ 1 \le k \le n\}$, given by the formula

$$\Psi_{u,x}(y) = -\nu(u,x,(-\log|y_k|)).$$

It is the largest negative plurisubharmonic function in D whose directional Lelong numbers at 0 coincide with those of u at x, $(dd^c\Psi_{u,x})^n = \tau(\Psi_{u,x},0)\,\delta(0)$, and finally,

$$\tau(u,x) \ge \tau(\Psi_{u,x},0), \tag{4}$$

so the singularity of u at x is controlled by its indicator $\Psi_{u,x}$.

Since $\tau(\Psi_{u,x},0) \ge [\nu(\Psi_{u,x},0)]^n = [\nu(u,x)]^n$, (4) is a refinement of (1). For the function u defined by (2), $\tau(\Psi_{u,0},0) = k_1k_2 = \tau(u,0) > [\nu(u,0)]^2$.

Being a function of quite a simple nature, the indicator can produce effective bounds for residual measures of plurisubharmonic functions. In Theorems 1–3 of the present paper we study the values $N(u,x) := \tau(\Psi_{u,x},0)$, the *Newton numbers* of u at x; the reason for this name is explained below. We obtain, in particular, the following bound for $\tau(u,x)$ (Theorem 4):

$$\tau(u,x) \ge \frac{[\nu(u,x,a)]^n}{a_1\ldots a_n} \quad \forall a \in \mathbb{R}^n_+;$$

it reduces to (1) when $a_1 = \ldots = a_n = 1$. For n plurisubharmonic functions $u_1,\ldots,u_n$ in general position (see the definition below), we estimate the measure $dd^c\Psi_{u_1,x}\wedge\ldots\wedge dd^c\Psi_{u_n,x}$ and prove a similar relation (Theorem 6):

$$dd^cu_1\wedge\ldots\wedge dd^cu_n|_{\{x\}} \ge \frac{\prod_j \nu(u_j,x,a)}{a_1\ldots a_n} \quad \forall a \in \mathbb{R}^n_+. \tag{5}$$

The main tool used to obtain these bounds is the Comparison Theorem due to Demailly. To formulate it we give the following

Definition 1. A q-tuple of plurisubharmonic functions $u_1,\ldots,u_q$ is said to be *in general position* if their unboundedness loci $A_1,\ldots,A_q$ satisfy the following condition: for all choices of indices $j_1 < \ldots < j_k$, $k \le q$, the $(2q-2k+1)$-dimensional Hausdorff measure of $A_{j_1}\cap\ldots\cap A_{j_k}$ equals zero.

Theorem A (Comparison Theorem, [2], Th. 5.9). *Let n-tuples of plurisubharmonic functions $u_1,\ldots,u_n$ and $v_1,\ldots,v_n$ be in general position on a neighbourhood of a point $x \in \mathbb{C}^n$. Suppose that $u_j(x) = -\infty$, $1 \le j \le n$, and*

$$\limsup_{z\to x}\frac{v_j(z)}{u_j(z)} = l_j < \infty.$$

Then

$$dd^c v_1 \wedge \ldots \wedge dd^c v_n|_{\{x\}} \le l_1 \ldots l_n\, dd^c u_1 \wedge \ldots \wedge dd^c u_n|_{\{x\}}.$$

We also obtain a geometric interpretation for the value $N(u,x)$ (Theorem 7). Let $\Theta_{u,x}$ be the set of points $b \in \overline{\mathbb{R}^n_+}$ such that $\nu(u,x,a) \ge \langle b,a\rangle$ for some $a \in \mathbb{R}^n_+$. Then

$$\tau(u,x) \ge N(u,x) = n!\,\mathrm{Vol}(\Theta_{u,x}). \tag{6}$$

In many cases the volume of $\Theta_{u,x}$ can be easily calculated, so (6) gives an effective formula for $N(u,x)$.

To illustrate these results, consider functions $u = \log|f|$, $f = (f_1,\ldots,f_n)$ being an equidimensional holomorphic mapping with an isolated zero at a point x. It is probably the only class of functions whose residual measures were studied in detail before. In this case, $\tau(u,x)$ equals m, the multiplicity of f at x, and

$$\nu(\log|f|,x,a) = I(f,x,a) := \inf\{\langle a,p\rangle : p \in \omega_x\} \tag{7}$$

where

$$\omega_x = \left\{ p \in \mathbb{Z}^n_+ : \sum_j \left| \frac{\partial^p f_j}{\partial z^p}(x) \right| \ne 0 \right\}$$

(see [13]). For polynomials $F : \mathbb{C}^n \to \mathbb{C}$, the value $I(F,x,a)$ is a known object (the *index* of F at x with respect to the weight a) used in number theory (see e.g. [11]).

Relation (4) gives us $m = \tau(\log|f|,x) \ge N(\log|f|,x)$. In general, the value $N(\log|f|,x)$ is not comparable to $m_1 \ldots m_n$ with m_j the multiplicity of the function f_j: for $f(z) = (z_1^2 + z_2, z_2)$ and $x = 0$, $m_1 m_2 = 1 < 2 = N(\log|f|,x) = m$ while for $f(z) = (z_1^2 + z_2, z_2^3)$, $N(\log|f|,x) = 2 < 3 = m_1 m_2 < 6 = m$. A sharper bound for m can be obtained from (5) with $u_j = \log|f_j|$, $1 \le j \le n$. In this case, the left-hand side of (5) equals m, and its right-hand side with $a_1 = \ldots = a_n$ equals $m_1 \ldots m_n$. For both the above examples of the mapping f, the supremum of the right-hand side of (5) over $a \in \mathbb{R}^n_+$ equals m. For $a_1,\ldots,a_n$ rational, relation (5) is a known bound for m via the multiplicities of weighted homogeneous initial Taylor polynomials of f_j with respect to the weights $(a_1,\ldots,a_n)$ ([1], Th. 22.7).

Recall that the convex hull $\Gamma_+(f,x)$ of the set $\bigcup_p \{p + \mathbb{R}^n_+\}$, $p \in \omega_x$, is called the *Newton polyhedron* of $(f_1,\ldots,f_n)$ at x, the union $\Gamma(f,x)$ of the compact faces of the boundary of $\Gamma_+(f,x)$ is called the *Newton boundary* of $(f_1,\ldots,f_n)$ at x, and the value $N_{f,x} = n!\,\mathrm{Vol}(\Gamma_-(f,x))$ with $\Gamma_-(f,x) = \{\lambda t : t \in \Gamma(f,x),\ 0 \le \lambda \le 1\}$ is called the *Newton number* of $(f_1,\ldots,f_n)$ at x (see [10], [1]). The relation

$$m \ge N_{f,x} \tag{8}$$

was established by A. G. Kouchnirenko [9] (see also [1], Th. 22.8). Since $\Theta_{\log|f|,x} = \Gamma_-(f,x)$, (8) is a particular case of (6). It is the reason for calling $N(u,x)$ the Newton number of u at x.

These observations show that the technique of plurisubharmonic functions (and local indicators in particular) is quite a powerful tool to produce, in a unified and simple way, sharp bounds for the multiplicities of holomorphic mappings.

Finally, we obtain a description for the indicator $\Psi_{u,x}(z)$ as the weak limit of the functions $m^{-1}u(x_1+z_1^m,\ldots,x_n+z_n^m)$ as $m\to\infty$ (Theorem 8), so $\Psi_{u,x}$ can be viewed as the tangent (in the logarithmic coordinates) for the function u at x. Using this approach we obtain a sufficient condition, in terms of $\mathcal{C}_{n-1}$-capacity, for the residual mass $\tau(u,x)$ to coincide with the Newton number of u at x (Theorem 9).

2. Indicators and their masses. We will use the following notations. For a domain Ω of $\mathbb{C}^n$, $\mathrm{PSH}(\Omega)$ will denote the class of all plurisubharmonic functions on Ω, $\mathrm{PSH}_-(\Omega)$ the subclass of nonpositive functions, and $\mathrm{PSH}(\Omega,x)=\mathrm{PSH}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega\setminus\{x\})$ with $x\in\Omega$.

Let $D=\{z\in\mathbb{C}^n : |z_k|<1,\ 1\le k\le n\}$ be the unit polydisk, $D^*=\{z\in D : z_1\cdot\ldots\cdot z_n\ne 0\}$, and $\mathbb{R}^n_\pm=\{t\in\mathbb{R}^n : \pm t_k>0\}$. By $\mathrm{CNVI}_-(\mathbb{R}^n_-)$ we denote the collection of all nonpositive convex functions on $\mathbb{R}^n_-$ increasing in each variable t_k. The mapping $\mathrm{Log} : D^*\to\mathbb{R}^n_-$ is defined as $\mathrm{Log}(z)=(\log|z_1|,\ldots,\log|z_n|)$, and $\mathrm{Exp} : \mathbb{R}^n_-\to D^*$ is given by $\mathrm{Exp}(t)=(\exp t_1,\ldots,\exp t_n)$.

A function u on D^* is called *n-circled* if

$$u(z)=u(|z_1|,\ldots,|z_n|), \tag{9}$$

i.e. if $\mathrm{Log}^*\,\mathrm{Exp}^*\,u=u$. Any n-circled function $u\in\mathrm{PSH}_-(D^*)$ has a unique extension to the whole polydisk D keeping the property (9). The class of such functions will be denoted by $\mathrm{PSH}^c_-(D)$. The cones $\mathrm{CNVI}_-(\mathbb{R}^n_-)$ and $\mathrm{PSH}^c_-(D)$ are isomorphic: $u\in\mathrm{PSH}^c_-(D)\Leftrightarrow\mathrm{Exp}^*\,u\in\mathrm{CNVI}_-(\mathbb{R}^n_-)$, $h\in\mathrm{CNVI}_-(\mathbb{R}^n_-)\Leftrightarrow\mathrm{Log}^*\,h\in\mathrm{PSH}^c_-(D)$.

DEFINITION 2 (see [15]). A function $\Psi\in\mathrm{PSH}^c_-(D)$ is called an *indicator* if its convex image $\mathrm{Exp}^*\,\Psi$ satisfies

$$\mathrm{Exp}^*\,\Psi(ct)=c\,\mathrm{Exp}^*\,\Psi(t)\qquad\forall c>0,\ \forall t\in\mathbb{R}^n_-. \tag{10}$$

The collection of all indicators will be denoted by I. It is a convex subcone of $\mathrm{PSH}^c_-(D)$, closed in $\mathcal{D}'$ (or equivalently, in $L^1_{\mathrm{loc}}(D)$). Moreover, if Ψ_1, $\Psi_2\in I$ then also $\sup\{\Psi_1,\Psi_2\}\in I$.

Every indicator is locally bounded in D^*. In what follows we will often consider indicators locally bounded in $D\setminus\{0\}$; the class of such indicators will be denoted by I_0: $I_0=I\cap\mathrm{PSH}(D,0)$.

An example of indicators can be given by the functions

$$\varphi_a(z) = \sup_k a_k \log|z_k|, \qquad a_k \ge 0,$$

("simple" indicators). If all $a_k > 0$, then $\varphi_a \in I_0$.

PROPOSITION 1. *Let* $\Psi \in I_0$, $\Psi \not\equiv 0$. *Then*

(a) *there exist reals* $\nu_1, \ldots, \nu_n > 0$ *such that*

$$\Psi(z) \ge \varphi_\nu(z) \qquad \forall z \in D \tag{11}$$

with φ_ν *the simple indicator corresponding to* $\nu = (\nu_1, \ldots, \nu_n)$;

(b) $\Psi \in C(\overline{D} \setminus \{0\})$, $\Psi|_{\partial D} = 0$;

(c) *the directional Lelong numbers* $\nu(\Psi, 0, a)$ *of* Ψ *at the origin with respect to* $a \in \mathbb{R}^n_+$ (see (3)) *are*

$$\nu(\Psi, 0, a) = -\Psi(\mathrm{Exp}(-a)), \tag{12}$$

and its Lelong number is $\nu(\Psi, 0) = -\Psi(e^{-1}, \ldots, e^{-1})$;

(d) $(dd^c\Psi)^n = 0$ *on* $D \setminus \{0\}$.

Proof. Let $\Psi_k(z_k)$ denote the restriction of the indicator $\Psi(z)$ to the disk $D^{(k)} = \{z \in D : z_j = 0\, \forall j \ne k\}$. By monotonicity of $\mathrm{Exp}^* \Psi$, $\Psi(z) \ge \Psi_k(z_k)$. Since Ψ_k is a nonzero indicator in the disk $D^{(k)} \subset \mathbb{C}$, $\Psi_k(z_k) = \nu_k \log|z_k|$ with some $\nu_k > 0$, and (a) follows.

As $\mathrm{Exp}^* \Psi \in C(\mathbb{R}^n_-)$, we have $\Psi \in C(D^*)$. Its continuity in $D \setminus \{0\}$ can be shown by induction on n. For $n = 1$ it is obvious, so assuming it for $n \le l$, consider any point $z^0 \ne 0$ with $z^0_j = 0$. Let $z^s \to z^0$; then the points $\widetilde{z}^s$ with $\widetilde{z}^s_j = 0$ and $\widetilde{z}^s_m = z^s_m$, $m \ne j$, also tend to z^0, and by the induction hypothesis, $\Psi(\widetilde{z}^s) \to \Psi(\widetilde{z}^0) = \Psi(z^0)$. So, $\liminf_{s\to\infty} \Psi(z^s) \ge \lim_{s\to\infty} \Psi(\widetilde{z}^s) = \Psi(z^0)$, i.e. Ψ is lower semicontinuous and hence continuous at z^0. Continuity of Ψ up to ∂D and the boundary condition follow from (11).

Equality (12) is an immediate consequence of the definition of the directional Lelong numbers (3) and the homogeneity condition (10). The relation $\nu(u, x) = \nu(u, x, (1, \ldots, 1))$ [5] gives us the desired expression for $\nu(\Psi, 0)$.

Finally, statement (d) follows from the homogeneity condition (10) (see [15], Proposition 4).

For functions $\Psi \in I_0$, the complex Monge–Ampère operator $(dd^c\Psi)^n$ is well defined and gives a nonnegative measure on D. By Proposition 1,

$$(dd^c\Psi)^n = \tau(\Psi)\delta(0)$$

with some constant $\tau(\Psi) \ge 0$ which is strictly positive unless $\Psi \equiv 0$. In this section, we will study the value $\tau(\Psi)$.

An upper bound for $\tau(\Psi)$ is given by

PROPOSITION 2. *For every* $\Psi \in I_0$,

$$\tau(\Psi) \le \nu_1 \ldots \nu_n \tag{13}$$

with $\nu_1, \ldots, \nu_n$ *as in Proposition* 1(a).

Proof. Since all $\nu_k > 0$, the simple indicator φ_ν is in I_0, and (11) implies

$$\limsup_{z\to 0} \frac{\Psi(z)}{\varphi_\nu(z)} \le 1,$$

so (13) follows by Theorem A.

To obtain a lower bound for $\tau(\Psi)$, we need a relation between $\Psi(z)$ and $\Psi(z^0)$ for $z, z^0 \in D$. Define

$$\Phi(z, z^0) = \sup_k \frac{\log|z_k|}{|\log|z_k^0||}, \qquad z \in D,\ z^0 \in D^*.$$

When considered as a function of z with z^0 fixed, $\Phi(z, z^0)$ is in I_0.

PROPOSITION 3. *For any* $\Psi \in I$, *we have* $\Psi(z) \le |\Psi(z^0)|\Phi(z, z^0)$ *for all* $z \in D$, $z^0 \in D^*$.

Proof. For a fixed $z^0 \in D^*$ and $t^0 = \mathrm{Log}(z^0)$, define $u = |\Psi(z^0)|^{-1}\,\mathrm{Exp}^*\,\Psi$ and $v = \mathrm{Exp}^*\,\Phi = \sup_k t_k/|t_k^0|$. It suffices to establish the inequality $u(t) \le v(t)$ for all $t \in \mathbb{R}^n_-$ with $t_k^0 < t_k < 0$, $1 \le k \le n$. Given such a t, define $\lambda_0 = [1 + v(t)]^{-1}$. Since $\{t^0 + \lambda(t - t^0) : 0 \le \lambda \le \lambda_0\} \subset \overline{\mathbb{R}^n_-}$, the functions $u_t(\lambda) := u(t^0 + \lambda(t - t^0))$ and $v_t(\lambda) := v(t^0 + \lambda(t - t^0))$ are well defined on $[0, \lambda_0]$. Furthermore, u_t is convex and v_t is linear there, $u_t(0) = v_t(0) = -1$, $u_t(\lambda_0) \le v_t(\lambda_0) = 0$. This implies $u_t(\lambda) \le v_t(\lambda)$ for all $\lambda \in [0, \lambda_0]$. In particular, as $\lambda_0 > 1$, $u(t) = u_t(1) \le v_t(1) = v(t)$, which completes the proof.

Consider now the function

$$P(z) = -\prod_{1\le k\le n} |\log|z_k||^{1/n} \in I. \tag{14}$$

THEOREM 1. *The Monge–Ampère mass* $\tau(\Psi)$ *of any indicator* $\Psi \in I_0$ *has the bound*

$$\tau(\Psi) \ge \left|\frac{\Psi(z^0)}{P(z^0)}\right|^n \qquad \forall z^0 \in D^* \tag{15}$$

where the function P *is defined by* (14).

Proof. By Proposition 3,

$$\frac{\Psi(z)}{\Phi(z, z^0)} \le |\Psi(z^0)| \qquad \forall z \in D,\ z^0 \in D^*.$$

By Theorem A,

$$(dd^c\Psi)^n \le |\Psi(z^0)|^n (dd^c\Phi(z, z^0))^n,$$

and the statement follows from the fact that

$$(dd^c\Phi(z,z^0))^n = \prod_{1\le k\le n} |\log|z_k^0||^{-1} = |P(z^0)|^{-n}.$$

REMARKS. 1. One can consider the value

$$A_\Psi = \sup_{z\in D}\left|\frac{\Psi(z)}{P(z)}\right|^n; \tag{16}$$

by Theorem 1,

$$\tau(\Psi) \ge A_\Psi. \tag{17}$$

2. Let $I_{0,M} = \{\Psi \in I_0 : \tau(\Psi) \le M\}$, $M > 0$. Then (15) gives a lower bound for the class $I_{0,M}$:

$$\Psi(z) \ge M^{1/n} P(z) \quad \forall z \in D,\ \forall \Psi \in I_{0,M}.$$

Let now $\Psi_1,\ldots,\Psi_n \in I$ be in general position in the sense of Definition 1. Then the current $\bigwedge_k dd^c\Psi_k$ is well defined, as is $(dd^c\Psi)^n$ with $\Psi = \sup_k \Psi_k$. Moreover, we have

PROPOSITION 4. *If $\Psi_1,\ldots,\Psi_n \in I$ are in general position, then*

$$\bigwedge_k dd^c\Psi_k = 0 \quad \text{on } D\setminus\{0\}. \tag{18}$$

Proof. For $\Psi_1,\ldots,\Psi_n \in I_0$, the statement follows from Proposition 1(d) and the polarization formula

$$\bigwedge_k dd^c\Psi_k = \frac{(-1)^n}{n!}\sum_{j=1}^n (-1)^j \sum_{1\le i_1<\ldots<i_j\le n}\Big(dd^c\sum_{k=1}^j \Psi_{j_k}\Big)^n. \tag{19}$$

When the only condition on $\{\Psi_k\}$ is to be in general position, we can replace $\Psi_k(z)$ with $\Psi_{k,N}(z) = \sup\{\Psi_k(z), N\sup_j \log|z_j|\} \in I_0$ for which $\bigwedge_k dd^c\Psi_{k,N} = 0$ on $D\setminus\{0\}$. Since $\Psi_{k,N} \searrow \Psi_k$ as $N\to\infty$, this gives us (18).

The mass of $\bigwedge_k dd^c\Psi_k$ will be denoted by $\tau(\Psi_1,\ldots,\Psi_n)$.

THEOREM 2. *Let $\Psi_1,\ldots,\Psi_n \in I$ be in general position, $\Psi = \sup_k \Psi_k$. Then*

(a) $\tau(\Psi) \le \tau(\Psi_1,\ldots,\Psi_n)$;

(b) $\tau(\Psi_1,\ldots,\Psi_n) \ge |P(z^0)|^{-n}\prod_k |\Psi_k(z^0)|$ *for all $z^0 \in D^*$, the function P being defined by* (14).

Proof. Since

$$\frac{\Psi(z)}{\Psi_k(z)} \le 1 \quad \forall z \ne 0,$$

statement (a) follows from Theorem A.

Statement (b) results from Proposition 3 exactly as the statement of Theorem 1 does.

3. Geometric interpretation. In this section we study the masses $\tau(\Psi)$ of indicators $\Psi \in I_0$ by means of their convex images $\operatorname{Exp}^* \Psi \in \mathrm{CNVI}_-(\mathbb{R}^n_-)$.

Let $V \in \mathrm{PSH}^c_-(rD) \cap C^2(rD)$, $r < 1$, and $v = \operatorname{Exp}^* V \in \mathrm{CNVI}_-((\mathbb{R}_- + \log r)^n)$. Since

$$\frac{\partial^2 V(z)}{\partial z_j \partial \overline{z}_k} = \frac{1}{4 z_j \overline{z}_k} \cdot \frac{\partial^2 v(t)}{\partial t_j \partial t_k}\bigg|_{t=\mathrm{Log}(z)}, \qquad z \in rD^*,$$

we have

$$\det\left(\frac{\partial^2 V(z)}{\partial z_j \partial \overline{z}_k}\right) = 4^{-n} |z_1 \ldots z_n|^{-2} \det\left(\frac{\partial^2 v(t)}{\partial t_j \partial t_k}\right)\bigg|_{t=\mathrm{Log}(z)}.$$

By setting $z_j = \exp\{t_j + i\theta_j\}$, $0 \le \theta \le 2\pi$, we get $\beta_n(z) = |z_1 \ldots z_n|^2 dt\, d\theta$, so

$$(20)\quad (dd^c V)^n = n! \left(\frac{2}{\pi}\right)^n \det\left(\frac{\partial^2 V}{\partial z_j \partial \overline{z}_k}\right) \beta_n = \frac{n!}{(2\pi)^n} \det\left(\frac{\partial^2 v}{\partial t_j \partial t_k}\right) dt\, d\theta.$$

Every function $U \in \mathrm{PSH}^c_-(D) \cap L^\infty(D)$ is the limit of a decreasing sequence of functions $U_l \in \mathrm{PSH}^c_-(E) \cap C^2(E)$ on an n-circled domain $E \subset\subset D$, and by the convergence theorem for the complex Monge–Ampère operators,

$$(21)\qquad (dd^c U_l)^n|_E \to (dd^c U)^n|_E.$$

On the other hand, for $u_l = \operatorname{Exp}^* U_l$ and $u = \operatorname{Exp}^* U$,

$$(22)\qquad \det\left(\frac{\partial^2 u_l}{\partial t_j \partial t_k}\right) dt\bigg|_{\mathrm{Log}(D^* \cap E)} \to \mathcal{MA}[u]\bigg|_{\mathrm{Log}(D^* \cap E)},$$

the *real* Monge–Ampère operator of u (see [16]).

Since $(dd^c U_l)^n$ and $(dd^c U)^n$ cannot charge pluripolar sets, (20) with $V = U_l$ and (21), (22) imply

$$(dd^c U)^n(E) = n!\,(2\pi)^{-n} \mathcal{MA}[u]\, d\theta\, (\mathrm{Log}(E) \times [0, 2\pi]^n)$$

for any n-circled Borel set $E \subset D$, i.e.

$$(23)\qquad (dd^c U)^n(E) = n!\, \mathcal{MA}[u](\mathrm{Log}(E)).$$

This relation allows us to calculate $\tau(\Psi)$ by using the technique of real Monge–Ampère operators in $\mathbb{R}^n$ (see [16]).

Let $\Psi \in I$. Consider the set

$$(24)\qquad B_\Psi = \{a \in \mathbb{R}^n_+ : \langle a, t\rangle \le \operatorname{Exp}^* \Psi(t)\ \forall t \in \mathbb{R}^n_-\}$$

and define

$$(25)\qquad \Theta_\Psi = \overline{\mathbb{R}^n_+ \setminus B_\Psi}.$$

Clearly, the set B_Ψ is convex, so $\mathrm{Exp}^*\Psi$ is the restriction of its support function to $\mathbb{R}^n_-$. If $\Psi \in I_0$, the set Θ_Ψ is bounded. Indeed, $a \in \Theta_\Psi$ if and only if $\langle a, t^0\rangle \ge \mathrm{Exp}^*\Psi(t^0)$ for some $t^0 \in \mathbb{R}^n_-$, which implies $|a_j| \le |\mathrm{Exp}^*\Psi(t^0)/t^0_j|$ for all j. By Proposition 1(a), $|\mathrm{Exp}^*\Psi(t^0)| \le \nu_j|t_j|$ and therefore $|a_j| \le \nu_j$ for all j.

Given a set $F \subset \mathbb{R}^n$, we denote its Euclidean volume by $\mathrm{Vol}(F)$.

THEOREM 3. *For any indicator $\Psi \in I_0$, we have the relation*

$$\tau(\Psi) = n!\,\mathrm{Vol}(\Theta_\Psi) \tag{26}$$

with the set Θ_Ψ given by (24) *and* (25).

Proof. Define $U(z) = \sup\{\Psi(z), -1\} \in \mathrm{PSH}^c_-(D) \cap C(D)$, $u = \mathrm{Exp}^* U \in \mathrm{CNVI}_-(\mathbb{R}^n_-)$. Since $U(z) = \Psi(z)$ near ∂D,

$$\tau(\Psi) = \int_D (dd^cU)^n.$$

Furthermore, as $(dd^cU)^n = 0$ outside the set $E = \{z \in D : \Psi(z) = -1\}$,

$$\tau(\Psi) = \int_E (dd^cU)^n. \tag{27}$$

In view of (23),

$$\int_E (dd^cU)^n = n! \int_{\mathrm{Log}(E)} \mathcal{MA}[u]. \tag{28}$$

As was shown in [16], for any convex function v in a domain $\Omega \subset \mathbb{R}^n$,

$$\int_F \mathcal{MA}[v] = \mathrm{Vol}(\omega(F, v)) \quad \forall F \subset \Omega, \tag{29}$$

where

$$\omega(F, v) = \bigcup_{t^0 \in F} \{a \in \mathbb{R}^n : v(t) \ge v(t^0) + \langle a, t - t^0\rangle \ \forall t \in \Omega\}$$

is the gradient image of the set F for the surface $\{y = v(x) : x \in \Omega\}$.

We claim that

$$\Theta_\Psi = \omega(\mathrm{Log}(E), u). \tag{30}$$

Observe that

$$\Theta_\Psi = \{a \in \overline{\mathbb{R}^n_+} : \sup_{\psi(t)=-1} \langle a, t\rangle \ge -1\} \quad \text{where} \quad \psi = \mathrm{Exp}^*\Psi.$$

If $a \in \omega(\mathrm{Log}(E), u)$, then for some $t^0 \in \mathbb{R}^n_-$ with $\psi(t^0) = 1$ we have $\langle a, t^0\rangle \ge \langle a, t\rangle$ for all $t \in \mathbb{R}^n_-$ such that $\psi(t) < -1$. Taking here $t_j \to -\infty$ we get $a_j \ge 0$, i.e. $a \in \overline{\mathbb{R}^n_+}$. Moreover, $\langle a, t^0\rangle \ge \langle a, t\rangle - 1 - \psi(t)$ for all $t \in \mathbb{R}^n_-$ with $\psi(t) > -1$, and letting $t \to 0$ we derive $\langle a, t^0\rangle \ge -1$. Therefore, $a \in \Theta_\Psi$ and $\Theta_\Psi \supset \omega(\mathrm{Log}(E), u)$.

Now we prove the converse inclusion. If $a \in \Theta_\Psi \cap \mathbb{R}^n_+$, then

$$\sup\{\langle a, t^0\rangle : t^0 \in \mathrm{Log}(E)\} \ge -1.$$

Let t be such that $\psi(t) = -\delta > -1$. Then $t/\delta \in \mathrm{Log}(E)$ and thus

$$\begin{aligned}\langle a, t\rangle - 1 - \psi(t) &= \delta\langle a, t/\delta\rangle - 1 + \delta \le \delta \sup_{t^0\in\mathrm{Log}(E)} \langle a, t^0\rangle - 1 + \delta \\ &\le \sup_{t^0\in\mathrm{Log}(E)} \langle a, t^0\rangle = \sup_{z^0\in E}\langle a, \mathrm{Log}(z^0)\rangle.\end{aligned}$$

Since E is compact, the latter supremum is attained at some point $\widehat{z}^0$. Furthermore, $\widehat{z}^0 \in E \cap D^*$ because $a_k \ne 0$, $1 \le k \le n$. Hence $\sup_{t^0\in\mathrm{Log}(E)}\langle a, t^0\rangle = \langle a, \widehat{t}^0\rangle$ with $\widehat{t}^0 = \mathrm{Log}(z^0) \in \mathbb{R}^n_-$, so that $a \in \omega(\mathrm{Log}(E), u)$ and $\Theta_\Psi \cap \mathbb{R}^n_+ \subset \omega(\mathrm{Log}(E), u)$. Since $\omega(\mathrm{Log}(E), u)$ is closed, this implies $\Theta_\Psi = \omega(\mathrm{Log}(E), u)$, and (30) follows.

Now relation (26) is a consequence of (27)–(30). The theorem is proved.

Note that the value $\tau(\Psi_1, \ldots, \Psi_n)$ can also be expressed in geometric terms. Namely, if $\Psi_1, \ldots, \Psi_n \in I_0$, the polarization formula (19) gives us, by Theorem 3,

$$\tau(\Psi_1, \ldots, \Psi_n) = (-1)^n \sum_{j=1}^n (-1)^j \sum_{1\le i_1<\ldots<i_j\le n} \mathrm{Vol}(\Theta_{\sum_k \Psi_{j_k}}).$$

We can also give an interpretation for the bound (17). Write A_Ψ from (16) as

$$A_\Psi = \sup_{a\in\mathbb{R}^n_+} \frac{|\psi(-a)|^n}{a_1\ldots a_n} = \sup_{a\in\mathbb{R}^n_+} |\psi(-a/a_1)\ldots\psi(-a/a_n)|, \tag{31}$$

where $\psi = \mathrm{Exp}^*\Psi$. For any $a \in \mathbb{R}^n_+$, the point $a^{(j)}$ whose jth coordinate equals $|\psi(-a/a_j)|$ and the others are zero, has the property $\langle a^{(j)}, -a\rangle = \psi(-a)$. This remains true for every convex combination $\sum \varrho_j a^{(j)}$, and thus $r\sum \varrho_j a^{(j)} \in \Theta_\Psi$ with any $r \in [0, 1]$. Since $(n!)^{-1}|\psi(-a/a_1)\ldots\psi(-a/a_n)|$ is the volume of the simplex generated by the points $0, a^{(1)}, \ldots, a^{(n)}$, we see from (31) that $(n!)^{-1}A_\Psi$ is the supremum of the volumes of all simplices contained in Θ_Ψ.

Moreover, $(n!)^{-1}[\nu(\Psi, 0)]^n$ is the volume of the simplex

$$\{a \in \overline{\mathbb{R}^n_+} : \langle a, (1, \ldots, 1)\rangle \le \nu(\Psi, 0)\} \subset \Theta_\Psi.$$

This is a geometric description for the "standard" bound $\tau(\Psi) \ge [\nu(\Psi, 0)]^n$.

4. Singularities of plurisubharmonic functions. Let u be a plurisubharmonic function in a domain $\Omega \subset \mathbb{C}^n$, and $\nu(u, x, a)$ be its directional Lelong number (3) at $x \in \Omega$ with respect to $a \in \mathbb{R}^n_+$. Fix a point x. It is known [5] that the function $a \mapsto \nu(u,x,a)$ is concave on $\mathbb{R}^n_+$. So, the function

$$\psi_{u,x}(t) := -\nu(u,x,-t), \qquad t \in \mathbb{R}^n_-,$$

belongs to $\mathrm{CNVI}_-(\mathbb{R}^n_-)$ and thus

$$\Psi_{u,x} := \mathrm{Log}^* \psi_{u,x} \in \mathrm{PSH}^c_-(D).$$

Moreover, due to the positive homogeneity of $\nu(u,x,a)$ in a, $\Psi_{u,x} \in I$. The function $\Psi_{u,x}$ was introduced in [15] and called the (*local*) *indicator* of u at x. According to (3),

$$\begin{aligned}\Psi_{u,x}(z) &= \lim_{R\to\infty} R^{-1}\sup\{u(y) : |y_k - x_k| \le |z_k|^R,\ 1 \le k \le n\}\\ &= \lim_{R\to\infty} R^{-1}\frac{1}{(2\pi)^n}\int\limits_{[0,2\pi]^n} u(x_k + |z_k|^R e^{i\theta_k})\, d\theta_1\ldots d\theta_n.\end{aligned}$$

Clearly, $\Psi_{u,x} \equiv 0$ if and only if $\nu(u,x) = 0$. It is easy to see that $\Psi_{\Phi,0} = \Phi$ for any $\Phi \in I$. In particular,

$$\nu(u,x,a) = \nu(\Psi_{u,x},0,a) = -\Psi_{u,x}(\mathrm{Exp}(-a)) \quad \forall a \in \mathbb{R}^n_+. \tag{32}$$

So, the results of the previous sections can be applied to study directional Lelong numbers of arbitrary plurisubharmonic functions.

PROPOSITION 5 (cf. [7], Prop. 5.3). *For any* $u \in \mathrm{PSH}(\Omega)$,

$$\nu(u,x,a) \ge \nu(u,x,b)\min_k \frac{a_k}{b_k} \qquad \forall x \in \Omega,\ \forall a,b \in \mathbb{R}^n_+.$$

Proof. In view of (32), this follows from Proposition 3.

For $r \in \mathbb{R}^n_+$ and $z \in \mathbb{C}^n$, we set $r^{-1} = (r_1^{-1},\ldots,r_n^{-1})$ and $r\cdot z = (r_1z_1,\ldots,r_nz_n)$.

PROPOSITION 6 ([15]). *Any function* $u \in \mathrm{PSH}(\Omega)$ *has the bound*

$$u(z) \le \Psi_{u,x}(r^{-1}\cdot z) + \sup\{u(y) : y \in D_r(x)\} \tag{33}$$

for all $z \in D_r(x) = \{y : |y_k - x_k| \le r_k,\ 1 \le k \le n\} \subset\subset \Omega$.

Proof. Assume for simplicity $x = 0$, $D_r(0) = D_r$.

Consider the function $v(z) = u(r\cdot z) - \sup\{u(y) : y \in D_r\} \in \mathrm{PSH}_-(D)$. The function $g_v(R,t) := \sup\{v(z) : |z_k| \le \exp\{Rt_k\},\ 1 \le k \le n\}$ is convex in $R > 0$ and $t \in \mathbb{R}^n_-$, so as $R \to \infty$,

$$\frac{g_v(R,t) - g_v(R_1,t)}{R - R_1} \nearrow \psi_{v,0}(t), \tag{34}$$

where $\psi_{v,0} = \mathrm{Exp}^*\Psi_{v,0}$.

For $R = 1$ and $R_1 \to 0$, (34) gives us $g_v(1,t) \le \psi_{v,0}(t)$ and thus (33). The proposition is proved.

Let $\Omega_k(x)$ be the connected component of the set $\Omega\cap\{z\in\mathbb{C}^n : z_j = x_j\ \forall j\neq k\}$ containing the point x. If for some $x\in\Omega$, $u|_{\Omega_k(x)}\not\equiv-\infty$ for all k, then $\Psi_{u,x}\in I_0$. For example, this is fulfilled for $u\in \mathrm{PSH}(\Omega,x)$.

If $u\in\mathrm{PSH}(\Omega,x)$, the measure $(dd^cu)^n$ is defined on Ω. Its residual mass at x will be denoted by $\tau(u,x)$:

$$\tau(u,x)=(dd^cu)^n|_{\{x\}}.$$

The indicator $\Psi_{u,x}$ of such a function belongs to the class I_0. Define

$$N(u,x)=\tau(\Psi_{u,x}).$$

PROPOSITION 7 ([15], Th. 1). *If* $u\in\mathrm{PSH}(\Omega,x)$, *then* $\tau(u,x)\geq N(u,x)$.

Proof. Inequality (33) implies

$$\limsup_{z\to x}\frac{\Psi_{u,x}(r^{-1}\cdot(z-x))}{u(z)}\leq 1,$$

and since

$$\lim_{y\to 0}\frac{\Psi_{u,x}(r^{-1}\cdot y))}{\Psi_{u,x}(y)}=1\qquad\forall r\in\mathbb{R}^n_+,$$

the statement follows from Theorem A.

So, to estimate $\tau(u,x)$ we may apply the bounds for $\tau(\Psi_{u,x})$ from the previous section.

THEOREM 4. *If* $u\in\mathrm{PSH}(\Omega,x)$, *then*

$$\tau(u,x)\geq\frac{[\nu(u,x,a)]^n}{a_1\ldots a_n}\qquad\forall a\in\mathbb{R}^n_+;$$

in other words, $\tau(u,x)\geq A_{u,x}$ *where* $A_{u,x}=A_{\Psi_{u,x}}$ *is defined by* (16).

Proof. The result follows from Theorem 1 and Proposition 7.

Let now $u_1,\ldots,u_n\in\mathrm{PSH}(\Omega)$ be in general position in the sense of Definition 1. Then the current $\bigwedge_k dd^cu_k$ is defined on Ω ([2], Th. 2.5); denote its residual mass at a point x by $\tau(u_1,\ldots,u_n;x)$. Moreover, the n-tuple of their indicators $\Psi_{u_k,x}$ is also in general position, which implies

$$\bigwedge_k dd^c\Psi_{u_k,x}=\tau(\Psi_{u_1,x},\ldots,\Psi_{u_n,x})\,\delta(0)$$

(Proposition 4).

In view of Theorem A and Proposition 6 we have

THEOREM 5. *The residual mass* $\tau(u_1,\ldots,u_n;x)$ *of the current* $\bigwedge_k dd^cu_k$ *has the bound* $\tau(u_1,\ldots,u_n;x)\geq\tau(\Psi_{u_1,x},\ldots,\Psi_{u_n,x})$.

Now Theorems 2 and 5 give us

THEOREM 6.

$$\tau(u_1,\ldots,u_n;x)\ge\frac{\prod_j \nu(u_j,x,a)}{a_1\ldots a_n}\quad \forall a\in\mathbb{R}_+^n. \tag{35}$$

REMARK. For $a_1=\ldots=a_n$, inequality (35) is proved in [2], Cor. 5.10.

Finally, by combination of Proposition 7 and Theorem 3 we get

THEOREM 7. *For any function* $u\in \mathrm{PSH}(\Omega,x)$,

$$\tau(u,x)\ge N(u,x)=n!\,V(\Theta_{u,x}) \tag{36}$$

with

$$\Theta_{u,x}=\{b\in\mathbb{R}_+^n:\sup_{\sum a_k=1}[\nu(u,x,a)-\langle b,a\rangle]\ge 0\}.$$

Remark on holomorphic mappings. Let $f=(f_1,\ldots,f_n)$ be a holomorphic mapping of a neighbourhood Ω of the origin into $\mathbb{C}^n$ and $f(0)=0$ be its isolated zero. Then in a subdomain $\Omega'\subset\Omega$ the zero sets A_j of the functions f_j satisfy the conditions

$$A_1\cap\ldots\cap A_n\cap\Omega'=\{0\},\quad \operatorname{codim} A_{j_1}\cap\ldots\cap A_{j_k}\cap\Omega'\ge k$$

for all choices of indices $j_1<\ldots<j_k$, $k\le n$. Set $u=\log|f|$, $u_j=\log|f_j|$. It is known that $\tau(u,0)=\tau(u_1,\ldots,u_n;0)=m_f$, the multiplicity of f at 0. For $a=(1,\ldots,1)$, $\nu(u_j,0,a)$ equals m_j, the multiplicity of f_j at 0. Therefore, (35) with $a=(1,\ldots,1)$ gives us the standard bound $m_f\ge m_1\ldots m_n$.

For a_j rational, (35) is the known estimate of m_f via the multiplicities of weighted homogeneous initial Taylor polynomials for f_j (see e.g. [1], Th. 22.7). Indeed, due to the positive homogeneity of the directional Lelong numbers, we can take $a_j\in\mathbb{Z}_+^n$. Then by (7), $\nu(u_j,0,a)$ is equal to the multiplicity of the function $f_j^{(a)}(z)=f_j(z^a)$.

We also mention that (35) gives a lower bound for the Milnor number $\mu(F,0)$ of a singular point 0 of a holomorphic function F (i.e. for the multiplicity of the isolated zero of the mapping $f=\operatorname{grad}F$ at 0) in terms of the indices $I(F,0,a)$ (see (7)) of F. Since $I(\partial F/\partial z_k,0,a)\ge I(F,0,a)-a_k$, we have

$$\mu(F,0)\ge\prod_{1\le k\le n}\left(\frac{I(F,0,a)}{a_k}-1\right).$$

Finally, it follows from (7) that the set $\overline{\mathbb{R}_+^n\setminus\Theta_{u,0}}$ is the Newton polyhedron for the system $(f_1,\ldots,f_n)$ at 0 (see Introduction). Therefore, $n!\,V(\Theta_{u,0})$ is the Newton number of $(f_1,\ldots,f_n)$ at 0, and (36) becomes the bound for m_f due to A. G. Kouchnirenko (see [1], Th. 22.8). So, for any plurisubharmonic function u, we will call the value $N(u,x)$ the *Newton number* of u at x.

5. Indicators as logarithmic tangents. Let $u \in \mathrm{PSH}(\Omega, 0)$, $u(0) = -\infty$. We will consider the following problem: under what conditions on u, does its residual measure equal its Newton number?

Of course, the relation

$$\lim_{z\to 0} \frac{u(z)}{\Psi_{u,0}(z)} = 1 \tag{37}$$

is sufficient, but it seems to be too restrictive. On the other hand, as the example $u(z) = \log(|z_1 + z_2|^2 + |z_2|^4)$ shows, the condition

$$\lim_{\lambda\to 0} \frac{u(\lambda z)}{\Psi_{u,0}(\lambda z)} = 1 \quad \forall z \in \mathbb{C}^n \setminus \{0\}$$

does not guarantee the equality $\tau(u, 0) = N(u, 0)$.

To weaken (37) we first give another description for the local indicators. In [6], a compact family of plurisubharmonic functions

$$u_r(z) = u(rz) - \sup\{u(y) : |y| < r\}, \quad r > 0,$$

was considered and the limit sets, as $r \to 0$, of such families were described. In particular, the limit set need not consist of a single function, so a plurisubharmonic function can have several (and thus infinitely many) tangents. Here we consider another family generated by a plurisubharmonic function u.

Given $m \in \mathbb{N}$ and $z \in \mathbb{C}^n$, write $z^m = (z_1^m, \dots, z_n^m)$ and set

$$T_m u(z) = m^{-1} u(z^m).$$

Clearly, $T_m u \in \mathrm{PSH}(\Omega \cap D)$ and $T_m u \in \mathrm{PSH}_-(\overline{D}_r)$ for any $r \in \mathbb{R}_+^n \cap D^*$ (i.e. $0 < r_k < 1$) for all $m \ge m_0(r)$.

PROPOSITION 8. *The family* $\{T_m u\}_{m \ge m_0(r)}$ *is compact in* $L^1_{\mathrm{loc}}(D_r)$.

Proof. Let $M(v, \varrho)$ denote the mean value of a function v over the set $\{z : |z_k| = \varrho_k,\ 1 \le k \le n\}$, $0 < \varrho_k \le r_k$. Then $M(T_m u, \varrho) = m^{-1} M(u, \varrho^m)$. The relation

$$m^{-1} M(u, \varrho^m) \nearrow \Psi_{u,0}(\varrho) \quad \text{as } m \to \infty \tag{38}$$

implies $M(T_m u, \varrho) \ge M(T_{m_0} u, \varrho)$. Since $T_m u \le 0$ in D_r, this proves the compactness.

THEOREM 8. (a) $T_m u \to \Psi_{u,0}$ *in* $L^1_{\mathrm{loc}}(D)$;

(b) *if* $u \in \mathrm{PSH}(\Omega, 0)$ *then* $(dd^c T_m u)^n \to \tau(u, 0)\,\delta(0)$.

Proof. Let g be a limit point of the sequence $T_m u$, that is, $T_{m_s} u \to g$ as $s \to \infty$ for some sequence m_s. For the function $v(z) = \sup\{u(y) : |y_k| \le |z_k|,\ 1 \le k \le n\}$ and any $r \in \mathbb{R}_+^n \cap D^*$ we have, by (33),

$$T_m u(z) \le (T_m v)(z) \le \Psi_{u,0}(r^{-1} \cdot z)$$

and thus

$$g(z) \le \Psi_{u,0}(z) \quad \forall z \in D. \tag{39}$$

On the other hand, the convergence of $T_{m_s}u$ to g in L^1 implies $M(T_{m_s}u,r) \to M(g,r)$ ([3], Prop. 4.1.10). By (38), $M(T_{m_s}u,r) \to \Psi_{u,0}(r)$, so $M(g,r) = \Psi_{u,0}(r)$ for every $r \in \mathbb{R}_+^n \cap D^*$. Comparison with (39) gives us $g \equiv \Psi_{u,0}$, and the statement (a) follows.

To prove (b) we observe that for each $\alpha \in (0,1)$,

$$\int\limits_{\alpha D} (dd^c T_m u)^n = \int\limits_{\alpha^m D} (dd^c u)^n \to \tau(u,0)$$

as $m \to \infty$, and for $0 < \alpha < \beta < 1$,

$$\lim_{m\to\infty} \int\limits_{\beta D\setminus\alpha D} (dd^c T_m u)^n = \lim_{m\to\infty} \Big[\int\limits_{\beta^m D} (dd^c u)^n - \int\limits_{\alpha^m D} (dd^c u)^n \Big] = 0.$$

The theorem is proved.

So, Theorem 8 shows us that $\tau(u,0) = N(u,0)$ if and only if $(dd^c T_m u)^n \to (dd^c \Psi_{u,0})^n$. Now we are going to find conditions for this convergence.

Recall the definition of the inner $\mathcal{C}_{n-1}$-capacity introduced in [17]: for any Borel subset E of a domain ω,

$$\mathcal{C}_{n-1}(E,\omega) = \sup \Big\{ \int\limits_E (dd^c v)^{n-1} \wedge \beta_1 : v \in \mathrm{PSH}(\omega),\ 0 < v < 1 \Big\}.$$

It was shown in [17] that convergence of uniformly bounded plurisubharmonic functions v_j to v in $\mathcal{C}_{n-1}$-capacity implies $(dd^c v_j)^n \to (dd^c v)^n$. In our situation, neither $T_m u$ nor $\Psi_{u,0}$ are bounded, so we will modify the construction from [17].

Set

$$E(u,m,\delta) = \Big\{ z \in D \setminus \{0\} : \frac{T_m u(z)}{\Psi_{u,0}(z)} > 1 + \delta \Big\}, \qquad m \in \mathbb{N},\ \delta > 0.$$

THEOREM 9. *Let $u \in \mathrm{PSH}(\Omega,0)$, $\varrho \in (0,1/4)$, $N > 0$, and a sequence $m_s \in \mathbb{N}$ be such that*

1) *$u(z) > -Nm_s$ on a neighbourhood of the sphere $\partial B_{\varrho^{m_s}}$, for each s;*
2) *$\lim_{s\to\infty} \mathcal{C}_{n-1}(B_\varrho \cap E(u,m_s,\delta), D) = 0$ for all $\delta > 0$.*

Then $(dd^c T_m u)^n \to (dd^c \Psi_{u,0})^n$ on D.

Proof. Without loss of generality we can take $u \in \mathrm{PSH}_-(D,0)$. Consider the functions $v_s(z) = \max\{T_{m_s}u(z), -N\}$ and $v = \max\{\Psi_{u,0}(z), -N\}$. We have $v_s = T_{m_s}u$ and $v = \Psi_{u,0}$ on a neighbourhood of ∂B_ϱ, $v_s = v = -N$ on a neighbourhood of 0, $v_s \le v$ on B_ϱ, and $v_s \ge (1+\delta)v$ on $B_\varrho \setminus E(u,m_s,\delta)$.

We will prove that

$$(dd^c v_s)^k \wedge (dd^c v)^l \to (dd^c v)^{k+l} \tag{40}$$

for $k = 1, \dots, n$, $l = 0, \dots, n-k$. This will give us the statement of the theorem. Indeed, by Theorem 8,

$$\int\limits_{B_\varrho} (dd^c v_s)^n = \int\limits_{B_\varrho} (dd^c T_{m_s} u)^n \to \tau(u,0)$$

while

$$\int\limits_{B_\varrho} (dd^c v)^n = \int\limits_{B_\varrho} (dd^c \Psi_{u,0})^n = N(u,0),$$

and (40) with $k = n$ proves the coincidence of the right-hand sides of these relations and thus the convergence of $(dd^c T_m u)^n$ to $(dd^c \Psi_{u,0})^n$.

We prove (40) by induction on k. Let $k = 1$, $0 \le l \le n-1$, $\delta > 0$. For any test form $\phi \in \mathcal{D}_{n-l-1,n-l-1}(B_\varrho)$,

$$\begin{aligned}
&\Big| \int dd^c v_s \wedge (dd^c v)^l \wedge \phi - \int (dd^c v)^{l+1} \wedge \phi \Big| \\
&\quad = \Big| \int (v - v_s)(dd^c v)^l \wedge dd^c \phi \Big| \le C_\phi \int\limits_{B_\varrho} (v - v_s)(dd^c v)^l \wedge \beta_{n-l} \\
&\quad = C_\phi \Big[\int\limits_{B_\varrho \setminus E_{s,\delta}} + \int\limits_{B_\varrho \cap E_{s,\delta}} \Big] (v - v_s)(dd^c v)^l \wedge \beta_{n-l} = C_\phi [I_1(s,\delta) + I_2(s,\delta)],
\end{aligned}$$

where, for brevity, $E_{s,\delta} = E(u, m_s, \delta)$.

We have

$$I_1(s,\delta) \le \delta \int\limits_{B_\varrho} |v| (dd^c v)^l \wedge \beta_{n-l} \le C\delta$$

with a constant C independent of s, and

$$\begin{aligned}
I_2(s,\delta) &\le N \int\limits_{B_\varrho \cap E_{s,\delta}} (dd^c v)^l \wedge \beta_{n-l} \\
&\le C(N, \varrho, l) \cdot \mathcal{C}_{n-1}(B_\varrho \cap E_{s,\delta}, D) \to 0.
\end{aligned}$$

Since $\delta > 0$ is arbitrary, this proves (40) for $k = 1$.

Suppose that we have (40) for $k = j$ and $0 \le l \le n - j$. For $\phi \in \mathcal{D}_{n-l-j,n-l}(B_\varrho)$,

$$\begin{aligned}
\int (dd^c v_s)^{j+1} \wedge (dd^c v)^l \wedge \phi &= \int (dd^c v_s)^j \wedge (dd^c v)^{l+1} \wedge \phi \\
&\quad + \int [(dd^c v_s)^{j+1} \wedge (dd^c v)^l - (dd^c v_s)^j \wedge (dd^c v)^{l+1}] \wedge \phi.
\end{aligned}$$

The first integral on the right-hand side converges to $\int (dd^c v)^{l+j+1} \wedge \phi$ by the induction assumption. The second integral can be estimated similarly to the case $k = 1$:

$$\Big|\int[(dd^c v_s)^{j+1}\wedge(dd^c v)^l - (dd^c v_s)^j\wedge(dd^c v)^{l+1}]\wedge\phi\Big|$$
$$\le C_\phi\Big[\int\limits_{B_\varrho\setminus E_{s,\delta}} + \int\limits_{B_\varrho\cap E_{s,\delta}}\Big](v-v_s)(dd^c v_s)^j(dd^c v)^l\wedge\beta_{n-j-l}$$
$$= C_\phi[I_3(s,\delta)+I_4(s,\delta)].$$

Since $(dd^c v_s)^j\wedge(dd^c v)^l\to(dd^c v)^{j+l}$, we have

$$\int(dd^c v_s)^j(dd^c v)^l\wedge\beta_{n-j-l}\le C\qquad\forall s$$

and

$$I_3(s,\delta)\le\delta\int\limits_{B_\varrho}|v|(dd^c v_s)^j(dd^c v)^l\wedge\beta_{n-j-l}\le CN\delta.$$

Similarly,

$$I_4(s,\delta)\le N\int\limits_{B_\varrho\cap E_{s,\delta}}(dd^c v_s)^j(dd^c v)^l\wedge\beta_{n-j-l}$$
$$\le C(N,\varrho,j,l)\cdot\mathcal{C}_{n-1}(B_\varrho\cap E_{s,\delta},D)\to 0,$$

and (40) is proved.

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