

On Cohen's proof of the Factorization Theorem

by JAN KISYŃSKI (Lublin)

Dedicated to the memory of Mieczysław Altman

Abstract. Various proofs of the Factorization Theorem for representations of Banach algebras are compared with its original proof due to P. Cohen.

Various proofs of the Cohen–Hewitt Factorization Theorem for representations of Banach algebras are presented and discussed in [H–R;II], pp. 263–290, [D–W], pp. 93–106 and 248–251, and [Pal], pp. 534–538.

The purpose of the present paper is to stress that all these proofs, and also some others, essentially rely upon a lemma which is implicitly contained in Cohen's paper [C]. This lemma appears below as Lemma 1 of Section 2.

Our discussion of various proofs of the Cohen–Hewitt Factorization Theorem is contained in Sections 2 and 3.2–3.4. Section 3.1 concerns the earlier factorization results of R. Salem, A. Zygmund and W. Rudin. In Sections 3.5–3.7 some papers are mentioned containing either more elaborate factorization theorems of the Cohen–Hewitt type or factorization results of another nature.

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1. The Cohen–Hewitt Factorization Theorem. The *left approximate identity* in a Banach algebra A is, by definition, a net $(e_\iota)_{\iota \in I} \subset A$ such that

$$(1) \quad \lim_{\iota} \|e_\iota a - a\|_A = 0 \quad \text{for every } a \in A.$$

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A left approximate identity $(e_\iota)_{\iota \in I} \subset A$ is said to be *bounded* if

$$\sup_{\iota \in I} \|e_\iota\|_A < \infty.$$

If A is a Banach algebra with bounded left approximate identity $(e_\iota)_{\iota \in I}$ and T is a continuous representation of A on a Banach space X , then

$$(2) \quad \lim_{\iota} \|T(e_\iota)y - y\|_X = 0 \quad \text{for every } y \in \overline{\text{span}} T(A)X.$$

This follows from the fact that $(T(e_\iota))_{\iota \in I}$ is a bounded net in $L(X)$ such that $\lim_{\iota} \|T(e_\iota)y - y\| = 0$ for every $y \in T(A)X$ ⁽¹⁾.

THE FACTORIZATION THEOREM (P. Cohen [C], 1959; E. Hewitt [H], 1964). *If A is a Banach algebra with bounded left approximate identity $(e_\iota)_{\iota \in I}$ and T is a continuous representation of A on a Banach space X , then $T(A)X$ is a closed subspace of X . Furthermore, for every $y \in T(A)X$ and every $\varepsilon > 0$ there are $a \in A$ and $x \in \overline{T(A)}y$ such that $T(a)x = y$, $\|x - y\| \leq \varepsilon$, and*

$$(3) \quad a = \sum_{n=1}^{\infty} p_n e_{\iota_n} \quad \text{where } \iota_n \in I, p_n > 0 \text{ for } n = 1, 2, \dots \text{ and } \sum_{n=1}^{\infty} p_n = 1.$$

The assertions of the Factorization Theorem may be expressed by the single condition

$$(4) \quad \text{for every } y \in \overline{\text{span}} T(A)X \text{ and every } \varepsilon > 0 \text{ there are } a \in A \text{ and } x \in \overline{T(A)}y \text{ such that } T(a)x = y, \|x - y\| \leq \varepsilon \text{ and (3) is satisfied.}$$

2. Cohen's proof of the Factorization Theorem. Let A be a Banach algebra, and T a continuous representation of A on a Banach space X . Suppose that there is a bounded left approximate identity in A . Then Cohen's proof of the existence of a factorization

$$y = T(a)x, \quad a \in A, x \in X,$$

of an element y of $\overline{\text{span}} T(A)X$ consists in approximation by a sequence of factorizations

$$y = \tilde{T}(a_n)x_n, \quad a_n \in G, x_n = \tilde{T}(a_n^{-1})y, \quad n = 1, 2, \dots,$$

where G is the set of invertible elements of the unitization A_u of A , and \tilde{T} is a representation of A_u on X extending the representation T .

⁽¹⁾ From (2) it follows at once that $\overline{\text{span}} T(A)X = \overline{T(A)X}$, i.e. $\overline{T(A)X}$ is a closed linear subspace of X . The factorization theorem says more: $T(A)X$ itself is a closed linear subspace. This last is almost trivial when A contains a unit e . Indeed, since $T(e) = T(e \cdot e) = T(e)T(e)$, it follows that $T(e)$ is a projector and hence $T(e)X$ is a closed subspace. Furthermore, $T(A)X = T(eA)X = T(e)T(A)X \subset T(e)X \subset T(A)X$, so that $T(A)X = T(e)X$.

The *unitization* A_u of A is the Banach algebra with unit such that

1° as a linear space, A_u is equal to the direct sum $\mathbb{K} + A$, where \mathbb{K} is the field of scalars of A ,

2° $A_u = \mathbb{K} + A$ is equipped with the norm $\| \cdot \|_{A_u}$ such that $\| \lambda + a \|_{A_u} = |\lambda| + \|a\|_A$ for every $\lambda \in \mathbb{K}$ and $a \in A$,

3° the multiplication in A_u is defined by

$$(\lambda + a)(\mu + b) = \lambda\mu + (\mu a + \lambda b + ab),$$

where $\lambda, \mu \in \mathbb{K}$ and $a, b \in A$.

The unit in A_u is $1 = 1 + 0 \in \mathbb{K} + A$ ⁽²⁾. Let

$$\varphi : A_u \rightarrow \mathbb{K}, \quad \pi : A_u \rightarrow A$$

be the projectors corresponding to the splitting $A_u = \mathbb{K} + A$. From 3° it follows that φ is a multiplicative functional on A_u . Let \tilde{T} be the representation of A_u on X such that

$$\tilde{T}(a) = \varphi(a) + T(\pi a) \quad \text{for every } a \in A_u.$$

The following lemma is implicitly contained in Cohen's paper [C].

LEMMA 1. *Under the assumptions of the Factorization Theorem, let*

$$\mathcal{E} = \{e_\iota : \iota \in I\}, \quad M = \sup_{e \in \mathcal{E}} \|e\|_A.$$

Take any $y \in \overline{\text{span}} T(A)X$, $a \in G$, $\gamma \in (0, 1/(M + 1))$, $\delta > 0$. Then there exists an $\tilde{a} \in G$ satisfying the three conditions:

- (i) $\varphi(\tilde{a}) = \varphi(a) - \gamma\varphi(a)$,
- (ii) $\pi\tilde{a} = \pi a + \gamma\varphi(a)e$ for some $e \in \mathcal{E}$,
- (iii) $\|\tilde{T}(\tilde{a}^{-1})y - \tilde{T}(a^{-1})y\|_X \leq \delta$.

Note that (i)&(ii) may be written as one condition

$$(iv) \quad \tilde{a} = a + \gamma\varphi(a)(e - 1),$$

and that (i)&(ii) implies

$$(v) \quad \|\pi\tilde{a} - \pi a\|_A \leq \gamma M |\varphi(a)| = M |\varphi(\tilde{a}) - \varphi(a)|.$$

Lemma 1 implies the Factorization Theorem. Let

$$p_n = \lambda_{n-1} - \lambda_n \quad \text{for } n = 1, 2, \dots,$$

where $\lambda_0, \lambda_1, \dots$ is a positive strictly decreasing sequence such that

$$(5) \quad \lambda_0 = 1, \quad \lim_{n \rightarrow \infty} \lambda_n = 0, \\ 1 - \frac{\lambda_n}{\lambda_{n-1}} = \gamma_n \in \left(0, \frac{1}{M + 1}\right) \quad \text{for } n = 1, 2, \dots$$

⁽²⁾ If A has a unit e , then A_u makes sense, but e is no longer a unit in A_u .

Then

$$(6) \quad \begin{aligned} \lambda_n &= (1 - \gamma_n)\lambda_{n-1}, \\ p_n &= \gamma_n\lambda_{n-1} > 0 \quad \text{for } n = 1, 2, \dots, \quad \text{and} \quad \sum_{n=1}^{\infty} p_n = 1. \end{aligned}$$

The concrete form of the sequence $\lambda_0, \lambda_1, \dots$ is inessential. One can follow [C] and take $\lambda_n = (1 - \gamma)^n$, $\gamma = \text{const} \in (0, 1/(M + 1))$, so that $\gamma_n = \gamma$, or $\lambda_n = (1 + M)/(1 + M + n)$, so that $\gamma_n = 1/(1 + M + n)$. From (5), (6) and (i)&(v)&(iii) it follows that for every $y \in \overline{\text{span}} T(A)X$ and every $\varepsilon > 0$ there exists a sequence a_0, a_1, \dots of elements of G such that $a_0 = 1$ and

$$\begin{aligned} (i)_0 & \quad \varphi(a_n) = \lambda_n, \\ (v)_0 & \quad \|\pi a_n - \pi a_{n-1}\|_A \leq Mp_n, \\ (iii)_0 & \quad \|\tilde{T}(a_n^{-1})y - \tilde{T}(a_{n-1}^{-1})y\|_X \leq \varepsilon p_n \end{aligned}$$

for $n = 1, 2, \dots$. From (i)₀&(v)₀&(iii)₀ it follows that $\lim_{n \rightarrow \infty} \|a_n - a\|_{A_u} = 0$ for some $a \in A$ such that

$$\|a\|_A = \left\| \pi \sum_{n=1}^{\infty} (a_n - a_{n-1}) \right\|_A \leq M \sum_{n=1}^{\infty} p_n = M,$$

and, if $x_n = \tilde{T}(a_n^{-1})y$, then $\lim_{n \rightarrow \infty} \|x_n - x\|_X = 0$ for some $x \in X$ such that

$$\|x - y\|_X = \|x - x_0\|_X \leq \sum_{n=1}^{\infty} \|x_n - x_{n-1}\|_X \leq \varepsilon \sum_{n=1}^{\infty} p_n = \varepsilon.$$

Since $y = \tilde{T}(a_n)x_n$ for every $n = 0, 1, \dots$, a passage to the limit implies that $y = \tilde{T}(a)x = T(a)x$. By (2), $y = \lim_i T(e_i)y \in \overline{T(A)y}$, whence

$$x_n = \tilde{T}(a_n^{-1})y = \varphi(a_n^{-1})y + T(\pi a_n^{-1})y \in \overline{T(A)y}$$

for $n = 0, 1, \dots$, and so $x = \lim_{n \rightarrow \infty} x_n \in \overline{T(A)y}$. Thus (i)&(v)&(iii) implies

$$(4)_M \quad \text{for every } y \in \overline{\text{span}} T(A)X \text{ and every } \varepsilon > 0 \text{ there are } a \in A \text{ and } x \in \overline{T(A)y} \text{ such that } \|a\|_A \leq M, \|x - y\|_X \leq \varepsilon \text{ and } T(a)x = y.$$

The statement (4)_M is a weakened version of (4): the condition (3) is replaced by $a \in A$, $\|a\|_A \leq M$. In order to prove (4), instead of (i)&(v)&(iii) one has to use (iv)&(iii). Indeed, (iv)&(iii) implies that for every $y \in \overline{\text{span}} T(A)X$ and every $\varepsilon > 0$ there exists a sequence a_0, a_1, \dots of elements of G such that $a_0 = 1$, the conditions (i)₀ and (iii)₀ are satisfied, and

$$(iv)_0 \quad \text{for every } n = 1, 2, \dots \text{ there is } e_n \in \mathcal{E} \text{ such that}$$

$$a_n = a_{n-1} + \gamma_n \varphi(a_{n-1})(e_n - 1) = a_{n-1} + p_n(e_n - 1).$$

Since (iv)₀ implies (v)₀, the preceding argument leading to (4)_M remains in

force. Furthermore, from (iv)₀ it follows that $a_n = \lambda_n + p_1e_1 + \dots + p_n e_n$ for every $n = 1, 2, \dots$ and hence, by (5) and (6), $\lim_{n \rightarrow \infty} \|a_n - a\|_{A_u} = 0$, where $a = \sum_{n=1}^{\infty} p_n e_n$.

Proof of Lemma 1 ⁽³⁾. For every $\iota \in I$ define

$$\tilde{a}_\iota = a + b_\iota, \quad b_\iota = \gamma\varphi(a)(e_\iota - 1).$$

Lemma 1 will follow once it is shown that

(I) there is $\iota_0 \in I$ such that $\tilde{a}_\iota \in G$ whenever $\iota_0 \prec \iota$,

and

(II) $\lim_{\iota} \|\tilde{T}(\tilde{a}_\iota^{-1})y - \tilde{T}(a^{-1})y\|_X = 0$.

In the proof of (I) we will use the equality

$$\tilde{a}_\iota = (1 + b_\iota a^{-1})a.$$

Notice that

$$\begin{aligned} b_\iota a^{-1} &= \gamma\varphi(a)(e_\iota - 1)[\varphi(a^{-1}) + \pi a^{-1}] \\ &= \gamma(e_\iota - 1) + \gamma\varphi(a)(e_\iota - 1)\pi a^{-1}, \end{aligned}$$

whence

(7) $\|b_\iota a^{-1}\|_{A_u} \leq \gamma(M + 1) + \gamma|\varphi(a)| \cdot \|(e_\iota - 1)\pi a^{-1}\|_A$.

Since $\gamma \in (0, 1/(M + 1))$, one can choose θ such that

$$\gamma(M + 1) < \theta < 1.$$

By (1) and (7), there is $\iota_0 \in I$ such that

$$\|b_\iota a^{-1}\|_{A_u} \leq \theta \quad \text{whenever } \iota_0 \prec \iota.$$

An application of the C. Neumann series shows that then $1 + b_\iota a^{-1} \in G$ and $\|(1 + b_\iota a^{-1})^{-1}\|_{A_u} \leq (1 - \theta)^{-1}$. In consequence, whenever $\iota_0 \prec \iota$, then $\tilde{a}_\iota = (1 + b_\iota a^{-1})a \in G$ ⁽⁴⁾ and

(8) $\|\tilde{a}_\iota^{-1}\|_{A_u} \leq \|a^{-1}\|_{A_u}(1 - \theta)^{-1}$.

For the proof of (II) observe that if $\iota_0 \prec \iota$, then

$$\tilde{a}_\iota^{-1} - a^{-1} = \tilde{a}_\iota^{-1}(a - \tilde{a}_\iota)a^{-1} = -\tilde{a}_\iota^{-1}b_\iota a^{-1} = -\gamma\varphi(a)\tilde{a}_\iota^{-1}(e_\iota - 1)a^{-1}.$$

As a consequence, by (8),

$$\begin{aligned} &\|\tilde{T}(\tilde{a}_\iota^{-1})y - \tilde{T}(a^{-1})y\|_X \\ &\leq \gamma|\varphi(a)| \cdot \|\tilde{T}\|_{L(A_u; L(X))} \|a^{-1}\|_{A_u} (1 - \theta)^{-1} \|[T(e_\iota) - 1]\tilde{T}(a^{-1})y\|_X. \end{aligned}$$

⁽³⁾ This proof differs from the argument in [C]; see Section 3.2.

⁽⁴⁾ This proof of the invertibility of \tilde{a}_ι follows [P;3], pp. 283–284, and [P–P], p. 136. The elements b , b' and c of A_u appearing in [P;3] correspond to our a^{-1} , \tilde{a}_ι^{-1} and $(1 + b_\iota a^{-1})^{-1}$. In [P–P] the elements of A_u are written in the form $a + r$, $a \in A$, $r \in \mathbb{K}$, so that our πa^{-1} corresponds to $(a + r)^{-1} - r^{-1}$ in [P–P].

From this estimate and from (2) the equality (II) follows, because $\widetilde{T}(a^{-1})y = \varphi(a^{-1})y + T(\pi a^{-1})y \in \overline{\text{span}} T(A)X$. ■

3. Forerunners of the Cohen–Hewitt Factorization Theorem, variants of the proof, and references to other factorization results

3.1. Forerunners of the Cohen–Hewitt theorem. Let G be a locally compact group with a fixed left-invariant Haar measure μ , and let $L^1(G)$ be the space of the equivalence classes of functions on G integrable with respect to μ . The convolution $a * b$ of two elements, a and b , of $L^1(G)$ is defined by

$$(a * b)(g) = \int_G a(h)b(h^{-1}g) \mu(dh) \quad \text{for a.e. } g \in G.$$

Then $L^1(G)$ is a convolution Banach algebra. Cohen’s paper [C] contains the following results which follow from the Factorization Theorem applied to $A = L^1(G)$, $X = L^1(G)$, or $X = C(G)$ if G is compact, and to the representation of A on X defined by $T(a)x = a * x$:

(a) $L^1(G) = L^1(G) * L^1(G)$ for every locally compact group G . This was proved earlier in particular cases: for $G = \mathbb{T}$, the circle group, by R. Salem and A. Zygmund, and for $G = \mathbb{R}$ and G a locally euclidian abelian group by W. Rudin [R;1], [R;2].

(b) $C(G) = L^1(G) * C(G)$ for every compact group G . In the particular case of $G = \mathbb{T}$ this was proved earlier by R. Salem and A. Zygmund.

(c) Strictly positive factorization: if G is a compact group, $y \in C(G)$ and $\inf_G y > 0$, then there are $a \in L^1(G)$ and $x \in C(G)$ such that $\text{ess inf}_G a > 0$, $\inf_G x > 0$ and $a * x = y$. In the deduction of this result from the Factorization Theorem the conditions (3) and $\|x - y\| \leq \varepsilon$ are essential. Note that Cohen [C] proved that an analogous non-negative factorization does not hold.

In their proofs of the particular cases of (a) and (b), R. Salem, A. Zygmund and W. Rudin used the Fourier methods which are not applicable in the general situation. Nevertheless it seems interesting to compare the condition (3) from the factorization theorem with the formulas labelled below by (*) and (*_*), appearing in the proof of the factorization $L^1(\mathbb{R}) = L^1(\mathbb{R}) * L^1(\mathbb{R})$ presented in [R;2]. The approximate identity in $L^1(\mathbb{R})$ appears there in the form $e_t = K_t$ for $t \in (0, \infty)$, where

$$K_t(x) = \frac{t}{2\pi} \left(\frac{\sin(tx/2)}{tx/2} \right)^2$$

is the Fejér kernel. The Fourier transform of K_t is

$$\widehat{K}_t(\xi) = (1 - |\xi|/t)^+.$$

For given $f \in L^1(\mathbb{R})$ Rudin proves that $f = a * g$ where $a \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ are defined as the integrals of continuous $L^1(\mathbb{R})$ -valued functions

$$(*) \quad a = \int_0^{\infty} t\lambda''(t)K_t dt,$$

$$(*) \quad g = f + \int_0^{\infty} \alpha(t)(f - K_t * f) dt.$$

Here $\alpha \in C[0, \infty)$ and $\lambda \in C^2[0, \infty)$ are positive functions such that

$$\int_0^{\infty} \alpha(t) dt = \infty, \quad \int_0^{\infty} \alpha(t) \left(\|K_t * f - f\| + \frac{1}{t} \right) dt < \infty, \quad \lambda(t) = \frac{1}{\Phi(t)},$$

where

$$\Phi(t) = 1 + \int_0^t \alpha(s) ds + t \int_t^{\infty} \frac{\alpha(s)}{s} ds.$$

These definitions imply that $g \in L^1(\mathbb{R})$, λ'' is positive and

$$(**) \quad \int_0^{\infty} t\lambda''(t) dt = 1,$$

so that $a \in L^1(\mathbb{R})$, and furthermore, $\widehat{a}(\xi) = \lambda(|\xi|)$ and $\widehat{g}(\xi) = \Phi(|\xi|)\widehat{f}(\xi)$ ⁽⁵⁾, whence $\widehat{a}\widehat{g} = \widehat{f}$, and so $a * g = f$.

Discrete forerunners of $(*)$ and $(**)$ appear in the book of A. Zygmund [Z;I] as (1.7) on p. 183, (4.2) on p. 93, and Theorem 1.5 on p. 183 (attributed on p. 378 to H. W. Young (1913) and A. N. Kolmogorov (1923)). A discrete forerunner of $(*)$ is the second formula in line 111³ of R. Salem's paper [S], related to a limit passage in (3) on p. 110 of [S]. A. Zygmund [Z;I], p. 378₁₁₋₆, points out that the theorems from [S] together with Theorem 1.5 of Chap. V of [Z;I], p. 183, lead to the factorization results for $G = \mathbb{T}$ mentioned in (a) and (b). A discrete analogue of the whole elegant Rudin construction is used in Sec. 7.5 of the book of R. E. Edwards [E;I] for reproving the Salem–Zygmund results. Remarks on the Fourier factorization may be found in Sec. 3.1.1(c) and 7.5.2–4 of [E;I] and Sec. 11.4.18(6) of [E;II].

⁽⁵⁾ Indeed, $\Phi'(t) = \int_t^{\infty} (\alpha(s)/s) ds$, $\Phi''(t) = -\alpha(t)/t \leq 0$, $\lambda'(t) = -\Phi'(t)/[\Phi(t)]^2 \leq 0$, $\lambda''(t) = (2[\Phi'(t)]^2 - \Phi''(t)\Phi(t))/[\Phi(t)]^3 \geq 0$. Hence $0 \leq -t\lambda'(t) = t\Phi'(t)/[\Phi(t)]^2 \leq [\Phi(t)]^{-1}$, so $\lim_{t \rightarrow \infty} t\lambda'(t) = \lim_{t \rightarrow \infty} \lambda(t) = 0$ and $\int_0^{\infty} t\lambda''(t) dt = t\lambda'(t)|_{t=0}^{t=\infty} - \int_0^{\infty} \lambda'(t) dt = \lambda(0) = 1$. Moreover, $(*)$ implies $\widehat{a}(\xi) = \int_0^{\infty} t\lambda''(t)(1 - |\xi|/t)^+ dt = \int_{|\xi|}^{\infty} (t - |\xi|)\lambda''(t) dt = (t - |\xi|)\lambda'(t)|_{t=|\xi|}^{t=\infty} - \int_{|\xi|}^{\infty} \lambda'(t) dt = \lambda(|\xi|)$, and $\widehat{g}(\xi) = [1 + \int_0^{\infty} \alpha(t)(1 - \widehat{K}_t(\xi)) dt]\widehat{f}(\xi) = \Phi(|\xi|)\widehat{f}(|\xi|)$ by $(*)$.

3.2. *The original calculations of Cohen.* As already mentioned, Lemma 1 is implicitly contained in Cohen's paper [C]. The argument used in [C] for proving this implicit lemma was repeated in [V], [H], [C-FT], [J], [G-L-R], [T], [S-T], [H-R;II], [B-D], [A-S], [D-W], [G] and [Pal]. In our notation, this argument is as follows. For every $\iota \in I$ one has $\|\gamma(e_\iota - 1)\| \leq \gamma(M+1) = \vartheta < 1$, whence $1 - \gamma + \gamma e_\iota \in G$ and $\|(1 - \gamma + \gamma e_\iota)^{-1}\| \leq (1 - \vartheta)^{-1}$. The possibility of choosing $\iota_0 \in I$ so that

$$(I) \quad \tilde{a}_\iota = a + \gamma\varphi(a)(e_\iota - 1) \in G \quad \text{whenever } \iota_0 \prec \iota$$

is proved in [C] by means of the formulas

$$(9) \quad \tilde{a}_\iota = (1 - \gamma + \gamma e_\iota)a_\iota, \quad a_\iota = \varphi(a) + (1 - \gamma + \gamma e_\iota)^{-1}\pi a.$$

Since

$$\begin{aligned} \|a_\iota - a\| &= \|(1 - \gamma + \gamma e_\iota)^{-1}\pi a - \pi a\| \\ &= \|\gamma(1 - \gamma + \gamma e_\iota)^{-1}(1 - e_\iota)\pi a\| \\ &\leq \gamma(1 - \vartheta)^{-1}\|e_\iota\pi a - \pi a\|, \end{aligned}$$

it follows by (1) that

$$(10) \quad \lim_{\iota} \|a_\iota - a\| = 0.$$

Hence there is $\iota_0 \in I$ such that if $\iota_0 \prec \iota$, then $\|a_\iota - a\| \leq \frac{1}{2}\|a^{-1}\|^{-1}$, and so $a_\iota = [1 + (a_\iota - a)a^{-1}]a \in G$ and $\|a_\iota^{-1}\| \leq 2\|a^{-1}\|$. In consequence, if $\iota_0 \prec \iota$, then $\tilde{a}_\iota = (1 - \gamma + \gamma e_\iota)a_\iota \in G$, proving (I). The equality

$$(II) \quad \lim_{\iota} \|\tilde{T}(\tilde{a}_\iota^{-1})y - \tilde{T}(a^{-1})y\| = 0$$

is proved in [C] as follows. If $\iota_0 \prec \iota$, then

$$\begin{aligned} \tilde{a}_\iota^{-1} - a^{-1} &= (a_\iota^{-1} - a^{-1})(1 - \gamma + \gamma e_\iota)^{-1} + a^{-1}[(1 - \gamma + \gamma e_\iota)^{-1} - 1] \\ &= -a_\iota^{-1}(a_\iota - a)a^{-1}(1 - \gamma + \gamma e_\iota)^{-1} \\ &\quad - \gamma a^{-1}(1 - \gamma + \gamma e_\iota)^{-1}(e_\iota - 1), \end{aligned}$$

whence

$$\begin{aligned} &\|\tilde{T}(\tilde{a}_\iota^{-1})y - \tilde{T}(a^{-1})y\| \\ &\leq \|\tilde{T}\| \cdot \|a^{-1}\|(1 - \vartheta)^{-1}\{2\|a^{-1}\| \cdot \|y\| \cdot \|a_\iota - a\| + \gamma\|T(e_\iota)y - y\|\}. \end{aligned}$$

This estimate implies (II), by (10) and (2).

3.3. *Variants of Cohen's proof. The method of Koosis.* The reasoning presented in Section 3.2 becomes simpler when the elements $\tilde{a}_\iota = a + \gamma\varphi(a)(e_\iota - 1)$ satisfying (9) are replaced by

$$\hat{a}_\iota = (1 - \gamma + \gamma e_\iota)a.$$

Then, for every $\iota \in I$, $\varphi(\hat{a}_\iota) = (1 - \gamma)\varphi(a)$,

$\|\pi\hat{a}_l - \pi a\| = \|\gamma\varphi(a)e_l + \gamma(e_l - 1)\pi a\| \leq \gamma|\varphi(a)|M + \gamma\|e_l\pi a - \pi a\|$,
 $\hat{a}_l \in G$, and $\hat{a}_l^{-1} - a^{-1} = -\gamma a^{-1}(1 - \gamma + \gamma e_l)^{-1}(e_l - 1)$, whence
 $\|\tilde{T}(\hat{a}_l^{-1})y - \tilde{T}(a^{-1})y\| \leq \gamma\|\tilde{T}\| \cdot \|a^{-1}\|(1 - \vartheta)^{-1}\|T(e_l)y - y\|$, $\vartheta = \gamma(M+1)$.

This yields

LEMMA 2. *Under the notation of Sections 1 and 2, and the assumptions of the Factorization Theorem, let $y \in \overline{\text{span}} T(A)X$, $a \in G$, $\gamma \in (0, 1/(M+1))$, $\delta > 0$. Then there is $\hat{a} \in G$ such that $\hat{a} = (1 - \gamma + \gamma e)a$ for some $e \in \mathcal{E}$,*

$$\|\pi\hat{a} - \pi a\| \leq \gamma|\varphi(a)|M + \delta \quad \text{and} \quad \|\tilde{T}(\hat{a}^{-1})y - \tilde{T}(a^{-1})y\| \leq \delta.$$

For given $y \in \overline{\text{span}} T(A)X$, $\varepsilon > 0$ and $\gamma \in (0, 1/(M+1))$, using Lemma 2, one can define inductively e_1, e_2, \dots in \mathcal{E} so that the formulas

$$(11) \quad a_0 = 1, \quad a_n = (1 - \gamma + \gamma e_n)a_{n-1} \quad \text{for } n = 1, 2, \dots$$

define a sequence a_0, a_1, \dots in G such that $\varphi(a_n) = (1 - \gamma)^n$, $\|\pi a_n - \pi a_{n-1}\| \leq \gamma(1 - \gamma)^{n-1}(M + \varepsilon)$ and $\|\tilde{T}(a_n^{-1})y - \tilde{T}(a_{n-1}^{-1})y\| \leq \gamma(1 - \gamma)^{n-1}\varepsilon$ for $n = 1, 2, \dots$. This yields

$$(\hat{4})_M \quad \text{for every } y \in \overline{\text{span}} T(A)X \text{ and every } \varepsilon > 0 \text{ there are } \hat{a} \in A \text{ and } \hat{x} \in \overline{T(A)y} \text{ such that } \|\hat{a}\| \leq M + \varepsilon, \|\hat{x} - y\| \leq \varepsilon \text{ and } T(\hat{a})\hat{x} = y.$$

If, for given $y \in \overline{\text{span}} T(A)X$ and $\varepsilon > 0$, \hat{a} and \hat{x} satisfy $(\hat{4})_M$, then for $a = \frac{M}{M+\varepsilon}\hat{a}$ and $x = \frac{M+\varepsilon}{M}\hat{x}$ one has $\|a\| \leq M$ and

$$\|x - y\| \leq \|\hat{x} - y\| + \frac{\varepsilon}{M}\|\hat{x}\| \leq \|\hat{x} - y\| + \frac{\varepsilon}{M}\|y\| + \frac{\varepsilon}{M}\|\hat{x} - y\| \leq \left(1 + \frac{\|y\| + \varepsilon}{M}\right)\varepsilon.$$

Hence $(\hat{4})_M$ is equivalent to $(4)_M$ from Section 2.

It was P. Koosis [K] who showed that the elements $a \in A$ and $x \in \overline{T(A)y}$ satisfying $T(a)x = y$ for a given $y \in \overline{\text{span}} T(A)X$ may be determined by an approximation $a = \lim a_n$, $x = \lim \tilde{T}(a_n^{-1})y$, where $a_0 = 1 \in A_u$, $a_n = (1 - \gamma + \gamma e_n)a_{n-1}$ for $n = 1, 2, \dots$, $\gamma = \text{const} \in (0, 1/(M+1))$, and e_1, e_2, \dots is a suitable sequence of elements of \mathcal{E} . This idea was continued by M. Altman [A;1-5], R. S. Doran and J. Wichmann [D-W], pp. 97-100, J. Esterle [ES;4] and H. G. Feichtinger and M. Leinert [F-L], still for Banach algebras.

W. Żelazko [Ż], Sec. 6.4, pp. 24-26, showed that the Koosis-Altman formulas (11) may be replaced by

$$(12) \quad a_0 = 1, \quad a_n = \Phi_\gamma(e_n)a_{n-1} \quad \text{for } n = 1, 2, \dots,$$

where $\gamma > 0$ is a constant such that $\gamma(M-1) < 1$, and

$$\Phi_\gamma(e) = (1 + \gamma - \gamma e)^{-1}$$

for every $e \in \mathcal{E}$. Then

$$\begin{aligned} \|\Phi_\gamma(e)\| &\leq (1 + \gamma - \gamma M)^{-1}, & \Phi_\gamma(e) - 1 &= \gamma\Phi_\gamma(e)(e - 1), \\ [\Phi_\gamma(e)]^{-1} - 1 &= -\gamma(e - 1), \end{aligned}$$

which implies

LEMMA 3. *Under the notation of Sections 1 and 2, and the assumptions of the Factorization Theorem, let $y \in \overline{\text{span}} T(A)X$, $a \in G$, $\gamma > 0$, $\gamma(M - 1) < 1$, $\delta > 0$. Then there is $e \in \mathcal{E}$ such that if $\widehat{a} = \Phi_\gamma(e)a$, then $\widehat{a} \in G$,*

$$\|\pi\widehat{a} - \pi a\| = \|\widehat{a} - a\| - |\varphi(\widehat{a}) - \varphi(a)| \leq \frac{\gamma(1 + M)}{1 + \gamma - \gamma M}|\varphi(a)| + \delta - \frac{\gamma}{1 + \gamma}|\varphi(a)|$$

and

$$\|\widetilde{T}(\widehat{a}^{-1})y - \widetilde{T}(a)y\| \leq \delta.$$

For given $y \in \overline{\text{span}} T(A)X$, $\varepsilon > 0$ and $\gamma > 0$ such that $\gamma(M - 1) < 1$, using Lemma 3, one can define inductively e_1, e_2, \dots in \mathcal{E} so that the sequence a_0, a_1, \dots of elements of G defined by (12) satisfies

$$\begin{aligned} \varphi(a_n) &= (1 + \gamma)^{-n}, \\ \|\pi a_n - \pi a_{n-1}\| &\leq \left(\frac{\gamma(1 + M)}{1 + \gamma - \gamma M} - \frac{\gamma}{1 + \gamma} \right) (1 + \gamma)^{1-n} + \frac{\varepsilon}{2^{n+1}}, \end{aligned}$$

and $\|\widetilde{T}(a_n^{-1})y - \widetilde{T}(a_{n-1}^{-1})y\| \leq \varepsilon/2^n$. If $a = \lim a_n$ and $x = \lim \widetilde{T}(a_n^{-1})y$, then $a \in A$, $x \in \overline{T(A)y}$, $y = T(a)x$, $\|x - y\| \leq \varepsilon$ and

$$\|a\| \leq \sum_{n=1}^{\infty} \|\pi a_n - \pi a_{n-1}\| \leq \frac{(1 + \gamma)(1 + M)}{1 + \gamma - \gamma M} - 1 + \frac{\varepsilon}{2}.$$

For $\gamma > 0$ such that

$$\gamma(M - 1) < 1 \quad \text{and} \quad \frac{(1 + \gamma)(1 + M)}{1 + \gamma - \gamma M} \leq 1 + M + \frac{\varepsilon}{2},$$

one obtains $(\widehat{4})_M$, which is equivalent to (4)_M.

In [H-R;II], Sec. 32.50, pp. 287–288, an argument similar to that of Żelazko is carried out under the assumptions that $M = 1$ and the approximate identity is two-sided. Instead of (12) Hewitt and Ross use the formula $a_n = a_{n-1}\varphi(e_n)$ where $\varphi(e) = \frac{1}{3}[1 + \sum_{k=1}^{\infty} (\frac{1}{3}[1 + e])^k]$ is equal to our $\Phi_1(e) = (2 - e)^{-1} = \frac{1}{3}(1 - \frac{1}{3}[1 + e])^{-1}$. A construction similar to that of Żelazko also appears in [Es;1] in the proof of Theorem 2.8.

In Section 3.5 some papers will be mentioned in which the constructions of Koosis and Żelazko are applied to representations of some non-Banach algebras. These applications are possible thanks to the fact that these constructions lead to sequences of factorizations $y = \widetilde{T}(a_n)x_n$, $x_n = \widetilde{T}(a_n^{-1})y$,

for which the proofs of the invertibility of a_n 's and of the existence of $\lim \tilde{T}(a_n^{-1})y$ are simpler than in the case of Cohen's construction.

3.4. Deduction of the Cohen–Hewitt Factorization Theorem from some results of V. Pták and J. Esterle. In a somewhat weakened version, namely with the condition $a \in \overline{\text{conv}}\{e_\iota : \iota \in I\}$ but without (3) ⁽⁶⁾, the Factorization Theorem may be deduced from the “induction theorem” of V. Pták ([P;3], p. 280; [P–P], p. 5; [P;1]; [P;2]), and also from the theorem of J. Esterle about projective systems ([Es;4], p. 109, Th. 2.1). The applicability of these theorems to the factorization problem follows from Lemmas 1–3. The proofs of both theorems are similar to the part of Cohen's argument presented above as “Lemma 1 implies the Factorization Theorem”. Nevertheless, the approach to factorization via the theorem of Pták or that of Esterle is interesting because it emphasizes the geometric nature of Cohen's construction.

Under the notation of Sections 1 and 2, consider in A_u the subset

$$E = \text{conv}(\{1\} \cup \mathcal{E}) = \bigcup_{0 \leq \lambda \leq 1} (\lambda + (1 - \lambda)\text{conv } \mathcal{E}).$$

Take any $y \in \overline{\text{span}} T(A)X$ and $\varepsilon > 0$. For every $\lambda \in (0, 1]$ let

$$Z(\lambda) = \{(a, x) \in (E \cap G) \times X : \|x - y\| \leq \varepsilon(1 - \varphi(a)), \tilde{T}(a)x = y, 0 < \varphi(a) \leq \lambda\}.$$

Then $(1, y) \in Z(1)$ and $Z(\lambda') \subset Z(\lambda'') \subset (E \cap G) \times \overline{T(A)y}$ whenever $0 < \lambda' < \lambda'' \leq 1$. If $(a, x) \in \bigcap_{0 < \lambda \leq 1} Z(\lambda)$, then $a \in \overline{\text{conv}} \mathcal{E} \subset A$, $x \in \overline{T(A)y}$, $\|x - y\| \leq \varepsilon$, and $T(a)x = y$. Hence the weakened version of the statement (4) will follow once it is shown that

$$(*) \quad \bigcap_{0 < \lambda \leq 1} \overline{Z(\lambda)} \neq \emptyset.$$

A proof of (*) may be obtained by using Lemma 1 and applying Pták's or Esterle's theorem. Indeed, Lemma 1 implies

COROLLARY. Fix any $y \in \overline{\text{span}} T(A)X$ and $\varepsilon > 0$. Then for every $\lambda \in (0, 1]$, $(a, x) \in Z(\lambda)$ and $\gamma \in (0, 1/(M + 1))$ there exists $(\tilde{a}, \tilde{x}) \in Z((1 - \gamma)\lambda)$ such that $\varphi(\tilde{a}) = (1 - \gamma)\varphi(a)$ and

$$\frac{\|\pi\tilde{a} - \pi a\|}{M} \vee \frac{\|\tilde{x} - x\|}{\varepsilon} \leq \gamma\varphi(a) = \varphi(a) - \varphi(\tilde{a}).$$

Define in $A_u \times X$ the norm

$$\|(a, x)\| = \left(\frac{\|\pi a\|}{M} \vee \frac{\|x\|}{\varepsilon} \right) + |\varphi(a)|, \quad a \in A_u, x \in X.$$

⁽⁶⁾ Sometimes this may be an essential loss of information, see (c) in Section 3.1.

From the Corollary it follows that for every $\lambda \in (0, 1]$, $(a, x) \in Z(\lambda)$ and $\gamma \in (0, 1/(M+1))$ there is $(\tilde{a}, \tilde{x}) \in Z((1-\gamma)\lambda)$ such that $\|(\tilde{a}, \tilde{x}) - (a, x) \| \leq 2\gamma\lambda$. Since $(1, y) \in Z(1)$, it follows that $Z(\lambda) \neq \emptyset$ for every $\lambda \in (0, 1]$, and

$$(*) \quad \sup_{(a,x) \in Z(\lambda)} \text{dist}((a, x); Z((1-\gamma)\lambda)) \leq 2\gamma\lambda$$

for every $\lambda \in (0, 1]$ and $\gamma \in (0, 1/(M+1))$, where the distance is defined by the norm $\| \cdot \|$.

The implication $(*) \Rightarrow (*)$ is a consequence of Pták's theorem, and also of Esterle's. Indeed, if $\tilde{Z}(t) = Z(t/(2\gamma))$ for $t \in (0, 2\gamma]$, then $(*)$ takes the form

$$\sup_{(a,x) \in \tilde{Z}(t)} \text{dist}((a, x); \tilde{Z}(w(t))) \leq t \quad \text{for every } t \in (0, 2\gamma],$$

where $w(t) = (1-\gamma)t$. Since $w(t) + (w \circ w)(t) + \dots = \sum_{n=1}^{\infty} (1-\gamma)^n t < \infty$ for every $t \in (0, 2\gamma]$, the function w is a *rate of convergence* on $(0, 2\gamma]$ ([P-P], p. 2, Def. 1). Hence the theorem of Pták ([P-P], p. 5, Proposition 1.7) implies that $\bigcap_{0 < t < 2\gamma} \tilde{Z}(t) \neq \emptyset$, proving $(*)$.

In order to see that $(*) \Rightarrow (*)$ is a consequence of Esterle's theorem ([Es;4], p. 109, Th. 2.1), take any positive strictly decreasing sequence $\lambda_0, \lambda_1, \dots$ satisfying (5). Then, by $(*)$,

$$\sup_{(a,x) \in Z(\lambda_{n-1})} \text{dist}((a, x); Z(\lambda_n)) \leq 2\gamma\lambda_{n-1} = 2(\lambda_{n-1} - \lambda_n)$$

for $n = 1, 2, \dots$. Since $\sum_{n=1}^{\infty} 2(\lambda_{n-1} - \lambda_n) = 2 < \infty$, Esterle's theorem implies that $\bigcap_{n=0}^{\infty} \overline{Z(\lambda_n)} \neq \emptyset$, proving $(*)$.

Similar applications of the theorems of Pták and Esterle to the factorization problem are presented in [K-V]; [P-P], pp. 134–137; [D-W], pp. 100–104; [Es;4], pp. 115–119. In [P;3] a more sophisticated factorization theorem is deduced from the “induction theorem”.

3.5. Extensions of the Cohen–Hewitt theorem to representations of some non-Banach algebras. It was observed in [O] and [Cr] that the argument of Koosis, presented in Remark 3, works in every Fréchet algebra $(\overline{})$ with a *uniformly bounded left approximate identity* (u.b.l.a.i.). This idea was taken up in [W.H.S], [M.K.S], [Vo;1] and [Vo;2]. Examples of Fréchet algebras with u.b.l.a.i. are discussed in [O] and [Vo;1]. In [D] the u.b.l.a.i. is defined for any metrizable algebra, and the argument of Żelazko, described in our Sec. 3.3, is used for extending the factorization theorem to any *essential* continuous representation T of a complete metrizable locally convex algebra A with

⁽⁷⁾ That is, an algebra which is a complete metrizable l.c.v.s. and, moreover, is locally multiplicatively convex.

u.b.l.a.i. on a complete metrizable vector space X . In [AP] this last result is generalized to complete metrizable algebras which need not be locally convex, but are *fundamental* ([AP], p. 54, Def. 2.1).

3.6. Simultaneous factorization, factorization involving an analytic function, power factorization, and factorization by means of a one-parameter semigroup. The arguments discussed in Sections 2 and 3.1–4 were also used for proving theorems about *simultaneous factorization* of all the elements of a subset of $\overline{\text{span}}T(A)X$ by means of a single element of A . For the case of Banach algebras see [R;2]; [V]; [J]; [T]; [Ri]; [S–T]; [C–S]; [H–R;II], pp. 268–269, Theorem 32.23 and Corollary 32.24; [W.H.S]; [P1]; [B–D], p. 62, Corollary 12; [Ž], p. 23, Theorem 6.4; [Pal], pp. 538–539, Corollary 5.2.3. For extensions to Fréchet algebras (and more general algebras) see [O]; [Cr]; [M.K.S]; [Ž], p. 34, Theorem C.6.1; [Vo;1]; [Vo;2]; [D], Corollary 4.2.

Cohen's basic idea of imbedding a Banach algebra A into its unitization A_u and using an approximation in A_u also appears in [C–St] and [S;2] in connection with the factorization $y = T(f(a))x$ where f is an analytic function, in [A–S], [Es;1], [P;2] and [G] where the power factorization is considered, and in [S;1,3,4] and [Pal], Sec. 5.3, pp. 543–548, in connection with the factorization by means of a one-parameter semigroup. The factorization theorem of A. M. Sinclair [S;4], pp. 35–37, concerns a representation T of a complex Banach algebra A on a complex Banach space X , and states that if there is a countable bounded two-sided approximate identity in A , then for every $y \in \overline{\text{span}}T(A)X$ there are a holomorphic map $\mathbb{C} \ni z \mapsto x_z \in X$ and a holomorphic semigroup $\mathbb{C}^+ \ni z \mapsto a^z \in A$ defined on the complex right half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Re } z > 0\}$, such that $y = x_0 = T(a^z)x_z$ for every $z \in \mathbb{C}^+$, and $\lim_{U(\Psi) \ni z \rightarrow 0} \|a^z b - b\|_A = 0$ for every $b \in A$ and every sector $U(\Psi) = \{z \in \mathbb{C}^+ : |\text{Arg } z| \leq \Psi\}$, $0 < \Psi < \pi/2$. The growth properties of the holomorphic semigroups $\mathbb{C}^+ \ni z \mapsto a^z \in A$ influence the structure of a Banach algebra A . See [Es;2,3], [S;4], [W], [G–W], [G–R] and other papers quoted there.

3.7. Factorization without a bounded approximate identity. Not all factorization theorems for topological algebras and their representations are proved by arguments of Cohen's type, related to bounded approximate identities. Indeed, the papers [Pas], [L], [D–M], [P–V], [Vo;1] and [Ou] deal with factorization and weak factorization in Banach algebras, and non-Banach function algebras (with pointwise multiplication) and convolution algebras, without bounded approximate identities. Also the proofs of weak factorization theorems for general separable Banach algebras, presented in [Pal], Sec. 5.3, pp. 549–552, have nothing in common with Cohen's argument.

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Faculty of Electrical Engineering
Technical University of Lublin
Nadbystrzycka 38A, P.O. Box 189
20-618 Lublin, Poland
E-mail: kisynski.wzipt@antenor.pol.lublin.pl

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