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On the topological triviality along moduli of deformations of $J_{k,0}$ singularities

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Abstract. It is well known that versal deformations of nonsimple singularities depend on moduli. However they can be topologically trivial along some or all of them. The first step in the investigation of this phenomenon is to determine the versal discriminant (unstable locus), which roughly speaking is the obstacle to analytic triviality. The next one is to construct continuous *liftable* vector fields smooth *far* from the versal discriminant and to integrate them. In this paper we extend the results of J. Damon and A. Galligo, concerning the case of the Pham singularity ($J_{3,0}$ in Arnold's classification) (see [2, 3, 4]), and deal with deformations of general $J_{k,0}$ singularities.

1. Introduction. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function with an isolated critical point at the origin. Let

$$F : U \to \mathbb{C}, \quad (0,0) \in U \subset X \times \varLambda,\ X = \mathbb{C}^n,\ \varLambda = \mathbb{C}^\mu,$$

be an analytic deformation of f (i.e. $f(x)$ is the germ of $F_0(x) = F(x, 0)$ at the origin) which is miniversal for right equivalence. Obviously $F_\lambda(x) = F(x, \lambda)$ is versal for V-equivalence. Furthermore if λ_1 is a free term, i.e. $F(x, \lambda) = F'(x, \lambda') + \lambda_1$, where $\lambda = (\lambda_1, \lambda') \in \varLambda = \mathbb{C} \times \varLambda'$, and the domain U splits, $U = \mathbb{C} \times U'$, $U' \subset X \times \varLambda'$, then the unfolding

$$\mathcal{F} : (U', (0,0)) \to (\varLambda, 0), \quad \mathcal{F}(x, \lambda') = (-F'(x, \lambda'), \lambda'),$$

is right-left stable.

Let T be the moduli set, i.e. the subset of $\varLambda$ consisting of all λ such that $F_\lambda(x)$ has a critical point p of multiplicity $\mu = \mu(f)$ (Milnor number) and $F_\lambda(p) = 0$. Let $\pi : (\varLambda, 0) \to (T, 0)$ be an analytic projection (transversal to T).

We say that the deformation F (resp. the unfolding $\mathcal{F}$) is *topologically trivial* along T relative to π if there exists a projection $\pi_0 : \varLambda \to \pi^{-1}(0)$ such

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that $F(z,\lambda)$ (resp. $\mathcal{F}(z,\lambda)$) is topologically equivalent to the deformation $F(z,\pi_0(\lambda))$ (resp. to the unfolding $(\mathcal{F}(z,\pi_0(\lambda)),\pi(\lambda))$).

Now let the germ f be quasihomogeneous (weighted homogeneous). Then there exists a distinguished class of projections induced by quasihomogeneity. We consider a *quasihomogeneous* miniversal deformation:

$$F_\lambda(x) = F(x,\lambda) = f(x) + \sum_{i=1}^{\mu} \lambda_i e_i(x)$$

where $e_1,\ldots,e_\mu$ is a quasihomogeneous basis of the local algebra $\mathcal{O}_n/I_f$, where I_f is the ideal spanned by the partial derivatives $\partial f/\partial x_i$, $i=1,\ldots,n$ (compare [1], §8). We say that the parameter λ_i is *underdiagonal*, *overdiagonal* or *diagonal* if the quasidegree of e_i is less than, greater than or equal to the quasidegree of f respectively. We remark that the number of basis elements of a given quasidegree does not depend on the choice of a basis (see [1], §12.2). In this case the moduli set T is a linear subspace of the base $\Lambda = \mathbb{C}^\mu$ spanned by the overdiagonal and diagonal λ's. Moreover, there is a canonical projection π onto T, "forgetting" the underdiagonal λ's. For this projection the restriction F_λ, $\lambda \in \pi^{-1}(0)$, is the part of the deformation consisting of the underdiagonal terms, the so-called underdiagonal deformation (also called the deformation of negative weight). Since quasihomogeneous germs are germs of polynomials, F_λ is defined globally and we may put for example $U = \mathbb{C}^n \times \mathbb{C}^\mu$. Nevertheless since the topological triviality depends on the choice of domain where the deformation is defined, we restrict ourselves to domains U such that the sum of the Milnor numbers of the critical points of F_λ in $U_\lambda = \{x : (x,\lambda) \in U\}$ is constant.

The goal of this paper is to show that in general quasihomogeneous miniversal deformations of $J_{k,0}$ singularities are topologically trivial along the moduli. The only exceptions are when $k \geq 3$ and the singularity is harmonic or aharmonic. That is, except the above two cases:

1. The *underdiagonal* part of the quasihomogeneous miniversal deformations of the $J_{k,0}$ singularity is topologically V-versal; or
2. The analytic miniversal deformations of (different) singularities belonging to the family $J_{k,0}$ are topologically V-equivalent.

The author would like to mention here that the problem of topological triviality has also been investigated by A. du Plessis and C. T. C. Wall (see [5, 6]).

2. Notation

2.1. *Deformations and unfoldings.* In the mathematical literature one can find two different points of view on the theory of deformations of germs of analytic sets: analytical and geometrical. Since we are going to deal with both of them, we have to differentiate the notation.

Let $f(x) : (\mathbb{C}^n, x_0) \to (\mathbb{C}, z_0)$ be a germ of an analytic function. By a *deformation* of f we mean an analytic family of functions

$$F(x,\lambda) : (U, (x_0, \lambda_0)) \to (\mathbb{C}, z_0), \qquad U \subset \mathbb{C}^n \times \Lambda,$$

such that $f(x)$ is the germ of $F(x,\lambda_0)$ at x_0 (compare [1]).

By an *unfolding* we mean a corank one mapping

$$\mathcal{F}(x,\lambda) : (U', (x_0, \lambda'_0)) \to (\mathbb{C} \times \Lambda', (z_0, \lambda'_0)), \qquad U' \subset \mathbb{C}^n \times \Lambda',$$

such that $\mathcal{F}^{-1}(z_0) = f^{-1}(z_0) \times \{\lambda'_0\}$.

In the following the set Λ (resp. Λ') will be called the *base* of the deformation (resp. of the unfolding).

Let $\Pi : \mathbb{C}^n \times \Lambda \to \Lambda$ and $\Pi' : \mathbb{C}^n \times \Lambda' \to \Lambda'$ denote the projections onto the second factor.

Two deformations $F_i(x,\lambda) : (U_i, (x_{i,0}, \lambda_{i,0})) \to (\mathbb{C}, 0)$, $i = 1, 2$, *are topologically V-equivalent* if there exist functions

$$\begin{aligned} \Phi(x,\lambda) &: (U_2, (x_{2,0}, \lambda_{2,0})) \to (\mathbb{C}^n, 0), \\ \Theta(\lambda) &: (\Pi(U_2), \lambda_{2,0}) \to (\Pi(U_1), (x_{1,0}, \lambda_{1,0})), \\ H(x,\lambda) &: (U_2, (x_{2,0}, \lambda_{2,0})) \to \mathbb{C} \end{aligned}$$

such that $\Phi(x, \lambda_{2,0})$ and $\Theta(\lambda)$ are homeomorphisms, H and $1/H$ are locally bounded and

$$F_1(\Phi(x,\lambda), \Theta(\lambda)) = H(x,\lambda) F_2(x,\lambda).$$

Furthermore we say that two unfoldings

$$\mathcal{F}_i(x,\lambda) : (U'_i, (x_{i,0}, \lambda'_{i,0})) \to (\mathbb{C} \times \Lambda', (0, \lambda'_{i,0})), \qquad i = 1, 2,$$

are *topologically A-equivalent* if there exist two homeomorphisms

$$\begin{aligned} \Phi(x,\lambda) &: (U'_2, (x_{2,0}, \lambda'_{2,0})) \to (U'_1, (x_{1,0}, \lambda'_{1,0})), \\ \Theta(z,\lambda) &: (U_3, (0, \lambda'_{2,0})) \to (\mathbb{C} \times \Lambda', (0, \lambda'_{2,0})), \qquad \operatorname{Im} \mathcal{F}_2 \subset U_3 \subset \mathbb{C} \times \Lambda', \end{aligned}$$

such that

$$\mathcal{F}_1 \circ \Phi = \Theta \circ \mathcal{F}_2.$$

Now assume that the base of a deformation (resp. of an unfolding) splits, $\lambda = (\tau, s)$. We say that the deformation F (resp. unfolding $\mathcal{F}$) is *topologically trivial* along s if it is topologically equivalent to the deformation

$$G(x,\lambda) = F(x, \tau, s_0)$$

(resp. to the unfolding

$$\mathcal{G}(x,\lambda) = (\mathcal{F}(x, \tau, s_0), s)).$$

2.2. $J_{k,0}$ *singularities.* We consider quasihomogeneous analytic functions

$$f(x,y)=y^3+\beta yx^{2k}+\gamma x^{3k},\qquad k=2,3,\ldots,$$

of type $(1/(3k),1/3)$, where $4\beta^3+27\gamma^2\neq 0$. Each of them has an isolated singular point at the origin. In Arnold's classification (see [1], §15) such singularities are called $J_{k,0}$. We remark that they are classified (up to both right and V-equivalence) by the j-*invariant*:

$$j=\frac{4\beta^3}{4\beta^3+27\gamma^2}.$$

Note that the j-invariant has two branching values: $j=0$ (the harmonic case) and $j=1$ (the aharmonic case).

We remark that the Milnor number of a $J_{k,0}$ singularity equals $6k-2$ and the modality is $k-1$ (but only one modulus occurs in the quasihomogeneous part, the other are overdiagonal).

3. The main result. With the exception of the case $k=2$ the topological triviality is not possible for $j=0$ or $j=1$ since in these cases there exist nontypical decompositions (see [8, 9]).

Therefore we restrict ourselves to the case when $j\neq 0,1$ and put

$$f(x,y)=y^3+ux^{2k}y+x^{3k},\qquad 4u^3+27\neq 0\neq u.$$

We shall consider the following deformation of f:

$$F(x,y,\tau,s)=y^3+ux^{2k}y+x^{3k}+\sum_{i=0}^{k-2}s_iyx^{2k+i}+\sum_{i=0}^{3k-2}\tau_{0,i}x^i+\sum_{i=0}^{2k-1}\tau_{1,i}yx^i,$$

and the associated unfolding

$$\mathcal{F}(x,y,\tau',s)=(-F(x,y,0,\tau',s),\tau',s),$$

where $\tau=(\tau_{0,0},\tau')\in\mathbb{C}^{5k-1}$ and $s\in\mathbb{C}^{k-1}$. We remark that F becomes a quasihomogeneous function of x,y,τ,s if we put $\operatorname{qdeg}s_i=-i/(3k)$ and $\operatorname{qdeg}\tau_{j,i}=1-j/3-i/(3k)$.

THEOREM 1. *If $u\neq 0$ and $u^3+27\neq 0$ then there exist quasihomogeneous neighbourhoods U and U' of the origin such that the deformation F restricted to U and the unfolding $\mathcal{F}$ restricted to U' are topologically trivial along s. Moreover if u is real then the above neighbourhoods and trivializations can be chosen to be invariant under complex conjugation.*

REMARK. The case $k=2$ was proved by Looijenga [11], and the next one, $k=3$, by J. Damon and A. Galligo [2, 3, 4]. Furthermore, it follows from the results of Wirthmüller [12] that $\mathcal{F}$ and F are topologically trivial along s_{k-2}.

4. Liftable vector fields. The construction of a topological trivialization is based on integration of certain vector fields.

We recall the basic definitions. Let $F(x,\lambda)$ be an analytic deformation of $f(x) = F(x,0)$. Let λ_0 be a point from the fibre $\pi^{-1}(0)$, where $\pi : \Lambda \to T$ is the projection onto the moduli set. A continuous vector field

$$\eta = \sum \eta_i(\lambda)\frac{\partial}{\partial \lambda_i}$$

defined on some neighbourhood of λ_0 is called *liftable* if there exists a neighbourhood W of λ_0 and a continuous vector field ξ defined on $\Pi^{-1}(W) \cap U$ such that

$$\xi = \sum \xi_i(x,\lambda)\frac{\partial}{\partial x_i} + \eta$$

and $\xi(F) = AF$, where $A(x,\lambda)$ is a locally bounded function. We call such a vector field ξ *lifted* and denote its component tangent to Λ by $\Pi^{\#}(\xi)$. Note that ξ and η are linear combinations of analytic (holomorphic) vector fields with continuous coefficients. Thus they annihilate all antiholomorphic functions.

We denote by $\mathcal{M}(U)$ (respectively by $\mathcal{L}(U)$) the $\mathcal{O}(\Pi(U))$-module of analytic liftable (respectively lifted) vector fields and by $\mathcal{M}(\lambda)$ (respectively $\mathcal{L}(\lambda)$) the $\mathbb{C}$-linear space of their values at the point λ.

We recall that the tangent space to a V-equivalence stratum is spanned by liftable vector fields. Therefore we apply the following definition (compare the notion of a versality discriminant [2, 3, 4] and an instability locus [5]):

DEFINITION. A point $\lambda \in \pi^{-1}(0)$ belongs to *the versal discriminant relative to the projection* π if the module $\mathcal{M}$ of analytic liftable vector fields is not transversal to the fibre $\pi^{-1}(0)$ at λ:

$$\mathcal{M}(\lambda) \oplus T_\lambda \pi^{-1}(0) \neq T_\lambda \Lambda.$$

5. Regular domains

5.1. *Definition.* Let the set of germs of analytic functions $e_1(x), \ldots, e_\mu(x)$ be a basis of the local algebra of the germ at the origin of an analytic function $f(x)$. We consider its (right) miniversal deformation

$$F(x,\lambda) = f(x) + \sum_{i=1}^{\mu}(\lambda_i - \lambda_{0,i})e_i(x), \qquad (x,\lambda) \in U \subset X \times \Lambda,\ X = \mathbb{C}^n,\ \Lambda = \mathbb{C}^\mu.$$

Moreover we assume that U is open and contains $(0,\lambda_0)$.

We say that the domain U is *regular* if:

• the analytic functions $e_1(x), \ldots, e_\mu(x)$ span the factor algebra of analytic functions on U modulo the ideal I_F generated by the derivatives

$\partial F/\partial x_i$, as a module over the analytic functions on the projection $\Pi(U)$,

$$\mathcal{O}(U) = I_F \oplus \mathcal{O}(\Pi(U))\langle e_1(x), \ldots, e_\mu(x)\rangle,$$

• for every $\lambda \in \Pi(U)$ the analytic functions $e_1(x), \ldots, e_\mu(x)$ are $\mathbb{C}$-linearly independent in the factor algebra of analytic functions on $U \cap \Pi^{-1}(\lambda)$ modulo the ideal I_{F_λ}.

The first condition implies that the fibres $U \cap \Pi^{-1}(\lambda)$ are not too big, i.e. they do not contain "additional" critical points of F_λ (not converging to the origin as λ tends to 0). The second one means that these fibres are not too small, i.e. the sum of the Milnor numbers of critical points of F_λ is constant (equal to μ).

Now, let F be a miniversal deformation of a $J_{k,0}$ singularity defined as in Section 3. As an example of a regular domain we can take the set

$$\begin{aligned} U_{\delta,\epsilon,\omega} &= \{(x, y, \tau_{0,0}, \tau', s_0, s') : (x, y, \tau', s_0, s') \in U'_{\delta,\epsilon,\omega},\ \tau_{0,0} \in \mathbb{C}\}, \\ U'_{\delta,\epsilon,\omega} &= \{(x, y, \tau', s_0, s') : |s_0| < \delta, (|x|^{3k} + |y|^3) \cdot \|s'\| < \epsilon, \\ &\qquad\qquad \|\tau'\| \cdot \|s'\| < \epsilon \cdot \omega\}, \end{aligned}$$

where δ, ϵ, ω are sufficiently small positive constants, and

$$\begin{aligned} &\|s'\| = \max\{|s_i|^{3k/i} : i = 1, \ldots, k-2\} \quad \text{for } k > 2, \quad \|s'\| = 0 \quad \text{for } k = 2, \\ &\|\tau'\| = \max\{|\tau_{i,j}|^{3k/(3k-jk-i)} : (i,j) \neq (0,0)\}. \end{aligned}$$

5.2. *Properties.* We state some basic properties of a regular domain U.

LEMMA 1. *For every $\overline{\lambda} \in \Pi(U)$ the mapping*

$$(\eta_1, \ldots, \eta_\mu) \mapsto \left(\sum_{i=1}^{\mu} \eta_i \frac{\partial F}{\partial \lambda_i}\right)\bigg|_{\lambda=\overline{\lambda}} \bmod I_{F_{\overline{\lambda}}}$$

is a $\mathbb{C}$-linear isomorphism of the tangent space $T_{\overline{\lambda}}\Lambda$ and the factor algebra $\mathcal{O}(U \cap \Pi^{-1}(\overline{\lambda}))/I_{F_{\overline{\lambda}}}$.

Proof. Indeed, $\partial F/\partial \lambda_i = e_i(x)$ and the e_i form a basis of the above factor algebra.

Furthermore, there is a canonical $\mathcal{O}(\Pi(U))$-linear mapping

$$\Upsilon : \mathcal{L}(U) \to \mathcal{O}(U), \qquad \Upsilon(\xi) = \xi(F)/F.$$

LEMMA 2. *Υ is onto and $\ker \Upsilon \subset \ker \Pi^{\#}$.*

Proof. The first assertion follows from the first regularity condition and the second from the second one. Indeed, for every analytic function a on U we have a decomposition

$$a(x,\lambda)F(x,\lambda) = \sum \xi_i(x,\lambda)\frac{\partial F}{\partial x_i} + \sum \eta_i(\lambda)\frac{\partial F}{\partial \lambda_i}.$$

We put

$$\xi = \sum \xi_i(x,\lambda)\frac{\partial}{\partial x_i} + \sum \eta_i(\lambda)\frac{\partial}{\partial \lambda_i}.$$

Obviously $\xi(F) = aF$. On the other hand if ξ is as above and $\xi(F) = 0$ then

$$\sum \xi_i(x,\lambda)\frac{\partial F}{\partial x_i} + \sum \eta_i(\lambda)\frac{\partial F}{\partial \lambda_i} = 0.$$

But $\partial F/\partial\lambda_i = e_i(x)$ and for any fixed λ they are linearly independent modulo $\partial F/\partial x_i$'s. Therefore all η_i must be 0. Hence $\Pi^{\#}(\xi) = 0$.

LEMMA 3. *$\mathcal{M}(U)$ is an $\mathcal{O}(\Pi(U))$-module generated by the vector fields $\Pi^{\#}(\xi_i)$ where $\xi_i(F) = e_iF$.*

Proof. Let ξ be a lifted vector field, $\xi(F) = aF$. We decompose a as

$$a(x,\lambda) = \sum \beta_i(x,\lambda)\frac{\partial F}{\partial x_i} + \sum \alpha_i(\lambda)e_i(x).$$

We put

$$\xi^* = F\cdot \sum \beta_i(x,\lambda)\frac{\partial}{\partial x_i} + \sum \alpha_i(\lambda)\xi_i.$$

Obviously $\xi^*(F) = aF$. Therefore

$$\Pi^{\#}(\xi) = \Pi^{\#}(\xi^*) = \sum \alpha_i(\lambda)\Pi^{\#}(\xi_i).$$

We apply the above to a deformation F of a $J_{k,0}$ singularity restricted to a regular domain. Basing on the results of [9] we obtain:

PROPOSITION 1. *The versal discriminant of the deformation F consists of the parameters τ such that*

$$F_\tau = y^3 + ud(x)^2y + d(x)^3,$$

where $d(x)$ is a polynomial of degree k with at least one multiple root.

REMARKS.. • Since F_τ is a subfamily of F, d is monic and the sum of its roots is 0, $d = x^k + a_{k-2}x^{k-2} + \ldots$

• In [9] we also described the versal discriminant of the two cases omitted here, $j = 0, 1$.

Basing on Lemma 3 one can also prove the uniqueness of quasihomogeneity.

PROPOSITION 2. *Let*

$$F(x,\lambda), \qquad (x,\lambda)\in U\subset X\times\Lambda,\ X=\mathbb{C}^n,\ \Lambda=\mathbb{C}^\mu,$$

be a quasihomogeneous miniversal deformation of a quasihomogeneous or semiquasihomogeneous singularity. If U is a regular domain then the quasihomogeneous $\mathbb{C}^*$ action on the base Λ is invariant under analytic automorphisms of the deformation.

P r o o f. The $\mathbb{C}^*$ action on Λ is induced by the so-called *Euler vector field*

$$\varepsilon = \sum \operatorname{qdeg}(\lambda_i)\lambda_i \frac{\partial}{\partial \lambda_i}.$$

The orbits of the $\mathbb{R}^+$ and $\mathbb{S}^1$ actions are integral curves of the real and imaginary parts of ε.

Next, ε is liftable and for its lifting we can take the Euler vector field on $X \times \Lambda$,

$$\Xi_e = \varepsilon + \sum \operatorname{qdeg}(x_i)x_i \frac{\partial}{\partial x_i}.$$

Obviously $\Xi_e(F) = \operatorname{qdeg}(F)F$. The last condition is preserved by automorphisms:

$$\Phi^*(\Xi_e)(F) = \Phi^*(\Xi_e)(F \circ \Phi) = \Xi_e(F) = \operatorname{qdeg}(F)F.$$

Due to Lemmas 2 and 3 the difference $\Phi^*(\Xi_e) - \Xi_e$ is tangent to the fibres of the projection $\Pi : X \times \Lambda \to \Lambda$. Therefore $\Phi^*(\Xi_e)$ is also a lifting of ε. Thus for any automorphism Φ,

$$\Pi^{\#}(\Phi^*(\Xi_e)) = \varepsilon.$$

Hence ε is an invariant of the automorphisms. (For more details see [10].)

6. The structure of the versal discriminant

6.1. *Critical points.* For

$$F = y^3 + ud(x)^2y + d(x)^3,$$

we have

$$\frac{\partial F}{\partial y} = 3y^2 + ud(x)^2, \qquad \frac{\partial F}{\partial x} = (2uy + 3d(x))d(x)d'(x).$$

Let $d_1(x)$ be the greatest common divisor of d and d'. We put $d' = d_0d_1$ and $d = d_2d_1$.

There are two types of critical points:

- $d(x_0) = y_0 = 0$,
- $d(x_0) \neq 0$ but $d'(x_0) = 0$.

We remark that due to the nondegeneracy condition $4u^3 + 27 \neq 0$ the intersection of the zero sets of $\partial F/\partial y$ and of the first factor of $\partial F/\partial x$ consists only of the points $(x_0, 0)$ where $d(x_0) = 0$.

The roots of $d(x)$ give rise to singular points of $F^{-1}(0)$. Let

$$d(x) = \prod (x - x_i)^{\alpha_i},$$

where all x_i are different. In the local coordinate system $\{\widetilde{x}, \widetilde{y}\}$ in the neighbourhood of the point $(x_i, 0)$,

$$F = \widetilde{y}^3 + u\widetilde{x}^{2\alpha_i}\widetilde{y} + \widetilde{x}^{3\alpha_i}, \quad \text{where} \quad \widetilde{x} = (x - x_i)\Big(\prod_{j\neq i}(x - x_j)^{\alpha_j}\Big)^{1/\alpha_i}, \quad \widetilde{y} = y.$$

This gives the following singularity types:

- if $\alpha_i = 1$ then the point $(x_i, 0)$ has type D_4;
- if $\alpha_i = m$, $m \geq 2$, then it has type $J_{m,0}$, $j = 4u^3/(4u^3 + 27)$. (We remark that at least one α_i is greater than 1.)

Furthermore the sums of the Milnor numbers and the modality of these critical points are respectively $\mu_1 = 6k - 2\deg d_2$ and $m_1 = k - \deg d_2$, where d_2 is the product of all prime factors of d, i.e. $\deg d_2$ equals the number of different roots of d.

The roots of $d_0(x)$ give rise to the remaining critical points of F, which are all of type A_*.

6.2. *Parametrization of the stratum.* We recall that a subset of the base of a deformation is a *μ-constant stratum* if it is a connected component of the set of parameters λ such that the sum of the Milnor numbers of the singular points of the zero sets of the corresponding functions is equal to a given value.

We consider the μ-constant stratum V_F containing the function $F = y^3 + ud(x)^2y + d(x)^3$, which was investigated in the previous subsection. Near F, V_F is parametrized by the family

$$F_s = y^3 + (u + s(x))d(x)^2y + d(x)^3, \qquad s(x) = s_{k-2}x^{k-2} + \ldots + s_0.$$

Indeed:

- F_s has the same singular points as F, $(x_i, 0)$, where x_i is a root of $d(x)$. In the local coordinate system $\{\widetilde{x}, \widetilde{y}\}$ in the neighbourhood of the point $(x_i, 0)$,

$$F = \widetilde{y}^3 + (u + \widetilde{s}(\widetilde{x}))\widetilde{x}^{2\alpha_i}\widetilde{y} + \widetilde{x}^{3\alpha_i}, \qquad \text{where}$$
$$\widetilde{x} = (x - x_i)\Big(\prod_{j\neq i}(x - x_j)^{\alpha_j}\Big)^{1/\alpha_i}, \qquad \widetilde{y} = y.$$

Thus the whole family has the same singularity μ-type.

- The dimension of V_F equals

$$m_1 + \mu_2 = k - \deg d_2 + 2\deg d_2 - 2 = k - 2 + \deg d_2.$$

Hence it is equal to the number of parameters of the family F_s; there are $k-1$ coefficients of $c(x)$ and $\deg d_2$ different roots of $d(x)$, which fulfill only one condition: their sum (counted with multiplicities) equals 0.

We remark that F_s, $s \neq 0$, has more critical points than F but the additional ones lie outside the regular domain.

LEMMA 4. *The μ-constant stratum containing points from the versal discriminant is transversal to the fibres of the projection π.*

Proof. Since
$$\frac{\partial F_s}{\partial s_i} = d(x)^2 y x^i$$
the matrix of coefficients of the "vectors"
$$e_1, \ldots, e_{5k-2}, \frac{\partial F_s}{\partial s_0}, \ldots, \frac{\partial F_s}{\partial s_{k-2}}$$
in the base $1, x, \ldots, x^{3k-2}, y, yx, \ldots, yx^{3k-2}$ of the factor algebra $\mathcal{O}(U \cap \Pi^{-1}(\lambda))/I_{F_\lambda}$ is triangular. Since the mapping $\eta \mapsto \eta(F)$ is an isomorphism of the above factor algebra and the tangent space to Λ (see Lemma 1), the tangent space to the parametrization F_s is transversal to the fibres of π.

7. Multigerm splitting

7.1. *Deformations of a multigerm.* Let
$$F(x,\lambda), \qquad (x,\lambda) \in U \subset \mathbb{C}^n \times \Lambda,\ \Lambda = \mathbb{C}^m,$$
be an analytic family of holomorphic functions. Let p be an isolated critical point of $F(x,\lambda_0)$. Then the germ of $F(\cdot,\cdot)$ at (p,λ_0) is a deformation of the germ of $F_{\lambda_0} = F(\cdot,\lambda_0)$ at p. Hence it is right-equivalent to the deformation induced from the miniversal one (see [1], Vol. 1, §8). Furthermore we choose a normal form of the germ $(F_{\lambda_0})_p$, say $F_{\lambda_0}(\overline{g}_p(x))$, where $\overline{g}_p$ is a germ of a diffeomorphism and $\overline{g}_p(p) = 0$. In such a way we construct a germ of an analytic mapping Ψ_p from the parameter space Λ to the base Λ_p of the right miniversal deformation $G_p(x,\overline{\lambda})$ of the germ of $F_{\lambda_0}(\overline{g}_p(x))$ at the point p,
$$\Psi_p : (\Lambda,\lambda_0) \to (\Lambda_p, \overline{\lambda}_{p,0})$$
such that
$$F(x,\lambda) = G_p(g_p(x,\lambda), \Psi_p(\lambda))$$
in some neighbourhood of (p,λ_0), where g_p is a holomorphic germ, and $g_p(x,\lambda_0) = \overline{g}_p(x)$. Let Crit_λ denote the set of critical points of F_λ . We introduce a splitting mapping
$$\Psi = \bigtimes_{p\in \mathrm{Crit}_{\lambda_0}} \Psi_p : (\Lambda,\lambda_0) \to \bigtimes_{p\in \mathrm{Crit}_{\lambda_0}} (\Lambda_p, \overline{\lambda}_{p,0})$$
from the parameter space Λ to the base of a miniversal deformation of the multigerm $(F_p(x,\lambda_0))$, $p \in \mathrm{Crit}_{\lambda_0}$.

In the following we shall consider the case when F is a right miniversal deformation of a germ with an isolated critical point of multiplicity μ.

PROPOSITION 3. *If the domain U of the deformation is regular then the splitting mapping*

$$\Psi = \bigtimes_{p\in \mathrm{Crit}_{\lambda_0}} \Psi_p : (\Lambda, \lambda_0) \to \bigtimes_{p\in \mathrm{Crit}_{\lambda_0}} (\Lambda_p, \overline{\lambda}_{p,0})$$

is a germ of a diffeomorphism.

P r o o f. We put $f_p(x) = F(\overline{g}_p(x), \lambda_0)$. We recall that we may choose the following miniversal deformations of the germ of f_p at p:

$$f_p(x) - f_p(p) + \sum b_j e_j,$$

where b_j are complex parameters, e_j form the basis of the local algebra of f_p at p and $e_1 = 1$. We identify the tangent space to Λ_p with the local algebra Q_p. Hence the derivative $\mathcal{D}\Psi$ is the composition

$$T\Lambda \to \mathcal{O}_{\Pi^{-1}(\lambda_0),\lambda_0}/I_F \to \bigoplus_{p\in \mathrm{Crit}} Q_p,$$

$$\frac{\partial}{\partial\lambda_i} \mapsto \frac{\partial}{\partial\lambda_i} F \bmod (\partial_1 F, \ldots, \partial_n F)$$

$$\overset{\pi}{\mapsto} \left(\frac{\partial}{\partial\lambda_i} f_p \bmod (\partial_1 f_p, \ldots, \partial_n f_p)\mathcal{O}_p\right)_{p\in \mathrm{Crit}}.$$

The homomorphism π is an isomorphism of linear spaces. As a matter of fact it is a splitting of a multilocal algebra into local components ([1], §5.8). Hence $\mathcal{D}\Psi_{|\lambda_0}$ is also an isomorphism.

Thus Ψ is a germ of a diffeomorphism.

In the following sections we denote by the same symbol Ψ the analytic diffeomorphism which represents it.

EXAMPLE. Let λ be a point from the versal discriminant of a $J_{3,0}$ singularity such that the curve $F_\lambda = 0$ has two singular points $J_{10} = J_{2,0}$ and D_4. Then the 16-dimensional base of a right miniversal deformation of $J_{3,0}$ is in some neighbourhood of λ the Cartesian product of:

- a 10-dimensional base of $J_{2,0}$,
- a 4-dimensional base of D_4, and
- two one-dimensional bases of A_1 (there are two critical points with nonzero critical values).

Obviously the 3-dimensional stratum containing the versal discriminant is the product of:

- the 1-dimensional moduli set of $J_{2,0}$,
- the origin of the base of D_4, and
- the two one-dimensional bases of A_1.

7.2. *Splitting and liftable vector fields.* Let η_p's be continuous vector fields defined on some neighbourhoods of $\overline{\lambda}_{p,0}$ in Λ_p, $p \in \operatorname{Crit} F_{\lambda_0}$. If p is a singular point of the curve $F_{\lambda_0} = 0$ then we assume that η_p is liftable and ξ_p is the corresponding lifted vector field,

$$\xi_p(F_p) = A_p F_p.$$

We extend each η_p to an open subset of the Cartesian product $\bigtimes \Lambda_q$, $q \in \operatorname{Crit} F_{\lambda_0}$, $\eta_p \mapsto (0, \ldots, 0, \eta_p, 0, \ldots 0) \in T(\bigtimes \Lambda_q)$.

LEMMA 5. *The vector field $\eta = (\Psi^*)^{-1}(\sum \eta_p)$ restricted to some neighbourhood of λ_0 is liftable.*

Proof. We construct the lifted vector field ξ in the following way. In a suitable small neighbourhood U_p of a singular point p we put

$$\widehat{\xi}_p = (g_p^*, \Psi^*)^{-1}\Big(\xi_p + \sum_{p' \neq p} \eta_{p'}\Big).$$

We remark that for $(x, \lambda) \in U_p$ we have

$$\begin{aligned}\widehat{\xi}_p(F)|_{(x,\lambda)} &= \widehat{\xi}_p(G_p \circ (g_p, \psi_p))|_{(x,\lambda)} = \Big(\xi_p + \sum_{p' \neq p} \eta_{p'}\Big)(G_p)\Big|_{(g_p(x,\lambda), \psi_p(\lambda))} \\ &= (A_p G_p)|_{(g_p(x,\lambda), \psi_p(\lambda))} = A_p(g_p(x,\lambda), \psi_p(\lambda)) F(x, \lambda).\end{aligned}$$

The complement of the set of singular points is covered by $n + 1$ sets:

$$\begin{aligned}U_0 &= \{(x, \lambda) : F(x, \lambda) \neq 0\}, \\ U_i &= \Big\{(x, \lambda) : \frac{\partial F}{\partial x_i}(x, \lambda) \neq 0\Big\}, \qquad i = 1, \ldots, n.\end{aligned}$$

We put $\widehat{\xi}_0 = \eta$ on U_0, and on U_i, $i = 1, \ldots, n$,

$$\widehat{\xi}_i = \frac{-\eta(F)}{\partial F / \partial x_i} \frac{\partial}{\partial x_i} + \eta.$$

The above vector fields are local liftings of η and

$$\widehat{\xi}_0(F) = \frac{-\eta(F)}{F} F, \qquad \widehat{\xi}_i(F) = 0.$$

Next we glue $\widehat{\xi}$'s using a smooth partition of unity to obtain a *global* lifting of η:

$$\xi = \sum_{p \in \mathrm{Sing}} \zeta_p \widehat{\xi}_p + \sum_{i=0}^{n} \zeta_i \widehat{\xi}_i,$$

where

$$\sum_{p \in \mathrm{Sing}} \zeta_p + \sum_{i=0}^{n} \zeta_i = 1, \qquad \operatorname{supp} \zeta_p \subset U_p, \ \operatorname{supp} \zeta_i \subset U_i.$$

7.3. *Split-stratified vector fields.* The aim of this subsection is to define a category of continuous vector fields lifted from the base of a given (right) miniversal deformation (of an isolated singularity S) which is closed under addition and multiplication by smooth functions and furthermore contains only integrable vector fields.

We recall that the so-called *control functions* are the basic tools used to show the uniqueness of integral curves of continuous vector fields (compare [2, 3, 4, 12]).

We associate with every (right, analytic) type S of a critical point a fixed normal form f_S and a fixed miniversal deformation F_S; if the singularity is quasihomogeneous or semiquasihomogeneous then we choose a normal form with the respective property and a quasihomogeneous deformation,

$$F_S : U_S \to \mathbb{C}, \quad U_S \subset \mathbb{C}^n \times \Lambda_S, \quad f_S(x) = F_S(x, \lambda_0).$$

Let T_S be the moduli set, i.e. the subset of Λ consisting of λ such that $F_{S,\lambda}(x)$ has a critical point p of multiplicity $\mu = \mu(S)$ and $F_{S,\lambda}(p) = 0$.

We say that a smooth function

$$\delta_S : \Lambda \to \mathbb{R}^+ \cup \{0\}, \quad \delta_S^{-1}(0) = T_S,$$

is an *admissible control function* if, for all singularities except the type $J_{k,0}$, it is the sum of the squares of the moduli of holomorphic functions which generate the ideal of holomorphic functions vanishing on T_S, and for the type $J_{k,0}$ it is the sum of the squares of the moduli of quasihomogeneous holomorphic functions of the same positive quasidegree.

Let $F : U \to \mathbb{C}$, $U \subset \mathbb{C}^n \times \Lambda$, be an analytic family of holomorphic functions.

Definition. A lifted vector field

$$\xi = \sum \xi_i(x, \lambda) \frac{\partial}{\partial x_i} + \eta(\lambda), \quad \xi(F) = AF,$$

is *split-stratified* on U if for every μ-constant stratum E:

(1) η is tangent to E and smooth on E,

(2) ξ is tangent to $\operatorname{Sing} F$ and

$$\xi\left(\frac{\partial F}{\partial x_i}\right) = A_{i,0} F + \sum A_{i,j} \frac{\partial F}{\partial x_j}$$

where $A_{i,j}$ are locally bounded on $U \cap \Pi^{-1}(E)$,

(3) ξ restricted to $\Pi^{-1}(E) \setminus \operatorname{Sing} F$ is smooth,

(4) for every $\lambda \in E$, for every set of admissible control functions δ_p, $p \in \operatorname{Sing} F_\lambda$, and for every splitting mapping Ψ, there exists a constant C such that

$$\Big|\eta\Big(\sum_{p\in \mathrm{Sing}\, F_{\lambda_0}} \delta_p\circ \Psi\Big)\Big| \le C\Big(\sum_{p\in \mathrm{Sing}\, F_{\lambda_0}} \delta_p\circ \Psi\Big)$$

in some neighbourhood of λ.

REMARK. The above definition does not depend on the choice of a particular splitting mapping. Indeed, the composition of an admissible control function and an automorphism of the miniversal deformation is still an admissible control function (see Proposition 2 for the case of $J_{k,0}$).

LEMMA 6. *If a continuous lifted vector field* ξ *is split-stratified then* $\mathrm{Re}\,\xi$, $\mathrm{Im}\,\xi$, $\mathrm{Re}\,\eta$ *and* $\mathrm{Im}\,\eta$ *are locally integrable.*

Proof. From condition (4) it follows that any integral curve of $\mathrm{Re}\,\eta$ and $\mathrm{Im}\,\eta$ lies in just one μ-constant stratum (compare e.g. [2]). Hence together with (1) this implies the uniqueness of the integral curves of η's.

Integral curves of $\mathrm{Re}\,\xi$ and $\mathrm{Im}\,\xi$ lie over integral curves of $\mathrm{Re}\,\eta$ and $\mathrm{Im}\,\eta$. The uniqueness outside the set of singular points follows from (3) while the uniqueness at singular points comes from (2) and the fact that F_λ has only isolated singular points. Indeed, over integral curves of η's, $\mathrm{Sing}\, F$ is a disjoint union of smooth curves. Next we consider the control function

$$\varrho = |F|^2 + \sum \left|\frac{\partial F}{\partial x_i}\right|^2.$$

Then

$$\begin{aligned}\xi(\varrho) &= \xi(F)\overline{F} + \sum \xi\left(\frac{\partial F}{\partial x_i}\right)\overline{\frac{\partial F}{\partial x_i}}\\ &= A|F|^2 + \sum\left(A_{i,0}F + \sum A_{i,j}\frac{\partial F}{\partial x_j}\right)\overline{\frac{\partial F}{\partial x_i}}.\end{aligned}$$

Therefore, since $|a\overline{b}| = |a|\cdot|b| \le |a|^2 + |b|^2$ we get

$$|\xi(\varrho)| \le C\varrho,$$

where C is the sum of bounds of the coefficients A and $A_{i,j}$. Hence no integral curve of ξ can cross the set of singular points.

The above yields the continuous dependence on initial values, i.e. the continuity of the flow (see [7], Ch. V, Th. 2.1).

LEMMA 7. *A linear combination, whose coefficients are bounded smooth functions of* λ, *of lifted split-stratified vector fields is again split-stratified.*

Proof. The first three conditions are obvious. We deal with the last one. Let $\eta = \sum b_i(\lambda)\eta_i$, where $|b_i(\lambda)| \le M$. We have

$$|\eta(\delta)| = \Big|\sum b_i(\lambda)\eta_i(\delta)\Big| \le \sum |b_i(\lambda)|\cdot|\eta_i(\delta)| \le \Big(M\sum C_i\Big)\delta.$$

LEMMA 8. *Every analytic lifted vector field* ξ *is split-stratified.*

Proof. The first three conditions are obvious. We deal with the last one. If $\eta = \Pi^{\#}(\xi)$ is analytic then so is $\Psi^*(\eta)$. Let η_p be the component of $\Psi^*(\eta)$ tangent to Λ_p. Then

$$\eta\Big(\sum \delta_p \circ \Psi\Big) = (\Psi^*\eta)\Big(\sum \delta_p\Big) = \sum \eta_p(\delta_p).$$

If δ_p is the square of the euclidean distance from the moduli set T_p, to which η_p is tangent, then there is a local coordinate system $\{y_i : i = 1, \ldots, m\}$ such that $T = \{y_1 = \ldots = y_k = 0\}$ and

$$\delta_p = \sum_{i,j=1}^{k} \delta_{i,j}(y) y_i y_j, \quad \text{where} \quad \delta_{i,i}(0) = 1,\ \delta_{i,j}(0) = 0 \text{ for } i \neq j.$$

Since η_p is tangent to T,

$$\eta_p = \sum_{i,j=1}^{k} \eta_{i,j} y_i \frac{\partial}{\partial y_j} + \sum_{i=k+1}^{m} \eta_i \frac{\partial}{\partial y_i}.$$

Thus $\eta_p(\delta_p)$ belongs to the square of the ideal generated by $y_1, \ldots, y_k$ and $\eta_p(\delta_p)/\delta_p$ is bounded.

If p has type $J_{k,0}$ then η_p is a linear combination of quasihomogeneous liftable vector fields of nonnegative quasidegree with analytic coefficients (see Lemma 3):

$$\eta_p = \sum a_i \eta_i, \qquad \operatorname{qdeg}(\eta_i(\delta_p)) = \operatorname{qdeg}(\eta_i) + \operatorname{qdeg}(\delta_p) \geq \operatorname{qdeg}(\delta_p).$$

Indeed, if $\eta_i = \Pi^{\#}(\xi_i)$, where $\xi_i(F) = e_i F$, then $\operatorname{qdeg}(\eta_i) = \operatorname{qdeg}(e_i) \geq 0$. Therefore the quotients $\eta_i(\delta_p)/\delta_p$ are bounded. The coefficients a_i are also bounded hence so is the quotient $\eta_p(\delta_p)/\delta_p$.

8. Construction of families Ξ_k of vector fields. Let $f(x, y)$ and $F(x, y, \tau, s)$ be as in Section 3, and let Π and π be the projections, $\Pi(x, y, \tau, s) = (\tau, s)$, $\pi(\tau, s) = s$.

PROPOSITION 4. *There exists a $(k-1)$-tuple $\Xi_k = \{\xi_{k,i} : i = 0, 1, \ldots, k-2\}$ of quasihomogeneous split-stratified lifted vector fields defined on some quasihomogeneous neighbourhood of $\Pi^{-1}(\pi^{-1}(0))$ such that*

$$\Pi^{\#}(\xi_{k,i}) = \frac{\partial}{\partial s_i} + \sum \eta_{k,i,j,l}(\tau, s) \frac{\partial}{\partial \tau_{j,l}}, \qquad i = 0, 1, \ldots, k-2.$$

Proof. The construction of Ξ_k is by induction. We let Ξ_1 be empty, and assume that for all $m < k$ the families Ξ_m are already constructed.

We proceed in three steps. First we show that for every point $(\tau_0, 0) \in \Lambda$ lying on the quasihomogeneous "sphere" $\|\tau\| = 1$ there exists a neighbourhood $W_{\tau_0} \subset \Lambda$ such that one can construct $\xi_{k,i}$'s on $\Pi^{-1}(W_{\tau_0})$. Next we choose a finite covering and glue the vector fields using a partition of unity.

The last step is to extend the vector fields by quasihomogeneity to some neighbourhood of the set $\Pi^{-1}(\pi^{-1}(0)) = \{(x, y, \tau, 0)\}$.

STEP 1. If the point $(\tau_0, 0)$ does not belong to the versal discriminant V then for every i, $i = 1, \ldots, k-2$, we can find a neighbourhood W_{τ_0} and a lifted analytic quasihomogeneous vector field ξ_i with

$$\Pi^{\#}(\xi_i) = \frac{\partial}{\partial s_i} + \sum \eta_{i,j,l}(\tau, s) \frac{\partial}{\partial \tau_{j,l}}$$

defined on $\Pi^{-1}(W_{\tau_0})$.

Otherwise, when $(\tau_0, 0) \in V$, we apply induction. Let $V(\tau_0)$ be the μ-constant stratum containing $(\tau_0, 0)$ (obviously it also contains a stratum of the versal discriminant). First we show that the tangent space to $V(\tau_0)$ at $(\tau_0, 0)$ is spanned by quasihomogeneous split-stratified continuous liftable vector fields. We consider the multigerm splitting

$$\Psi : (\Lambda, (\tau_0, 0)) \to \bigtimes_{p \in \mathrm{Crit}_{\tau_0}} (\Lambda_p, \overline{\lambda}_{p,0}).$$

We notice that Ψ maps the μ-constant stratum $V(\tau_0)$ to the product of the overdiagonal and diagonal subspaces of Λ_p's for p singular (we denote them by T_p) and whole Λ_p's for the other critical points:

$$\Psi(V(\tau_0)) = \Big(\bigtimes_{p \in \mathrm{Sing}} T_p \Big) \times \Big(\bigtimes_{p \notin \mathrm{Sing}} \Lambda_p \Big).$$

The tangent space to T_p, $p \in \mathrm{Sing}$, at the origin is (by induction) spanned by continuous liftable vector fields such that their liftings are split-stratified. (If p has type D_4 then $T_p = \{0\}$.)

The tangent space to Λ_p for $p \notin \mathrm{Sing}$ is spanned by analytic vector fields.

The pull-backs of the above vector fields are liftable (see Lemma 5) and span the tangent space to $V(\tau_0)$ at $(\tau_0, 0)$. Furthermore their liftings are continuous and split-stratified.

Since the stratum $V(\tau_0)$ is transversal to the fibres of the projection $\pi : \Lambda \to T$, the vector fields we are looking for are linear combinations of the above with bounded continuous coefficients.

STEP 2. Since the quasihomogeneous sphere $\|\tau\| = 1$, $s = 0$, is compact, we may choose from the covering $\{W_\tau\}$ a finite subcovering $\{W_\tau\}$, $\tau \in I$. Let $\{\zeta_0, \zeta_\tau : \tau \in I\}$ be the corresponding smooth partition of unity:

$$\zeta_0 + \sum \zeta_\tau = 1, \quad \operatorname{supp} \zeta_\tau \subset W_\tau \quad \text{for } \tau \in I,$$
$$\zeta_0 = 0 \quad \text{in some neighbourhood of } \|\tau\| = 1.$$

Obviously the vector fields

$$\widetilde{\xi}_i = \sum_{\tau\in I} \zeta_\tau \xi_{\tau,i}, \qquad i = 0, 1, \ldots, k-2,$$

are continuous, lifted and split-stratified. Furthermore

$$\widetilde{\xi}_i(s_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

STEP 3. We restrict the vector fields $\widetilde{\xi}_i$ to the preimage of the set $\|\tau\| = 1$, $\|s\| \le \varepsilon$, where ε is positive and sufficiently small. Next we extend them by quasihomogeneity (preserving the quasidegree) to a quasihomogeneous neighbourhood of the origin minus the preimage of T. We obtain quasihomogeneous vector fields ξ_i such that

$$\eta_i = \Pi^{\#}(\xi_i) = \frac{\partial}{\partial s_i} + \sum \eta_{i,j,l} \frac{\partial}{\partial \tau_{j,l}}.$$

Moreover the quasidegree of η_i equals $-\operatorname{qdeg} s_i$, hence it is nonnegative. The quasidegrees of $\partial/\partial\tau_{j,l}$'s are negative and thus the quasidegrees of the corresponding coefficients are positive. Therefore these functions can be extended to continuous functions on the whole neighbourhood of the origin by putting 0 on T. We denote the extended continuous vector field by the same symbol η_i.

We cover the complement of the set $\{(0,0)\} \times T$ by three sets:

$$U_0 = U \setminus \Pi^{-1}(T), \quad U_1 = \left\{ F \neq 0,\ \frac{\partial F}{\partial x} \neq 0,\ \frac{\partial^2 F}{\partial y \partial x} \neq 0 \right\},$$
$$U_2 = \left\{ \frac{\partial F}{\partial y} \neq 0 \right\}.$$

U_0 is the domain of the vector fields ξ_i, $i = 0, \ldots, k-2$. We define

$$\xi_i' = \frac{-\eta_i(\partial F/\partial y)}{\partial^2 F/\partial x \partial y} \cdot \frac{\partial}{\partial x} + \eta_i \quad \text{on } U_1, \qquad \xi_i'' = \frac{-\eta_i(F)}{\partial F/\partial y} \cdot \frac{\partial}{\partial y} + \eta_i \quad \text{on } U_2.$$

Outside the set $\{(0,0)\} \times T$, we glue the three vector fields with the same index using the quasihomogeneous partition of unity $(\zeta_0, \zeta_1, \zeta_2)$. We obtain quasihomogeneous continuous vector fields defined outside $\{(0,0)\} \times T$. Next as above we extend them to the whole neighbourhood of the origin.

We thus obtain quasihomogeneous continuous vector fields $\xi_{k,i}$ such that $\xi_{k,i}(F) = A_i F$, where the A_i are bounded.

Furthermore the vector fields are split-stratified. Indeed, outside T we base on the fact that the conditions of split-stratification are invariant under analytic transformations of the deformation, hence also under quasihomogeneous $\mathbb{R}^+$ action. Thus they are valid for η_i's and ξ_i's. Next, since

$$\eta_i|_T = \frac{\partial}{\partial s_i},$$

each η_i is tangent to T. Thus condition (1) is fulfilled.

Condition (4) is a consequence of quasihomogeneity. Both the vector field η_i and the control function δ are quasihomogeneous. Furthermore $\operatorname{qdeg} \eta_i \geq 0$. Therefore

$$\operatorname{qdeg} \frac{\eta_i(\delta)}{\delta} \geq 0.$$

Hence the above quotient is bounded.

The coefficients of ξ_i' and ξ_i'' are smooth in x and y, hence condition (3) is valid for $\xi_{k,i}$. Furthermore

$$\xi_i'\left(\frac{\partial F}{\partial x}\right) = \frac{H_i'}{\partial F/\partial x} \cdot \frac{\partial F}{\partial x}, \qquad \xi_i''\left(\frac{\partial F}{\partial x}\right) = \frac{H_i''}{\partial F/\partial y} \cdot \frac{\partial F}{\partial y},$$

$$\xi_i\left(\frac{\partial F}{\partial x}\right) = A_1 F + A_2 \frac{\partial F}{\partial x} + A_3 \frac{\partial F}{\partial y}.$$

Thus, outside $\{(0,0)\} \times T$,

$$\xi_{k,i}\left(\frac{\partial F}{\partial x}\right) = \zeta_0 A_1 F + \left(\zeta_0 A_2 + \zeta_1 \frac{H_i'}{\partial F/\partial x}\right)\frac{\partial F}{\partial x} + \left(\zeta_0 A_3 + \zeta_2 \frac{H_i''}{\partial F/\partial y}\right)\frac{\partial F}{\partial y}.$$

Next

$$\xi_i'\left(\frac{\partial F}{\partial y}\right) = 0, \qquad \xi_i''\left(\frac{\partial F}{\partial y}\right) = \frac{H_i'''}{\partial F/\partial y} \cdot \frac{\partial F}{\partial y},$$

$$\xi_i\left(\frac{\partial F}{\partial y}\right) = B_1 F + B_2 \frac{\partial F}{\partial x} + B_3 \frac{\partial F}{\partial y}.$$

Thus, outside $\{(0,0)\} \times T$,

$$\xi_{k,i}\left(\frac{\partial F}{\partial x}\right) = \zeta_0 B_1 F + \zeta_0 B_2 \frac{\partial F}{\partial x} + \left(\zeta_0 B_3 + \zeta_2 \frac{H_i'''}{\partial F/\partial y}\right)\frac{\partial F}{\partial y}.$$

Note that since ζ_0 is zero on the set $\Pi^{-1}(T)$, ζ_1 is zero on the set $\partial_x F = 0$ and ζ_2 on the set $\partial_y F = 0$, and the coefficients of the above decompositions are quasihomogeneous with nonnegative weights, they are bounded on $\Pi^{-1}(T)$. Since for other μ-strata the coefficients A_i and B_i are locally bounded by construction, the coefficients of the above decompositions are bounded as well.

This proves condition (2) and finishes the proof of the proposition.

9. Proof of the Main Theorem

9.1. *Deformations.* To complete the proof of Theorem 2 for deformations it is enough to integrate the vector fields $\operatorname{Re} \xi_{k,i}$ and $\operatorname{Im} \xi_{k,i}$.

We remark that each $\xi_{k,i}$ is tangent to the subspace $s_{i+1} = \ldots = s_{k-2} = 0$. Thus:

- $\xi_{k,k-2}$ implies that F is equivalent to $F|_{s_{k-2}=0}$;
- $\xi_{k,k-3}$ implies that $F|_{s_{k-2}=0}$ is equivalent to $F|_{s_{k-3}=s_{k-2}=0}$; ...
- $\xi_{k,0}$ implies that $F|_{s_1=\ldots=s_{k-2}=0}$ is equivalent to $F|_{s_0=\ldots=s_{k-2}=0}$.

Hence $F(x, y, \tau, s)$ is equivalent to $F(x, y, \tau, 0)$.

Furthermore since the flows mentioned above are quasihomogeneous, the equivalence holds on some quasihomogeneous neighbourhood of the origin.

If the singularity is "'real", i.e. u belongs to $\mathbb{R}$, the vector fields can be constructed in such a way that they are invariant under complex conjugation, hence so are the flows.

9.2. *Unfoldings.* The unfolding $\mathcal{F}$ associated with the deformation F is the composition of two mappings

$$U' \xrightarrow{J} U \xrightarrow{\Pi} \Lambda,$$

where Π is the projection and $J(x, \tau') = (x, -F(x, 0, \tau'), \tau')$ is a parametrization of the set $F = 0$. Since the flows constructed in the previous section preserve the zero set of F, they induce the equivalence of unfoldings as well.

References

[1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps*, Vol. 1, Birkhäuser, 1985.

[2] J. Damon, *On the Pham example and the universal topological stratification of singularities*, in: Singularities, Banach Center Publ. 20, PWN–Polish Scientific Publ., Warszawa, 1988, 161–167.

[3] —, *A-equivalence and the equivalence of sections of images and discriminants*, in: Singularity Theory and its Applications, Part 1 (Coventry 1988/1989), Lecture Notes in Math. 1492, Springer, Berlin, 1991, 93–121.

[4] J. Damon and A. Galligo, *Universal topological stratification for the Pham example*, Bull. Soc. Math. France 121 (1993), 153–181.

[5] A. du Plessis and C. T. C. Wall, *Topological stability*, in: Singularities (Lille, 1991), London Math. Soc. Lecture Note Ser. 201, Cambridge Univ. Press, Cambridge, 1994, 351–362.

[6] —, —, *The Geometry of Topological Stability*, London Math. Soc. Monogr. (N.S.) 9, Oxford University Press, New York, 1995.

[7] P. Hartman, *Ordinary Differential Equations*, Birkhäuser, Boston, 1982.

[8] P. Jaworski, *Decompositions of hypersurface singularities of type $J_{k,0}$*, Ann. Polon. Math. 59 (1994), 117–131.

[9] —, *On the versal discriminant of the $J_{k,0}$ singularities*, ibid. 63 (1996), 89–99.

[10] —, *On the uniqueness of the quasihomogeneity*, in: Geometry and Topology of Caustics—Caustics '98, Banach Center Publ. 50, Inst. Math., Polish Acad. Sci., Warszawa, 1999, 163–167.

[11] E. Looijenga, *Semi-universal deformation of a simple elliptic hypersurface singularity, I: Unimodularity*, Topology 16 (1977), 257–262.
[12] K. Wirthmüller, *Universell topologische triviale Deformationen*, Ph.D. thesis, Univ. of Regensburg, 1979.

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