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The coincidence index for fundamentally contractible multivalued maps with nonconvex values

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Abstract. We study a coincidence problem of the form $A(x) \in \phi(x)$, where A is a linear Fredholm operator with nonnegative index between Banach spaces and ϕ is a multivalued A-fundamentally contractible map (in particular, it is not necessarily compact). The main tool is a coincidence index, which becomes the well known Leray–Schauder fixed point index when $A = \mathrm{id}$ and ϕ is a compact singlevalued map. An application to boundary value problems for differential equations in Banach spaces is given.

Introduction. When studying boundary value problems for differential equations or inclusions one usually encounters the question of solvability of an inclusion

$$N(x) \in G(x), \tag{1}$$

where $N : X \to Y$ is a bounded linear operator between Banach spaces and $G : \operatorname{cl} U \to Y$ is a single- or multivalued transformation defined on the closure of an open set $U \subset X$, subject to some "boundary" conditions of the form

$$l(x) \in B(x), \tag{2}$$

where $l : X \to E'$ is a bounded linear map into a Banach space E' and $B : \operatorname{cl} U \to E'$ is again a single- or multivalued transformation.

For instance, let $C = C^1([0,T], E)$ be the space of C^1-functions defined on the interval $[0,T]$ with values in a Banach space E and let $f : [0,T] \times E \times E \to E$ be a continuous map. If we study the existence of solutions to the general boundary value problem

$$\begin{cases} u'(t) = f(t, u(t), u'(t)), \\ l_1(u(0)) + l_2(u(T)) \in b(u), \end{cases} \tag{3}$$

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where $l_1, l_2 : E \to E'$ are bounded linear maps, and $b : C \multimap E'$ is a multivalued map (E' is a Banach space), then we reformulate it as

$$
\begin{cases}
y(t) = f\Big(t, z + \int\limits_0^t y(s)\,ds, y(t)\Big), \\
l_1(z) + l_2\Big(z + \int\limits_0^T y(s)\,ds\Big) \in b\Big(z + \int\limits_0^{\cdot} y(s)\,ds\Big).
\end{cases}
\tag{4}
$$

Obviously, if $(z, y) \in E \times C([0,T], E)$ is a solution to (4), then $u(t) = z + \int_0^t y(s)\,ds$ is a solution to (3).

If we set

$$
x = (z, y), \qquad N(z, y) = y,
$$

$$
G(z, y) = f\Big(\cdot, z + \int\limits_0^{\cdot} y(s)\,ds, y(\cdot)\Big), \qquad l(z, y) = l_1(z) + l_2(z),
$$

$$
B(z, y) = b\Big(z + \int\limits_0^{\cdot} y(s)\,ds\Big) - l_2\Big(\int\limits_0^T y(s)\,ds\Big),
$$

then problem (4) is equivalent to (1), (2).

Observe that in case $E = E'$, $l_1 = \mathrm{id}_E$, $l_2 = -\,\mathrm{id}_E$ and $b \equiv 0$, (3) becomes an ordinary periodic boundary value problem.

It is clear that without suitable assumptions concerning N, G, l and B it does not make sense to study the solvability of (1), (2). To see what type of hypotheses are most reasonable, let us further transform the problem considered. Namely, putting $A = (N, l) : X \to Y \oplus E'$ and $\phi = (G, B) : \mathrm{cl}\, U \multimap Y \oplus E'$, we arrive at a coincidence problem (a generalized fixed point problem) of the form

$$
A(x) \in \phi(x). \tag{5}
$$

Problems of this type have been intensively studied by many authors, especially when ϕ is a compact map and A is the identity (the Leray–Schauder fixed point theory) or A is a Fredholm operator of index 0 (e.g. Mawhin [17], Pruszko [18]) or of nonnegative index (Kryszewski [12]). However, boundary value problems of the form (3) translated into the form (5) rarely lead to a compact map ϕ. Hence, there is a need to develop a framework to study problem (5) also in the case when A is a Fredholm operator of nonnegative index and ϕ belongs to a more general class of nonlinear (single- or multivalued) transformations, e.g. condensing, ultimately compact or fundamentally contractible maps.

The paper is organized as follows. At the end of this section we introduce some notions and definitions. In Section 1 we describe u-fundamentally contractible maps and give some examples. Section 2 is technical—it prepares tools for Section 3, which is devoted to the construction of a coincidence index in the class of maps considered. Finally, in Section 4 we present an abstract existence result and illustrate it by a boundary value problem in Banach spaces.

All topological spaces considered are metric, and all singlevalued maps are continuous.

Let E, E' be Banach spaces with norms $\|\cdot\|_E$, $\|\cdot\|_{E'}$, respectively, and let $X \subset E$, $Y \subset E'$. By a *multivalued map* φ from X to Y (denoted by $\varphi : X \multimap Y$) we understand an upper semicontinuous transformation which assigns to any $x \in X$ a nonempty compact set $\varphi(x) \subset Y$. Recall that φ is *upper semicontinuous* (u.s.c.) if, for every closed $B \subset Y$,

$$\varphi^{-1}(B) = \{x \in X : \varphi(x) \cap B \neq \emptyset\}$$

is closed in X. As a general reference for multivalued maps we suggest [8] or [13].

Throughout the paper we use the following notation: if V is a subset of a Banach space E, then by $\mathrm{cl}\, V$ we denote the closure of V, by $\mathrm{bd}\, V$ the boundary of V, by $\mathrm{conv}(V)$ the convex hull of V, i.e. the set $\{x \in E : x = \sum_{i=1}^n t_i v_i$ and $v_i \in V,\ t_i \in [0,1],\ \sum_{i=1}^n t_i = 1\}$, and $\overline{\mathrm{conv}}(V) = \mathrm{cl}\,\mathrm{conv}(V)$. Moreover, $B_E(x_0, r) = \{x \in E : \|x_0 - x\|_E \leq r\}$.

1. u-Fundamentally contractible maps and morphisms. Let E, E' be Banach spaces, X, V be subsets of E such that $X \subset V$, and Y be a closed convex subset of E'. Moreover, let $u : V \to Y$ be a proper ([1]) continuous (singlevalued) map and $\varphi : X \multimap Y$ be a multivalued map.

Definition 1.1. A nonempty closed convex set $K \subset Y$ is called *u-fundamental* for φ provided

(i) $\varphi(u^{-1}(K) \cap X) \subset K$,
(ii) if $u(x) \in \overline{\mathrm{conv}}(\varphi(x) \cup K)$, then $u(x) \in K$.

Observe that if $E = E'$ and $u = \mathrm{id}_E$ is the identity on E, then K is nothing else but a fundamental set for φ in the sense of e.g. [3].

Properties of u-fundamental sets are summarized in the following result.

Theorem 1.2. (i) *If K is a u-fundamental set for φ, then $\{x \in X : u(x) \in \varphi(x)\} \subset u^{-1}(K)$.*

(ii) *If $K_1 \cap K_2 \neq \emptyset$ and K_1, K_2 are u-fundamental sets for φ, then so is $K = K_1 \cap K_2$.*

(iii) *If $P \subset K$ and K is a u-fundamental set for φ, then so is $K' = \overline{\mathrm{conv}}(\varphi(u^{-1}(K) \cap X) \cup P)$.*

([1]) That is, $u^{-1}(C)$ is compact for any compact set $C \subset Y$.

(iv) *If K is the intersection of all u-fundamental sets for φ, then* $K = \overline{\mathrm{conv}}(\varphi(u^{-1}(K)\cap X))$.

(v) *For any $A \subset Y$, there exists a u-fundamental set K such that* $K = \overline{\mathrm{conv}}(\varphi(u^{-1}(K)\cap X)\cup A)$.

Proof. Only part (v) requires a proof. It is obvious that Y is a u-fundamental set for φ. Put

$$K = \bigcap_{L\in\mathcal{K}} L,$$

where $\mathcal{K} = \{L \subset Y : L \text{ is a } u\text{-fundamental set for } \varphi \text{ and } A \subset L\}$. We see at once that K is a u-fundamental set for φ, $A \subset K$ and

$$K' = \overline{\mathrm{conv}}(\varphi(u^{-1}(K)\cap X)\cup A) \subset K.$$

But $K' \in \mathcal{K}$, which follows from part (iii). By the definition of K, $K \subset K'$, which completes the proof. ∎

Observe that the set K from (v) of the above theorem is the smallest u-fundamental set for φ containing A.

Definition 1.3. We say that φ is a *u-fundamentally contractible map* if there exists a compact u-fundamental set for φ.

Remark 1.4. Let E'' be a Banach space. Assume that $Z : E' \to E''$ is a continuous linear isomorphism. If $\varphi : X \multimap Y$ is u-fundamentally contractible, then $Z\circ\varphi$ is $(Z\circ u)$-fundamentally contractible. Indeed, if K is a compact u-fundamental set for φ, then $Z(K)$ is a compact $Z\circ u$-fundamental set for $Z\circ\varphi$.

It is clear that without additional knowledge about the structure of a mapping and its values no satisfactory approximation or algebraic methods are available. Observe that any multivalued map $\varphi : X \multimap Y$ may be represented by the formula

$$\varphi(x) = q_\varphi(p_\varphi^{-1}(x))$$

for $x \in X$, where

$$X \xleftarrow{p_\varphi} \mathrm{Gr}(\varphi) \xrightarrow{q_\varphi} Y,$$

$\mathrm{Gr}(\varphi) = \{(x,y) \in X\times Y : y \in \varphi(x)\}$ is the graph of φ and p_φ, q_φ are the projections. Note that, in view of the upper semicontinuity of φ, p_φ is proper as a closed surjection with compact fibers.

Clearly φ may admit other factorizations of the form $X \xleftarrow{p} W \xrightarrow{q} Y$ (i.e. $\varphi(x) = q(p^{-1}(x)),\ x\in X$) where p, q are no longer projections, but p is still a proper surjection.

As usual, we will need some assumptions concerning the values of the maps considered. In the described situation some additional assumptions on p (more precisely, on its fibers) imply suitable properties of the values of φ.

Therefore, from now on we will consider only those multivalued maps which can be factorized by means admissible cotriads (in the sense defined below).

DEFINITION 1.5. Let W be a space. We say that a cotriad $X \overset{p}{\leftarrow} W \overset{q}{\rightarrow} Y$, also denoted by (p,q), is *admissible* if

(i) q is a singlevalued map,
(ii) p is a singlevalued proper map,
(iii) $p^{-1}(x)$ is a cell-like set for every $x \in X$.

Recall that a set $A \subset W$ is called *cell-like* if it is compact and if for any embedding $e : A \hookrightarrow Y$ into an absolute neighborhood retract Y, the set $e(A)$ is contractible in Y (see [2] and [16]).

REMARK 1.6. Condition (iii) may be replaced by

(iii)′ $\sup_{x\in X} \dim p^{-1}(x) < \infty$ and for each $x \in X$, the fiber $p^{-1}(x)$ is acyclic with respect to Čech cohomology ([2]).

Of course, in this way we get a different class of multivalued maps, but it is also appropriate for all considerations that follow. In both classes, values of maps may be nonconvex.

Observe that each admissible cotriad (p,q) generates a u.s.c. multivalued map $\varphi_{(p,q)} : X \multimap Y$, $\varphi(x) = q(p^{-1}(x))$, so we can define the following notions.

DEFINITION 1.7. (i) A closed convex set K is called *u-fundamental* for an admissible cotriad (p,q) if it is u-fundamental for the multivalued map $\varphi_{(p,q)}$.
(ii) The cotriad (p,q) is called *u-fundamentally contractible* if so is the map $\varphi_{(p,q)}$.

DEFINITION 1.8. We denote by $\mathcal{D}(X,Y,u)$ the class of all admissible cotriads
$$X \overset{p}{\leftarrow} W \overset{q}{\rightarrow} Y$$
such that

(i) (p,q) is u-fundamentally contractible,
(ii) if K, K' are two compact disjoint u-fundamental sets for (p,q), then there exist a finite number of compact u-fundamental sets $K_1,\ldots,K_n$ such that $K\cap K_1 \neq \emptyset$, $K_n \cap K' \neq \emptyset$ and $K_i \cap K_{i+1} \neq \emptyset$.

Observe that if we know *a priori* that $\{x \in X : u(x) \in q(p^{-1}(x))\} \neq \emptyset$, then any two u-fundamental sets have nonempty intersection (see Theorem 1.2). However we need to impose condition (ii) in Definition 1.8 in

([2]) That is, $\mathcal{H}^*(p^{-1}(x)) = \mathcal{H}^*(\mathrm{pt})$, where pt is a one-point space and $\mathcal{H}^*$ denotes Čech cohomology.

order to have a relation between different u-fundamental sets. As shown by the examples given below, this condition does not restrict generality.

EXAMPLE 1.9 (Compact cotriads). If (p,q) is *compact* (i.e. $\operatorname{cl} q(p^{-1}(X))$ is compact), then $(p,q) \in \mathcal{D}(X,Y,u)$. Indeed, $K = \overline{\operatorname{conv}}(q(p^{-1}(X)))$ is a compact u-fundamental set for (p,q). If there exists a compact u-fundamental set K' such that $K \cap K' = \emptyset$, then the set

$$K'' = \overline{\operatorname{conv}}(q(p^{-1}(X)) \cup \{v\}),$$

where $v \in K'$, is a compact u-fundamental set for (p,q) (because in fact $K'' = \overline{\operatorname{conv}}(q(p^{-1}(u^{-1}(Y) \cap X)) \cup \{v\}))$ and it has nonempty intersections with K and K'. We denote by $\mathcal{D}_{\mathrm{c}}(X,Y,u)$ the subclass of $\mathcal{D}(X,Y,u)$ consisting of the compact cotriads.

EXAMPLE 1.10 (u-Condensing cotriads). Recall that a *measure of noncompactness* in a Banach space E' is a function $\mu : \mathcal{B} \to [0,\infty)$ defined on the family of all bounded subsets of E' and having the following properties:

(1) $\mu(A) = 0$ if and only if $\operatorname{cl} A$ is compact.

(2) μ is a "seminorm", i.e. $\mu(\lambda \cdot A) = |\lambda|\mu(A)$ and $\mu(A_1 + A_2) \le \mu(A_1) + \mu(A_2)$.

(3) If $A_1 \subset A_2$, then $\mu(A_1) \le \mu(A_2)$, $\mu(A_1 \cup A_2) \le \max\{\mu(A_1), \mu(A_2)\}$.

(4) $\mu(\operatorname{conv} A) = \mu(A)$ and $\mu(\operatorname{cl} A) = \mu(A)$.

(5) If a sequence $\{A_i\}_{i=1}^{\infty}$ is decreasing and $\lim_{i\to\infty} \mu(A_i) = 0$, then $A_\infty = \bigcap_{i=1}^{\infty} \operatorname{cl} A_i$ is a compact nonempty set.

Assume that both u and the map determined by (p,q) are bounded and that (p,q) is *u-condensing*, i.e. for any $A \subset X$, if $\mu(q(p^{-1}(A))) \ge \mu(u(A))$ then $\operatorname{cl} A$ is compact. Then $(p,q) \in \mathcal{D}(X,Y,u)$.

Indeed, let $y \in q(p^{-1}(X))$. By Theorem 1.2(v), there exists a u-fundamental set K such that $K = \overline{\operatorname{conv}}(q(p^{-1}(u^{-1}(K)) \cap X) \cup \{y\})$. Observe that K is bounded. Assume that K is not compact. Then

$$\begin{aligned}
\mu(K) &= \mu(\overline{\operatorname{conv}}(q(p^{-1}(u^{-1}(K) \cap X)) \cup \{y\})) \\
&= \mu(q(p^{-1}(u^{-1}(K) \cap X)) \cup \{y\}) \\
&= \mu(q(p^{-1}(u^{-1}(K) \cap X))) < \mu(u(u^{-1}(K) \cap X)) \le \mu(K),
\end{aligned}$$

a contradiction.

Now, for any two compact u-fundamental sets K_1, K_2 and $x \in K_1$, $y \in K_2$, there exists a u-fundamental set K_3 such that

$$K_3 = \overline{\operatorname{conv}}((q(p^{-1}(K_3) \cap X)) \cup \{x,y\}).$$

As above we prove that K_3 is compact.

EXAMPLE 1.11 (u-Set contractions). A cotriad (p,q) is called a *u-set contraction* if there exists $k \in (0,1)$ such that for any bounded $A \subset X$,

$\mu(q(p^{-1}(A))) \leq k\mu(u(A))$. Then, clearly, (p,q) is u-condensing and hence $(p,q) \in \mathcal{D}(X,Y,u)$.

EXAMPLE 1.12 (u-Limit compact cotriads). Let $Y = u(X)$ and $K_1 = \overline{\operatorname{conv}}(q(p^{-1}(X)))$, $K_i = \overline{\operatorname{conv}}(q(p^{-1}(u^{-1}(K_{i-1}) \cap X)))$ for $i > 1$. The cotriad (p,q) is *u-limit compact* if the set

$$K = \bigcap_{i=1}^{\infty} K_i$$

is nonempty and compact. Obviously, K is a u-fundamental set for (p,q). Assume that $L_0 \subset Y$ is another compact u-fundamental set for (p,q). Of course $u^{-1}(L_0) \cap X \neq \emptyset$. For $i = 1, 2, \ldots$, put

$$L_i = \overline{\operatorname{conv}}(q(p^{-1}(u^{-1}(L_{i-1}) \cap X))), \quad i \in \mathbb{N} \setminus \{0\},$$

and observe that

$$\emptyset \neq L_1 \subset \overline{\operatorname{conv}}(q(p^{-1}(X))) = K_1$$

and, for $i > 1$,

$$\emptyset \neq L_i \subset \overline{\operatorname{conv}}(q(p^{-1}(u^{-1}(K_{i-1}) \cap X))) = K_i \subset u(X).$$

Moreover $L_i \subset L_0 \cap K_i$, hence $L_0 \cap K \neq \emptyset$.

EXAMPLE 1.13. If $u = \mathrm{id}_X$ and $E = E'$, then every u-condensing cotriad is condensing, every u-set contraction is a k-set contraction and every u-limit compact map is limit compact.

EXAMPLE 1.14. First we generalize the definition of $\mathcal{K}_n$-operators (cf. [1]). Let $\mathcal{A} \subset 2^{E'}$. A multivalued map $\varphi : X \multimap Y$ is called an *$(u, \mathcal{A})$-operator* if for any $T \in \mathcal{A}$ and $Z \subset Y$, the following condition holds:

$$\text{if} \quad \overline{\operatorname{conv}}(\varphi(u^{-1}(Z) \cap X) \cup T) = Z, \quad \text{then} \quad Z \text{ is compact.}$$

Let $\mathcal{K}_n$ denote the family of all n-element subsets of Y, and $\mathcal{K}_\infty$, $\mathcal{K}_c$ the families of all finite and all compact subsets of Y, respectively.

If $\mathcal{A}$ is an arbitrary family among $\mathcal{K}_2, \ldots, \mathcal{K}_n, \ldots, \mathcal{K}_\infty, \mathcal{K}_{\mathrm{c}}$, and $\varphi_{(p,q)}$ is a $(u, \mathcal{A})$-operator, then $(p,q) \in \mathcal{D}(X,Y,u)$. The proof is easy if one recalls properties of u-fundamental sets (Theorem 1.2).

As mentioned earlier, a multivalued map can be factorized by different cotriads. To reduce this freedom we define the following equivalence relation in $\mathcal{D}(X,Y,u)$.

DEFINITION 1.15. We say that cotriads

$$X \xleftarrow{p_i} W_i \xrightarrow{q_i} Y, \quad i = 1, 2,$$

from $\mathcal{D}(X,Y,u)$ are *equivalent* (written $(p_1,q_1)\approx(p_2,q_2)$) if there exists a homeomorphism $f: W_1\to W_2$ such that the following diagram is commutative:

$$\begin{tikzcd} & W_1 \arrow[dl, "p_1"'] \arrow[dr, "q_1"] \arrow[dd, "f"] & \\ X & & Y \\ & W_2 \arrow[ul, "p_2"] \arrow[ur, "q_2"'] & \end{tikzcd}$$

Define

$$\mathcal{M}(X,Y,u)=\mathcal{D}(X,Y,u)/\approx .$$

Elements of $\mathcal{M}(X,Y,u)$ will be called *u-fundamentally contractible morphisms* and will be denoted by Greek capital letters $\Phi,\Psi,\ldots$

Any morphism $\Phi\in\mathcal{M}(X,Y,u)$ determines a multivalued map $\varphi_\Phi: X\multimap E'$ by $\varphi_\Phi(x)=q(p^{-1}(x))$, where $(p,q)\in\Phi$. The definition does not depend on the choice of (p,q), so φ_Φ is well defined. From now on we write $\Phi(x)$ for $\varphi_\Phi(x)$.

Remark 1.16. Assume that $(p_1,q_1)\approx(p_2,q_2)$. Then

(i) $\{x\in X: u(x)\in q_1(p_1^{-1}(x))\}=\{x\in X: u(x)\in q_2(p_2^{-1}(x))\}$,

(ii) if K is a u-fundamental set for (p_1,q_1), then it is a u-fundamental set for (p_2,q_2), so we can call K a u-fundamental set for the morphism $\Phi=[(p_1,q_1)]$ and, of course, K is a u-fundamental set for φ_Φ.

If $f: X\to Y$ is a u-fundamentally contractible singlevalued map, then we can identify f with a morphism $[(\mathrm{id}_X,f)]$ from $\mathcal{M}(X,Y,u)$. Let Φ be represented by the cotriad $Y\overset{p}{\leftarrow}W\overset{q}{\rightarrow}Z$. The composition $\Phi\circ f$ is the morphism represented by the cotriad

$$X\overset{\widetilde{p}}{\leftarrow}X\otimes W\xrightarrow{q\circ\widetilde{f}}Z,$$

where $X\otimes W=\{(x,w)\in X\times W: p(w)=f(x)\}$, $\widetilde{p}(x,w)=x$ and $\widetilde{f}(x,w)=w$.

Let $I=[0,1]$, $i_j: X\to X\times I$, $j=0,1$, be the embeddings given by $i_j(x)=(x,j)$, $x\in X$, and $v: X\times I\to Y$ be defined by $v(x,t)=u(x)$. For any fixed $t\in I$ we may identify $v(\cdot,t)$ and u.

Definition 1.17. Let K be a compact convex subset of Y. Morphisms $\Phi_0,\Phi_1\in\mathcal{M}(X,Y,u)$ are *(u,K)-homotopic* (written $\Phi_0\simeq_K\Phi_1$) if there exists a morphism $\Phi\in\mathcal{M}(X\times I,Y,v)$ such that for every $t\in I$ and $(P,Q)\in\Phi$, K is a u-fundamental set for $Q(P^{-1}(\cdot,t))$ and $\Phi\circ i_j=\Phi_j$, $j=0,1$.

DEFINITION 1.18. Morphisms $\Phi_0, \Phi_1 \in \mathcal{M}(X, Y, u)$ are *u-homotopic* (written $\Phi_0 \simeq \Phi_1$) if there exist a finite number of compact convex sets $K_1, \ldots, K_n \subset Y$ and morphisms $\Psi_1, \ldots, \Psi_{n-1}$ such that

$$\Phi_0 \simeq_{K_1} \Psi_1 \simeq_{K_2} \ldots \simeq_{K_{n-1}} \Psi_{n-1} \simeq_{K_n} \Phi_1.$$

We end this section with the notion of compact u-homotopy. Let $\mathcal{M}_c(X, Y, u) \subset \mathcal{M}(X, Y, u)$ be the set of compact morphisms, i.e. such that $\mathcal{M}_c(X, Y, u) = \mathcal{D}_c(X, Y, u)/\approx$ (see 1.9).

DEFINITION 1.19. Morphisms $\Phi_0, \Phi_1 \in \mathcal{M}_c(X, Y, u)$ are *compactly u-homotopic* if there exists a morphism $\Phi \in \mathcal{M}_c(X \times I, Y, v)$ such that $\Phi \circ i_j = \Phi_j$.

Of course, if morphisms are compactly u-homotopic, then they are (u, K)-homotopic with $K = \overline{\mathrm{conv}}(\varphi_\Phi(X \times I))$, hence simply u-homotopic.

2. Associated compact cotriads. Let E, E', X, Y, u be as in the previous section. Take $\Phi \in \mathcal{M}(X, Y, u)$ and a cotriad $X \xleftarrow{p} W \xrightarrow{q} Y$ such that $(p, q) \in \Phi$.

DEFINITION 2.1. Let K be a compact u-fundamental set for (p, q). A cotriad $X \xleftarrow{p} W \xrightarrow{\overline{q}} K$ such that $\overline{q}$ is a compact map and $\overline{q}_{|p^{-1}(u^{-1}(K) \cap X)} = q_{|p^{-1}(u^{-1}(K) \cap X)}$ is said to be *associated with* (p, q) *with respect to* K.

Obviously, the set K is u-fundamental for $(p, \overline{q})$, and $(p, \overline{q}) \in \mathcal{D}_c(X, Y, u)$.

It is easily seen that for any $(p, q) \in \Phi \in \mathcal{M}(X, Y, u)$ there exists a compact associated cotriad. Indeed, if K is a u-fundamental set for (p, q), then the map

$$q_{|p^{-1}(u^{-1}(K) \cap X)} : p^{-1}(u^{-1}(K) \cap X) \to K$$

admits a compact extension $q' : W \to K$ and (p, q') satisfies the above definition.

Let us collect some properties of associated cotriads.

LEMMA 2.2. (i) *Let* $(p, q) \in \mathcal{D}(X, Y, u)$, $K \subset Y$ *be a compact u-fundamental set for* (p, q), *and* $(p, \overline{q})$ *be a compact cotriad associated with* (p, q) *with respect to* K. *Then* $\{x \in X : u(x) \in q(p^{-1}(x))\} = \{x \in X : u(x) \in \overline{q}(p^{-1}(x))\}$.

(ii) *Let* K_1, K_2 *be compact u-fundamental sets for* (p, q) *and* $K_1 \subset K_2$. *If* $(p, \overline{q})$ *is a cotriad associated with* (p, q) *with respect to* K_2, *then* K_1 *is a u-fundamental set for* $(p, \overline{q})$.

(iii) *Morphisms* $[(p, q)]$ *and* $[(p, \overline{q})]$ *are u-homotopic. If, additionally,* $\{x \in X : u(x) \in q(p^{-1}(x))\} \cap \mathrm{bd}\, X = \emptyset$, *then the multivalued map determined by the u-homotopy joining these cotriads also has no coincidence points with u on* $\mathrm{bd}\, X$.

Proof. Parts (i) and (ii) are obvious.

To prove (iii), let $r : E' \to K$ be a retraction. Consider the cotriad $X \times I \overset{P}{\leftarrow} W \times I \overset{Q}{\rightarrow} Y$ given by

$$P(w,t) = (p(w),t),$$
$$Q(w,t) = \begin{cases} (1-2t)q(w) + 2t(r\circ q)(w) & \text{for } t \in [0,1/2], \\ (2-2t)(r\circ q)(w) + (2t-1)\overline{q}(w) & \text{for } t \in (1/2,1]. \end{cases}$$

An easy computation shows that $[(P,Q)] \circ i_0 = [(p,q)]$, $[(P,Q)] \circ i_1 = [(p,\overline{q})]$ and for every $t \in I$, K is a u-fundamental set for $Q(P^{-1}(\cdot,t))$ and $\{x \in X : u(x) \in Q(P^{-1}(x,t))\} \cap \operatorname{bd} X = \emptyset$.

Moreover, observe that if (p,q) and u have no coincidence points on X, then also (P,Q) has no coincidence points with $v(x,t) := u(x)$ on $X \times I$. ∎

From the above lemma it follows, in particular, that if

$$\{x \in X : u(x) \in q(p^{-1}(x))\} \cap \operatorname{bd} X = \emptyset,$$

then also

$$\{x \in X : u(x) \in \overline{q}(p^{-1}(x))\} \cap \operatorname{bd} X = \emptyset.$$

Let $\Phi \in \mathcal{M}(X,Y,u)$ and K be a u-fundamental set for Φ (see Remark 1.16). One can define a compact morphism $\overline{\Phi}$ associated with Φ with respect to K by putting $\overline{\Phi} = [(p,\overline{q})]$, where $(p,\overline{q})$ is a compact cotriad associated with $(p,q) \in \Phi$ with respect to K. The following theorems show that any two compact morphisms associated with Φ are compactly u-homotopic.

THEOREM 2.3. *Let $(p,\overline{q}_0),(p,\overline{q}_1)$ be compact cotriads associated with $(p,q) \in \mathcal{D}(X,Y,u)$ with respect to compact u-fundamental sets K_0, K_1, respectively. Assume that (p,q) and u have no coincidence points on* $\operatorname{bd} X$. *Then the morphisms $[(p,\overline{q}_0)],[(p,\overline{q}_1)]$ are compactly u-homotopic and the map determined by the homotopy has no coincidence points with u on* $\operatorname{bd} X$.

Proof. Without loss of generality assume that $K_2 = K_0 \cap K_1 \neq \emptyset$ (cf. Definition 1.8). Of course, K_2 is a compact u-fundamental set for (p,q). Let $r_i : E' \to K_i$, $i = 0,1,2$, be retractions. Consider the cotriad $X \times I \overset{P}{\leftarrow} W \times I \overset{Q}{\rightarrow} K_0 \cup K_1$ defined by

$$P(w,t) = (p(w),t),$$
$$Q(w,t) = \begin{cases} (1-4t)\overline{q}_0(w) + 4t(r_0\circ q)(w) & \text{for } t \in [0,1/4], \\ (2-4t)(r_0\circ q)(w) + (4t-1)(r_2\circ q)(w) & \text{for } t \in (1/4,1/2], \\ (3-4t)(r_2\circ q)(w) + (4t-2)(r_1\circ q)(w) & \text{for } t \in (1/2,3/4], \\ (4-4t)(r_1\circ q)(w) + (4t-3)\overline{q}_1(w) & \text{for } t \in (3/4,1]. \end{cases}$$

Obviously, $(P,Q) \in \mathcal{D}_{\mathrm{c}}(X \times I, Y, v)$, where, as above, $v : X \times I \to Y$ is given by $v(x,t) = u(x)$. We prove that, for any $t \in I$, $\{x \in \operatorname{bd} X : u(x) \in$

$Q(P^{-1}(x,t))\} = \emptyset$. Assume that $u(x) \in Q(P^{-1}(x,t)) = Q(\{(w,s) \in W \times I : p(w) = x,\ s = t\})$.

If $t \in [0, 1/4]$, then there exists $w \in W$ such that

$$u(x) = (1-4t)\overline{q}_0(w) + 4t(r_0 \circ q)(w) \quad \text{and} \quad p(w) = x.$$

By the above, $u(x) \in K_0$ and hence $x \in u^{-1}(K_0) \cap X$. Since $p(w) = x$, we have $w \in p^{-1}(u^{-1}(K_0) \cap X)$. This gives $\overline{q}_0(w) = q(w)$, and $q(w) = r_0 \circ q(w)$, because $q(w) \in K_0$. Therefore, $Q(w,t) = q(w)$ and, consequently, $x \notin \operatorname{bd} X$.

The proof for $t \in (3/4, 1]$ is similar.

If $t \in (1/4, 1/2]$, then there exists $w \in W$ such that

$$u(x) = (2-4t)(r_0 \circ q)(w) + (4t-1)(r_2 \circ q)(w) \quad \text{and} \quad p(w) = x.$$

As above, $u(x) \in K_0$, so $w \in p^{-1}(u^{-1}(K_0) \cap X)$. Hence $q(w) \in K_0$. This gives $r_0 \circ q(w) = q(w) \in q(p^{-1}(x))$. From this it follows that $u(x) \in \overline{\operatorname{conv}}(q(p^{-1}(x)) \cup K_2)$, because $r_2 \circ q(w) \in K_2$. Since K_2 is a u-fundamental set for (p,q), $u(x) \in K_2$. It follows that $x \in u^{-1}(K_2)$ and hence $w \in p^{-1}(u^{-1}(K_2) \cap X)$, which implies that $q(w) \in K_2$, and finally, that $r_2 \circ q(w) = q(w)$. Therefore $u(x) \in q(p^{-1}(x))$, and $x \notin \operatorname{bd} X$.

The same proof works for $t \in (1/2, 3/4]$.

It is easy to verify that the morphism $[(P,Q)]$ is a u-homotopy joining $[(p,\overline{q}_0)]$ to $[(p,\overline{q}_1)]$ with K_2 being a u-fundamental set for all maps $Q(P^{-1}(\cdot,t))$, $t \in I$. ∎

THEOREM 2.4. *Let* $X \xleftarrow{p_0} W_0 \xrightarrow{q_0} E'$, $X \xleftarrow{p_1} W_1 \xrightarrow{q_1} E'$, $(p_0,q_0), (p_1,q_1) \in \mathcal{D}(X,Y,u)$ *and* $(p_0,\overline{q}_0)$, $(p_1,\overline{q}_1)$ *be cotriads associated with* (p_0,q_0) *and* (p_1,q_1), *respectively. If* (p_0,q_0), (p_1,q_1) *are equivalent*, *then the morphisms* $[(p_0,\overline{q}_0)]$, $[(p_1,\overline{q}_1)]$ *are compactly* u*-homotopic.*

P r o o f. Since (p_0,q_0) and (p_1,q_1) are equivalent, there exists a homeomorphism $f : W_0 \to W_1$ such that $p_1 \circ f = p_0$ and $q_1 \circ f = q_0$ (see Definition 1.15) and we can assume that $(p_0,\overline{q}_0)$, $(p_1,\overline{q}_1)$ are associated with (p_0,q_0) and (p_1,q_1) with respect to the same u-fundamental set K (cf. Remark 1.16). Observe that $(p_1,\overline{q}_1)$, $(p_0,\overline{q}_1 \circ f)$ are equivalent in $\mathcal{D}_{\mathrm{c}}(X,Y,u)$. The morphism $[(P,Q)] \in \mathcal{M}_{\mathrm{c}}(X \times I, Y, v)$, where $v : X \times I \to Y$, $v(x,t) = u(x)$ and

$$X \times I \xleftarrow{P} W_0 \times I \xrightarrow{Q} K_0,$$

defined by

$$P(w,t) = (p_0(w), t), \qquad Q(w,t) = (1-t)(\overline{q}_1 \circ f)(w) + t\overline{q}_0(w),$$

gives a compact u-homotopy between $[(p_0,\overline{q}_1 \circ f)]$ and $[(p_0,\overline{q}_0)]$. But we have $[(p_0,\overline{q}_1 \circ f)] = [(p_1,q_1)]$ and the proof is complete. ∎

REMARK 2.5. If (p_0,q_0) and (p_1,q_1) are as in the above theorem, then $\{x\in X : u(x)\in q_0(p_0^{-1}(x))\}\cap \operatorname{bd} X=\emptyset$ if and only if $\{x\in X : u(x)\in q_1(p_1^{-1}(x))\}\cap \operatorname{bd} X=\emptyset$. Moreover, the homotopy defined in the proof of Theorem 2.4 and v have no coincidence point on $\operatorname{bd} X$.

3. The coincidence index. Let E, E' be Banach spaces and $F:E\to E'$ be a Fredholm linear operator of nonnegative index $k=\operatorname{ind}(F)$.

Recall that a bounded linear operator $F:E\to E'$ is called a *Fredholm operator* if $\dim\operatorname{Ker}(F)<\infty$ and $\dim\operatorname{Coker}(F)<\infty$. Here, $\operatorname{Ker}(F)=\{x\in E : F(x)=0\}$ is the null space of F and $\operatorname{Coker}(F)=E'/R(F)$ where $R(F):=F(E)$ is the range of F. Note that $R(F)$ is a closed subspace of E' (see IV.2.6 of [7]). The *index* of a Fredholm operator F is the number

$$\operatorname{ind}(F):=\dim\operatorname{Ker}(F)-\dim\operatorname{Coker}(F).$$

DEFINITION 3.1. We denote by $\mathcal{M}^F(E,E')$ the collection of pairs (Φ,U) such that U is an open bounded subset of E, $\Phi\in\mathcal{M}(\operatorname{cl}U,E',F|_{\operatorname{cl}U})$ and $F(x)\notin\Phi(x)$ for all $x\in\operatorname{bd}U$.

To simplify the notation, from now on, if $(\Phi,U)\in\mathcal{M}^F(E,E')$, then an $F|_{\operatorname{cl}U}$-fundamental set for Φ (resp. for $(p,q)\in\Phi$ or for φ_Φ) will be called an *F-fundamental set* for Φ (resp. for $(p,q)\in\Phi$ or for φ_Φ).

We say that the pairs (Φ_0,U), (Φ_1,U) are *F-homotopic* (resp. *compactly F-homotopic*) if they are u-homotopic (resp. compactly u-homotopic) with $u=F|_{\operatorname{cl}U}$ (see 1.18 and 1.19), and the u-homotopy has no coincidence points with F on the boundary of U.

In this section we construct a general coincidence index, which is a homotopy invariant count of coincidence points of F and multivalued maps determined by a morphism from $\mathcal{M}^F(E,E')$.

DEFINITION 3.2. By a *coincidence index* we understand a function ind_F which assigns to any pair (Φ,U) (Φ is a single- or multivalued map or morphism) an element of the kth stable homotopy group of spheres Π_k and has the following properties:

(i) (Existence) If $\operatorname{ind}_F(\Phi,U)\neq 0$, then there is $x_0\in U$ such that $F(x_0)\in\Phi(x_0)$.

(ii) (Localization) If $U'\subset U$ is open and $F(x)\notin\Phi(x)$ for $x\in\operatorname{cl}U\setminus U'$, then $\operatorname{ind}_F(\Phi,U)=\operatorname{ind}_F(\Phi|_{\operatorname{cl}U'},U')$.

(iii) (Additivity) If U_1,U_2 are open disjoint subsets of U and $F(x)\notin\Phi(x)$ for $x\in\operatorname{cl}U\setminus(U_1\cup U_2)$, then $\operatorname{ind}_F(\Phi,U)=\operatorname{ind}_F(\Phi|_{\operatorname{cl}U_1},U_1)+\operatorname{ind}_F(\Phi|_{\operatorname{cl}U_2},U_2)$.

(iv) (Homotopy) If (Φ_1,U) is F-homotopic to (Φ,U), then $\operatorname{ind}_F(\Phi,U)=\operatorname{ind}_F(\Phi_1,U)$.

(v) (Restriction) Let $G:E'\to E'$ be a bounded linear projection with $\operatorname{Ker}(G)=R(F)$. Suppose that $\Phi(\operatorname{cl}U)\subset T$, where T is a closed subspace of E'. Then $\mathrm{ind}_F(\Phi, U) = \mathrm{ind}_{F'}(\Phi', U \cap T')$, where $T' = F^{-1}(T + R(G))$, $\Phi' = \Phi|_{\mathrm{cl}\, U \cap T'}$ and $F' = F|_{T'}$.

In order to provide a construction of an index on $\mathcal{M}^F(E, E')$, we first recall it briefly in the case of compact maps (following [12]). Similar homotopy invariants (in a less general situation) were studied earlier in [17], [6], [4], [10], [9].

First, let $E = \mathbb{R}^m$, $E' = \mathbb{R}^n$, where $n \le m \le 2n-2$, and let $f : \mathrm{cl}\, U \to \mathbb{R}^n$ be a continuous (singlevalued) map such that $\{x \in \mathrm{bd}\, U : F(x) = f(x)\} = \emptyset$. We identify the unit sphere S^n with the one-point compactification of $\mathbb{R}^n$, where $0 \in \mathbb{R}^n$ is identified with the south pole s_{-1} of S^n. Since $S^n \setminus \{s_{-1}\}$ is an absolute retract, there exists a continuous map $g : S^m \setminus U \to S^n \setminus \{s_{-1}\}$ such that $g|_{\mathrm{bd}\, U} = (F - f)|_{\mathrm{bd}\, U}$. Define $g' : S^m \to S^n$ by the formula

$$g'(x) = \begin{cases} g(x) & \text{for } x \in S^m \setminus U, \\ F(x) - f(x) & \text{for } x \in U. \end{cases}$$

Take the homotopy class $[g']$ of g' in the mth homotopy group of S^n, denoted by $\pi_m(S^n)$.

Let $k = m - n$. If $k + 2 \le n$, then we have the suspension isomorphism $\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1})$, hence the kth stable homotopy group of spheres Π_k defined by

$$\Pi_k = \varinjlim_{n \ge 0} \pi_{n+k}(S^n)$$

is isomorphic to $\pi_{n+k}(S^n)$ for sufficiently large n. The index $\mathrm{ind}_F(f, U)$ is the element of Π_k corresponding to $[g']$ by an appropriate isomorphism.

The above procedure was suggested by [6] and introduced in [9]. It may be formulated in the language of cohomotopy groups (see [12]).

This index has all the standard properties (i)–(v) from 3.2. The proofs can be found in [12] and [6].

Now, let E be an arbitrary Banach space, E' be a Banach space with a given orientation (3) and let $f : \mathrm{cl}\, U \to E'$ be a compact map with $f(x) \ne F(x)$ for $x \in \mathrm{bd}\, U$. Using much more complex arguments, based on Gęba's infinite-dimensional cohomotopy theory (see [5]) with suitable modifications, Kryszewski [12] has defined an index

$$\mathrm{ind}_F(f, U) \in \Pi_k,$$

which has all properties from 3.2 and additionally the following one:

PROPERTY 3.3 (Boundary dependence). *If $f, g : \mathrm{cl}\, U \to E'$ and $f|_{\mathrm{bd}\, U} = g|_{\mathrm{bd}\, U}$, then $\mathrm{ind}_F(f, U) = \mathrm{ind}_F(g, U)$.*

(3) An orientation in an infinite-dimensional Banach space E' is a family $\mathcal{O} = \{\mathcal{O}_L\}_{L \in \Lambda}$, where Λ is the family of all finite-dimensional linear subspaces of E' and $\mathcal{O}_L$ is a fixed orientation on $L \in \Lambda$ (cf. [5]).

REMARK 3.4. If $E = E'$ and $F = \mathrm{id}_E$, then $\mathrm{ind}_{\mathrm{id}_E}(g, U) \in \Pi_0 = \mathbb{Z}$ is the Leray–Schauder fixed point index of g on U.

Now we can describe the multivalued situation starting with the compact case (cf. [12], [14], [15]). We obtain the following theorem, which is a corollary of Theorem 4.51 and Remark 4.52 of [12].

THEOREM 3.5. *Let $\Phi \in \mathcal{M}_{\mathrm{c}}(\mathrm{cl}\, U, E', F|_{\mathrm{cl}\, U})$ be such that $\{x \in \mathrm{bd}\, U : F(x) \in \Phi(x)\} = \emptyset$ and let $i : \mathrm{bd}\, U \to \mathrm{cl}\, U$ be the inclusion. Then there exists a* (*unique up to $F|_{\mathrm{bd}\, U}$-homotopy*) *compact singlevalued map $g : \mathrm{bd}\, U \to E'$ such that the morphism $[(\mathrm{id}_{\mathrm{bd}\, U}, g)]$ is $F|_{\mathrm{bd}\, U}$-compactly homotopic to the morphism $\Phi \circ i$ and the map determined by the homotopy has no coincidence points with F.*

The above theorem and Property 3.3 allow us to define a generalized coincidence index for compact morphisms.

DEFINITION 3.6. Assume that Φ satisfies the assumptions of Theorem 3.5. The *generalized coincidence index* of Φ on U with respect to F is defined by

$$\mathrm{ind}_F(\Phi, U) := \mathrm{ind}_F(\overline{g}, U),$$

where $\overline{g} : \mathrm{cl}\, U \to E'$ is an arbitrary compact extension of g.

THEOREM 3.7 (see [12]). *The index defined above has properties* (i)–(v) *of* 3.2.

Now, we are in a position to define a generalized coincidence index for morphisms from $\mathcal{M}^F(E, E')$.

DEFINITION 3.8. Assume that $(\Phi, U) \in \mathcal{M}^F(E, E')$. The *coincidence index of Φ on U with respect to F* is

$$\mathrm{ind}_F(\Phi, U) := \mathrm{ind}_F(\overline{\Phi}, U),$$

where $\overline{\Phi} = [(p, \overline{q})]$ and $(p, \overline{q})$ is a compact cotriad associated with $(p, q) \in \Phi$.

The correctness of this definition follows from Theorems 2.3 and 2.4 (the index does not depend on the choice of (p, q) and its associated cotriad). Obviously, $[(p, \overline{q})] \in \mathcal{M}_{\mathrm{c}}(\mathrm{cl}\, U, E', F|_{\mathrm{cl}\, U})$.

Observe that if $(\Phi, U) \in \mathcal{M}^F(E, E')$ and U' is an open subset of E such that $\mathrm{cl}\, U' \subset U$, then also $(\Phi|_{\mathrm{cl}\, U'}, U') \in \mathcal{M}^F(E, E')$. Indeed, any F-fundamental set for Φ is an F-fundamental set for $\Phi|_{U'}$. This allows us to state the following theorem.

THEOREM 3.9. *Let $(\Phi, U) \in \mathcal{M}^F(E, E')$, φ_Φ be the multivalued map determined by Φ and assume that φ_Φ has no coincidence points with F on the boundary of U. Then the index ind_F from Definition* 3.8 *has all the properties* 3.2(i)–(v).

P r o o f. As above, we denote by $\overline{\Phi}$ the morphism containing the compact cotriad associated with $(p,q)\in\Phi$.

(i) If $\mathrm{ind}_F(\Phi,U)\neq 0$, then $\mathrm{ind}_F(\overline{\Phi},U)\neq 0$. Hence there exists $x_0\in U$ such that $F(x_0)\in\overline{\Phi}(x_0)$. But the sets of coincidence points are identical for Φ and $\overline{\Phi}$, so $F(x_0)\in\Phi(x_0)$.

(ii) Observe that if $(p,\overline{q})$ is a compact cotriad associated with $(p,q)\in\Phi$, then $(p,\overline{q})|_{\mathrm{cl}\,U'}$ is a compact cotriad associated with $(p,q)|_{\mathrm{cl}\,U'}$, and $(p,\overline{q})|_{\mathrm{cl}\,U'}\in\overline{\Phi}|_{\mathrm{cl}\,U'}$. Hence

$$\mathrm{ind}_F(\Phi,U)=\mathrm{ind}_F(\overline{\Phi},U)=\mathrm{ind}_F(\overline{\Phi},U')=\mathrm{ind}_F(\Phi,U').$$

(iii) Similar to the above:

$$\begin{aligned}\mathrm{ind}_F(\Phi,U)&=\mathrm{ind}_F(\overline{\Phi},U)=\mathrm{ind}_F(\overline{\Phi},U_1)+\mathrm{ind}_F(\overline{\Phi},U_2)\\&=\mathrm{ind}_F(\Phi,U_1)+\mathrm{ind}_F(\Phi,U_2).\end{aligned}$$

(iv) Without loss of generality we can assume that there exists a morphism $\Psi\in\mathcal{M}(\mathrm{cl}\,U\times I,E',v)$ ($v:\mathrm{cl}\,U\times I\to E'$, $v(\cdot,t)=F|_{\mathrm{cl}\,U}$) with an F-fundamental set K (cf. 1.18) and such that $\Psi\circ i_0=\Phi$ and $\Psi\circ i_1=\Phi_1$. Let $(P,Q)\in\Psi$. Of course,

$$\begin{aligned}(p_0,q_0)&=(P|_{P^{-1}(\mathrm{cl}\,U\times\{0\})},Q|_{P^{-1}(\mathrm{cl}\,U\times\{0\})})\in\Phi,\\(p_1,q_1)&=(P|_{P^{-1}(\mathrm{cl}\,U\times\{1\})},Q|_{P^{-1}(\mathrm{cl}\,U\times\{1\})})\in\Phi_1.\end{aligned}$$

Consider

$$Q|_{P^{-1}(F^{-1}(K)\cap\mathrm{cl}\,U)}:P^{-1}(F^{-1}(K)\cap\mathrm{cl}\,U)\to K$$

and its extension $\overline{Q}:W\to K$. Then $(P,\overline{Q})\in\mathcal{M}(\mathrm{cl}\,U\times I,E',v)$ is a compact homotopy between

$$(p_0,\overline{q}_0)=(P|_{P^{-1}(\mathrm{cl}\,U\times\{0\})},\overline{Q}|_{P^{-1}(\mathrm{cl}\,U\times\{0\})})$$

and

$$(p_1,\overline{q}_1)=(P|_{P^{-1}(\mathrm{cl}\,U\times\{1\})},\overline{Q}|_{P^{-1}(\mathrm{cl}\,U\times\{1\})}).$$

It is easily seen that $(p_0,\overline{q}_0)$ and $(p_1,\overline{q}_1)$ are compact pairs associated with (p_0,q_0) and (p_1,q_1), respectively. Hence

$$\mathrm{ind}_F(\Phi,U)=\mathrm{ind}_F([(p_0,\overline{q}_0)],U)=\mathrm{ind}_F([(p_1,\overline{q}_1)],U)=\mathrm{ind}_F(\Phi_1,U).$$

(v) First, observe that $(\Phi',\mathrm{cl}\,U\cap T')\in\mathcal{M}^{F'}(T',T)$ and if K is an F-fundamental set for Φ, then $K\cap T$ is an F'-fundamental set for Φ'. Next, let $(p,q)\in\Phi$ and $(p,\overline{q})$ be its associated compact cotriad (with an F-fundamental set K). Observe that if $p'=p|_{p^{-1}(\mathrm{cl}\,U\cap T')}$ and $q'=q|_{p^{-1}(\mathrm{cl}\,U\cap T')}$, then $(p',q')\in\Phi'$. Consider the cotriad (p',r) such that $r=\overline{q}|_{p^{-1}(\mathrm{cl}\,U\cap T')}$. Observe that it is a compact cotriad associated with (p',q'). Indeed, r is a compact extension of

$$q'|_{p^{-1}(F'^{-1}(K\cap T)\cap\mathrm{cl}\,U\cap T')}:p^{-1}(F'^{-1}(K\cap T)\cap\mathrm{cl}\,U\cap T')\to K\cap T.$$

Hence,

$$\operatorname{ind}_F(\Phi, U) = \operatorname{ind}_F([(p, \overline{q})], U), \qquad \operatorname{ind}_F([(p', r)], U) = \operatorname{ind}_F(\Phi', U).$$

But from the restriction property of the generalized coincidence index for compact maps, $\operatorname{ind}_F([(p, \overline{q})], U) = \operatorname{ind}_F([(p', r)], U)$, which ends the proof. ∎

4. Applications. In this section we present conditions sufficient for the existence of solutions to an abstract inclusion

$$A(x) \in \psi(x), \tag{6}$$

where $A : E \to E'$ is a Fredholm linear operator of nonnegative index k (E, E' are Banach spaces) and ψ is a multivalued map. Next, we apply this result to a concrete boundary value problem in Banach spaces.

Of course, if ψ is determined by an A-fundamentally contractible morphism (Ψ, U) such that $\operatorname{ind}_A(\Psi, U)$ is not a trivial element of Π_k, then the inclusion (6) has a solution. But it is not simple to verify it when E is not finite-dimensional.

Now we reformulate (6). Since A is a Fredholm operator, there are two bounded linear projections $P : E \to E$ and $Q : E' \to E'$ such that $R(P) = \operatorname{Ker}(A)$ and $\operatorname{Ker}(Q) = R(A)$. Observe that $\operatorname{Ker}(P) \oplus \operatorname{Ker}(A) = E$ and $R(Q) \oplus R(A) = E'$. Moreover, $\dim R(P), \dim R(Q) < \infty$ and $\dim R(P) - \dim R(Q) = k$.

The restriction of A to $\operatorname{Ker}(P)$ is a linear homeomorphism onto $R(A)$, hence it admits a right inverse $K_P : R(A) \to E$ defined by $K_P(y) = x$ if and only if $x \in \operatorname{Ker}(P)$ and $A(x) = y$. K_P is a continuous linear map. Let $K_{PQ} : E' \to E$ be the generalized inverse of A, i.e. $K_{PQ} = K_P \circ (I - Q)$. Clearly,

$$R(K_{PQ}) = \operatorname{Ker}(P), \qquad A \circ K_{PQ} = \operatorname{id}_{E'} - Q, \qquad K_{PQ} \circ A = \operatorname{id}_E - P.$$

Denote by J an arbitrary (but fixed) injective linear map from $R(Q)$ to $R(P)$. Of course, J is continuous. Observe that $x \in \operatorname{cl} U$ is a solution to the inclusion (6) if and only if

$$F(x) \in \phi(x),$$

where

$$F = \operatorname{id}_E - P : E \to \operatorname{Ker}(P) \oplus R(J)$$

and

$$\phi = (K_{PQ} + J \circ Q) \circ \psi(x) : \operatorname{cl} U \multimap \operatorname{Ker}(P) \oplus R(J).$$

Indeed, if $y = A(x) \in \psi(x)$, then $Q(y) = Q \circ A(x) = 0$ and $F(x) = K_{PQ} \circ A(x) = (K_{PQ} + J \circ Q)(y) \in \phi(x)$. On the other hand, if $F(x) \in \phi(x)$, then $F(x) = (K_{PQ} + J \circ Q)(y)$, where $y \in \psi(x)$. It follows that $Q(y) = 0$ and hence $A(x) = A \circ K_{PQ}(y) = y \in \psi(x)$. Moreover, F is a Fredholm operator of index k (equal to the index of A).

THEOREM 4.1. *Assume that* $U = E$ *and*

(i) $\operatorname{ind} A = k \geq 0$;

(ii) *there exists a metric space* W *and a cotriad* $E \xleftarrow{p} W \xrightarrow{q} E'$ *such that* $\psi(x) = q(p^{-1}(x))$, $([(p|_{p^{-1}(\mathrm{cl}\, V)}, q|_{p^{-1}(\mathrm{cl}\, V)})], V) \in \mathcal{M}^A(E, E')$ *for every open bounded* V, *and some* A*-fundamental set for* $(p|_{p^{-1}(\mathrm{cl}\, V)}, q|_{p^{-1}(\mathrm{cl}\, V)})$ *contains* 0;

(iii) *there exists a constant* $M > 0$ *such that for any* $x \in E$, *if* $y \in (\mathrm{id}_{E'} - Q) \circ \psi(x)$, *then* $\|y\|_{E'} < M$.

Additionally, *if* $R(Q) \neq 0$, *assume that there exists a constant* $R > 0$ *such that*

(iv) *if* $\|P(x)\|_E \geq R$ *and* $\|x - P(x)\|_E \leq \|K_P\| M$, *then* $0 \notin Q \circ \psi(x)$;

(v) $\operatorname{ind}_{\mathcal{O}}(-Q \circ \psi|_{B_E(0,R) \cap R(P)}, B_E(0,R) \cap R(P)) \in \Pi_k$ *is nontrivial* (*here* $\mathcal{O} : R(P) \to R(Q)$ *is the zero operator*).

Then the inclusion (6) *has a solution.*

Proof. First, suppose that $R(Q) \neq 0$. If $x \in E$ is a solution to (6), then there is $y \in \psi(x)$ such that $Q(y) = 0$ and $x - P(x) = K_{PQ}(y)$. Therefore, by (iii), $\|x - P(x)\|_E < \|K_P\| M$ and, by (iv), $\|P(x)\|_E < R$. Hence each solution x is contained in the open bounded set $V = \{x \in E : \|x - P(x)\|_E < \|K_P\| M,\ \|P(x)\|_E < R\}$ and the map determined by $[(p|_{p^{-1}(\mathrm{cl}\, V)}, q|_{p^{-1}(\mathrm{cl}\, V)})]$ has no coincidence points with A on $\mathrm{bd}\, V$.

To simplify the notation put $p' = p|_{p^{-1}(\mathrm{cl}\, V)}$ and $q' = q|_{p^{-1}(\mathrm{cl}\, V)}$. Let $(p', \overline{q}')$ be a compact cotriad associated with (p', q'). Observe that

$$K_{PQ} + J \circ Q : E' \to \mathrm{Ker}(P) \oplus R(J)$$

is a continuous linear isomorphism. By Remark 1.4 and Theorem 2.2, the morphisms $[(p', (K_{PQ} + J \circ Q) \circ q')]$ and $[(p', (K_{PQ} + J \circ Q) \circ \overline{q}')]$ are F-homotopic without coincidence points on $\mathrm{bd}\, V$. Moreover, the morphisms $[(p', (K_{PQ} + J \circ Q) \circ \overline{q}')]$ and $[(p', J \circ Q \circ \overline{q}')]$ are compactly F-homotopic (hence F-homotopic) without coincidence points on $\mathrm{bd}\, V$. Indeed, consider the following homotopy $\mathrm{cl}\, V \times I \xleftarrow{R} W \times I \xrightarrow{S} \mathrm{Ker}(P) \oplus R(J)$:

$$R(w, \lambda) = (p'(w), \lambda), \qquad S(w, \lambda) = (\lambda K_{PQ} + J \circ Q) \circ \overline{q}'(w).$$

Of course, it has no coincidence points with F on $\mathrm{bd}\, V$.

Hence, by the homotopy property of the index,

$$\operatorname{ind}_F([(p', (K_{PQ} + J \circ Q) \circ \overline{q}')], V) = \operatorname{ind}_F([(p', J \circ Q \circ \overline{q}')], V).$$

But if we put $\Phi = [(p', J \circ Q \circ \overline{q}')]$, $T = R(J)$, $T' = F^{-1}(R(J)) = R(P)$, $F' = F|_{T'} \equiv 0$ and $\Phi' = \Phi|_{T' \cap V}$, then, by the restriction property of the index and assumption (v), $\operatorname{ind}_F([(p', J \circ Q \circ \overline{q}')], V)$ is a nontrivial element of Π_k, because $\overline{q}'|_{\mathrm{cl}\, V \cap R(P)} = q|_{\mathrm{cl}\, V \cap R(P)}$. Hence the inclusion $F(x) \in \phi(x)$

has a solution, which implies the existence of a solution to the inclusion $A(x) \in \psi(x)$.

If $Q \equiv 0$, then $K_{PQ} = K_P : E' \to \mathrm{Ker}(P)$ is a continuous linear isomorphism and by assumption (iii) for all $y \in \psi(E)$, $\|y\|_{E'} \le M$.

Let $V = \{x \in E : \|x\|_E < \|K_P\|M\}$. Observe that, by Remark 1.4, $([(p|_{p^{-1}(\mathrm{cl}\,V)}, K_P \circ q|_{p^{-1}(\mathrm{cl}\,V)})], V) \in \mathcal{M}^F(E,E)$, because, as is easily seen, $([(p|_{p^{-1}(\mathrm{cl}\,V)}, q|_{p^{-1}(\mathrm{cl}\,V)})], V) \in \mathcal{M}^A(E,E')$. Without loss of generality we can assume that a compact F-fundamental set of the multivalued map η, determined by the cotriad $(p|_{p^{-1}(\mathrm{cl}\,V)}, K_P \circ q|_{p^{-1}(\mathrm{cl}\,V)})$, is contained in the convex set V. Hence, its associated cotriad $(p|_{p^{-1}(V)}, \overline{q})$ determines a compact map $\overline{\eta}$ from V to V and by the Schauder Fixed Point Theorem there is $x \in V$ such that $x \in \overline{\eta}(x)$. Of course, then $x \in \eta(x)$ and hence $F(x) \in \eta(x)$. Finally, the inclusion $F(x) \in \phi(x)$ has a solution and the proof is complete. ■

Remark 4.2. Observe that if $Q \not\equiv 0$ then instead of (iii) and (iv) in the above theorem we can assume that there is an open bounded $V \subset E$ such that there exists a compact A-fundamental set $K \subset E'$ for $\psi|_V$, $Q(\psi(V)) \subset K$ and F has no coincidence point with $(\lambda K_{PQ} + J \circ Q) \circ \psi|_{\mathrm{cl}\,V}$ on $\mathrm{bd}\,V$ for every $\lambda \in [0,1]$.

Indeed, it is easy to verify that then $K' = (K_{PQ} + J \circ Q)(K) \supset J \circ Q(\psi(V))$ and K' is an F-fundamental set for $(\lambda K_{PQ} + J \circ Q) \circ \psi$ for any $\lambda \in [0,1]$. Hence the morphisms $[(p', (K_{PQ} + J \circ Q) \circ q')]$ and $[(p', J \circ Q \circ q')]$ are F-homotopic with an F-fundamental set K'. The rest of the proof is identical.

If $Q \equiv 0$, instead of (iii) we can assume that there exists a bounded convex set $Z \subset \mathrm{Ker}(P)$ such that $K_P \circ \psi(Z) \subset Z$.

Below we illustrate the above result by a boundary value problem.

Let E, E' be Banach spaces, $J = [0,T] \subset \mathbb{R}$, χ, χ' be the Hausdorff measures of noncompactness on E and E' respectively, and let ξ be the Hausdorff measure of noncompactness on the space $C = C(J,E)$ of continuous functions from J to E.

Recall that if $\mathcal{B}$ is the family of all bounded subsets of E, then the *Hausdorff measure of noncompactness* $\chi : \mathcal{B} \to [0,\infty)$ is given by the formula $\chi(B) = \inf\{r > 0 : B \text{ has a finite } r\text{-net } (^4)\}$.

Let Z be the set of all positive numbers k such that the Fredholm linear operator $D : E \to E'$ is a (k,χ,χ')-set contraction ([5]). Following [1] we define

([4]) That is, a finite number of points $x_1, \ldots, x_k \in E$ such that $B \subset \bigcup_{i=1}^k B_E(x_i, r)$.

([5]) That is, for any bounded set $B \subset E$, the set $D(B)$ is bounded and $\chi'(D(B)) \le k\chi(B)$.

$$\|D\|^{(\chi,\chi')} := \inf Z.$$

Observe that $\|D\|^{(\chi,\chi')} \le \|D\|$.

Let $g : J \times E \to E$ and $h : J \times E \times E \to E$ be continuous (singlevalued) maps such that

(f_1) g is completely continuous and uniformly continuous on bounded sets,

(f_2) there exists a continuous function $\varrho : J \to [0,1)$ such that

$$\|h(t,u_1,v_1) - h(t,u_2,v_2)\|_E \le \varrho(t)(\|u_1 - u_2\|_E + \|v_1 - v_2\|_E)$$

for any $t \in J$, $u_1,u_2,v_1,v_2 \in E$.

Suppose that g, h have sublinear growth, i.e.

(f_3) there exist continuous functions $m,n : J \to [0,\infty)$ such that

$$\|g(t,u) + h(t,u,v)\|_E \le m(t) + n(t)\|u\|_E \quad \text{for } (t,u,v) \in J \times E \times E.$$

Consider the following boundary value problem:

$$\begin{cases} u'(t) = f(t,u(t),u'(t))), \\ L_1(u(0)) + L_2(u(T)) = b(u(0)), \end{cases} \tag{7}$$

where $f(t,u,v) = g(t,u) + h(t,u,v)$, $b : E \to E'$ is a compact continuous map and $L_1, L_2 : E \to E'$ are linear operators such that $L = L_1 + L_2$ is a Fredholm operator of nonnegative index.

We can rewrite problem (7) as follows:

$$A(z,y) = \psi(z,y), \tag{8}$$

where $A, \psi : E \times C \to E' \times C$ and

$$A(z,y) = (L(z), y),$$

$$\psi(z,y) = \Big(b(z) - L_2\Big(\int_0^T y(s)\,ds\Big), f\Big(\cdot, z + \int_0^{\cdot} y(s)\,ds, y(\cdot)\Big)\Big).$$

It is easily seen that if (z,y) is a solution of (8), then u defined by $u(t) = z + \int_0^t y(s)\,ds$ is a solution of (7).

As usual we equip the spaces $E \times C$ and $E' \times C$ with the max-norms, i.e. for $z \in E$, $c \in E'$ and $y \in C$, $\|(z,y)\|_1 = \max(\|z\|_E, \|y\|_C)$, $\|(c,y)\|_2 = \max(\|c\|_{E'}, \|y\|_C)$. Denote the Hausdorff measures of noncompactness in $E \times C$ and $E' \times C$ by μ and μ', respectively. Let π_E, π_C (resp. $\pi_{E'}$, π'_C) be projections of $E \times C$ (resp. $E' \times C$) onto E and C (resp. onto E' and C). Observe that if S is a bounded subset of $E \times C$, then $\mu(S) = \max(\chi(\pi_E(S)), \xi(\pi_C(S)))$.

Denote by P_L, Q_L, K_{P_L} the respective projections and the right inverse for L. Moreover, let $M = \sup_{t\in J} m(t)$, $N = \sup_{t\in J} n(t)$, $N_1 = \sup_{t\in J} tn(t)$.

THEOREM 4.3. *Assume that* f, b, L *are as above*, $R(L_2) \subset R(L)$, $Q_L \not\equiv 0$ *and additionally*:

(f_4) $\sup_{t\in J}(\varrho(t)(t+1)) < 1$ *and* $\|K_{P_L}\|^{(\chi,\chi')} < \dfrac{1-\sup_{t\in J}\varrho(t)(t+1)}{\sup_{t\in J}(\varrho(t))}$,

(f_5) $\|L_2\| \cdot T < 1$,

(f_6) $Ne^{N_1} < 1$,

(f_7) *there exists* $R > 0$ *such that for all* z *if* $\|P(z)\|_E > R$, *then* $Q_L(b(z)) \neq 0$ *and* $\mathrm{ind}_{\mathcal{O}}(Q_L \circ b, B(0,R) \cap R(P))$ *is a nontrivial element of* Π_k.

Then problem (7) *has a solution.*

Proof. We prove that problem (8), equivalent to (7), has a solution.

Observe that A is a Fredholm linear operator and its index is equal to the index of L.

STEP 1. We prove that if V is an open bounded subset of $E \times C$, then $\psi(V)$ is a bounded set and $\psi|_{\mathrm{cl}\,V}$ is an $A|_{\mathrm{cl}\,V}$-condensing (so $A|_{\mathrm{cl}\,V}$-fundamentally contractible) map.

Take an arbitrary subset S of V. We have to prove that $\mu'(\psi(S)) < \mu'(A(S))$. Let $\chi(\pi_E(S)) = \varepsilon$ and $\xi(\pi_C(S)) = \delta$. Then

$$\mu'(A(S)) = \max(\chi'(\pi_{E'}(A(S))), \xi(\pi_C(A(S)))) = \max(\chi'(\pi_{E'}(A(S))), \delta).$$

Observe that, since $\mathrm{Ker}(L) = R(P_L)$ is of finite dimension,

$$\chi(\pi_E(S)) = \chi((\mathrm{id}_E - P_L) \circ \pi_E(S)) = \chi(K_{P_L} \circ L \circ \pi_E(S)).$$

But

$$\chi(K_{P_L} \circ L \circ \pi_E(S)) \le \|K_{P_L}\|^{(\chi,\chi')} \chi'(L \circ \pi_E(S))$$

and $L(\pi_E(S)) = \pi_{E'}(A(S))$, hence

$$\chi'(\pi_{E'}(A(S))) \ge \frac{\chi(\pi_E(S))}{\|K_{P_L}\|^{(\chi,\chi')}} = \frac{\varepsilon}{\|K_{P_L}\|^{(\chi,\chi')}}.$$

Finally we get

$$\mu'(A(S)) \ge \max\left(\frac{\chi(\pi_E(S))}{\|K_{P_L}\|^{(\chi,\chi')}}, \delta\right).$$

Now we estimate $\mu(\psi(S))$. Obviously,

$$\mu'(\psi(S)) = \max\Big(\chi'\Big(\Big\{b(z) - L_2\Big(\int_0^T y(s)\,ds\Big) : (z,y) \in S\Big\}\Big),$$
$$\xi\Big(\Big\{f\Big(\cdot, z + \int_0^{\cdot} y(s)\,ds, y(\cdot)\Big) : (z,y) \in S\Big\}\Big)\Big).$$

Observe that

$$\chi'\Big(\Big\{b(z) - L_2\Big(\int_0^T y(s)\,ds\Big) : (z,y) \in S\Big\}\Big) \le \chi'\Big(\Big\{L_2\Big(\int_0^T y(s)\,ds\Big) : y \in \pi_C(S)\Big\}\Big),$$

because $\chi'(\{b(z) : z \in \pi_E(S)\}) = 0$. Now, for arbitrary $\delta_1 > 0$, take an element $\overline{y}$ from a $(\delta + \delta_1)$-net of $\pi_C(S)$. If $\|\overline{y} - y\|_C < \delta + \delta_1$, then

$$\Big\|L_2\Big(\int_0^T \overline{y}(s)\,ds\Big) - L_2\Big(\int_0^T y(s)\,ds\Big)\Big\|_E \le \|L_2\| \int_0^T \|\overline{y}(s) - y(s)\|_E\,ds < \|L_2\|T(\delta + \delta_1).$$

Hence $\chi'(\{L_2(\int_0^T y(s)\,ds) : y \in \pi_L(S)\}) \le \|L_2\|T\delta < \delta$.

Consider the family of functions

$$F_1(S) = \Big\{g_{z,y} \in C : g_{z,y}(t) = g\Big(t, z + \int_0^t y(s)\,ds\Big),\ (z,y) \in S\Big\}.$$

Since S is bounded and g is completely continuous, the sets $\{g_{z,y}(t) : (z,y) \in S\}$ are compact for any $t \in J$. Moreover, the family $F_1(S)$ is uniformly equicontinuous. Hence, by Ascoli's Theorem (cf. [11]), $\xi(F_1(S)) = 0$. This gives

$$\xi\Big(\Big\{f\Big(\cdot, z + \int_0^{\cdot} y(s)\,ds, y(\cdot)\Big) : (z,y) \in S\Big\}\Big) \le \xi\Big(\Big\{h\Big(\cdot, z + \int_0^{\cdot} y(s)\,ds, y(\cdot)\Big) : (z,y) \in S\Big\}\Big).$$

If we take arbitrary $\varepsilon_1 > 0$ and $\delta_1 > 0$, $\overline{z}$ from an $(\varepsilon + \varepsilon_1)$-net of $\pi_E(S)$, $\overline{y}$ from a $(\delta + \delta_1)$-net of $\pi_C(S)$ and if $\|\overline{z} - z\|_E < \varepsilon + \varepsilon_1$ and $\|\overline{y} - y\|_C < \delta + \delta_1$, then

$$\begin{aligned}
&\Big\|h\Big(t, \overline{z} + \int_0^t \overline{y}(s)\,ds, y(t)\Big) - h\Big(t, z + \int_0^t y(s)\,ds, y(t)\Big)\Big\|_E \\
&\quad \le \varrho(t)\Big(\|\overline{z} - z\|_E + \int_0^t \|\overline{y}(s) - y(s)\|_E\,ds + \|\overline{y}(t) - y(t)\|_E\Big) \\
&\quad < \varrho(t)(\varepsilon + \varepsilon_1 + t(\delta + \delta_1) + \delta + \delta_1) \le \sup_{t \in J} \varrho(t)(\varepsilon + \varepsilon_1 + (t+1)(\delta + \delta_1)).
\end{aligned}$$

Therefore $\xi(\pi'_C(\psi(S))) \le \sup_{t \in J} \varrho(t)(\varepsilon + (t+1)\delta)$.

Now, it is easy to check that assumptions $(f_4)-(f_5)$ imply that $\mu'(\psi(S)) < \mu'(A(S))$.

STEP 2. Since b is a compact map, there exists $Z > 0$ such that $b(E) \subset B_{E'}(0,Z)$. Denote by P, Q, K_P and K_{PQ} the respective projections, the right inverse and the generalized inverse for A (see the beginning of this section).

Now, we introduce a set V satisfying the assumptions of Remark 4.2. Let

$$V = \{(z,y) \in E \times C : z = z_0 + z_1,\ z_0 \in B_E(0,R) \cap R(P_L),\\ z_1 \in B_E(0,R_1) \cap \operatorname{Ker}(P_L),\ y \in B_C(0,R_2)\},$$

where R is as in assumption (f_7),

$$R_1 > \frac{\|K_{P_L}\|(Z + Me^{N_1} + NRe^{N_1})}{1 - Ne^{N_1}}, \qquad R_2 = (M + N(R+R_1))e^{N_1}.$$

We can find a compact A-fundamental set for $\psi|_{\operatorname{cl} V}$ containing $Q \circ \psi(V)$ (since it is a bounded subset of the finite-dimensional space $R(Q)$, see Example 1.10).

We will show that there is no solution of the problem $F(z,y) = (\lambda K_{PQ} + J \circ Q) \circ \psi(z,y)$ on the boundary of the set V for every $\lambda \in [0,1]$.

Assume that $(z,y) \in \operatorname{cl} V$ is such a solution for some $\lambda \in [0,1]$. Let $z = z_0 + z_1$ where $z_0 \in R(P_L)$, $z_1 \in \operatorname{Ker}(P_L)$. Then

$$(z_1, y) = \Big(\lambda K_{P_L} \circ (\mathrm{id}_{E'} - Q_L) \circ b(z) - \lambda K_{P_L} \circ L_2\Big(\int_0^T y(s)\,ds\Big),\\ f\Big(\cdot, z + \int_0^{\cdot} y(s)\,ds, y(\cdot)\Big)\Big) + J(Q_L \circ b(z), 0).$$

Hence

$$Q(b(z)) = 0, \tag{9}$$

$$z_1 = \lambda K_{P_L}(\mathrm{id}_{E'} - Q)(b(z)) - \lambda K_{P_L}\Big(L_2\Big(\int_0^T y(s)\,ds\Big)\Big), \tag{10}$$

$$y = \Big(\cdot, z + \int_0^{\cdot} y(s)\,ds, y(\cdot)\Big). \tag{11}$$

Equality (9) and assumption (f_7) imply at once $\|z_0\| < R$. Moreover, by (11),

$$\|y(t)\|_E = \Big\| f\Big(t, z_0 + z_1 + \int_0^t y(s)\,ds, y(t)\Big)\Big\|_E \\ \le m(t) + n(t)\Big(\|z_0\|_E + \|z_1\|_E + \int_0^t \|y(s)\|_E\,ds\Big)$$

and by the Gronwall inequality

$$\|y(t)\|_E \leq (m(t) + n(t)(\|z_0\|_E + \|z_1\|_E))e^{N_1}.$$

Hence assumption (f_6) gives

$$\|y\|_C = \sup_{t\in J} \|y(t)\|_E \leq (M + N(\|z_0\|_E + R_1))e^{N_1} < (M + N(R + R_1))e^{N_1} = R_2.$$

Finally, if $\lambda = 0$, then (10) gives $\|z_1\|_E = 0 < R_1$. If $\lambda \in (0,1]$, then by (10) and assumption (f_5),

$$\|z_1\|_E \leq \lambda\|K_{P_L}\|(Z + \|y\|_C) < \|K_{P_L}\|(Z + (M + N(R + R_1))e^{N_1}) < R_1.$$

Step 3. Observe that assumption (v) of Theorem 4.1 is also satisfied. Indeed, assumption (f_7) implies the nontriviality of the index, because

$$Q \circ \psi|_{B(0,R)\cap R(P)}(z, y) = Q(b(z), 0) = Q_L(b(z)).$$

Now, by Theorem 4.1 with Remark 4.2 the proof is complete. ∎

Remark 4.4. If $Q_L \equiv 0$ then we also get an existence result for problem (7) if we replace assumptions (f_6) and (f_7) by

$$\|K_{P_L}\| \cdot \|L_2\| + TN < 1.$$

Indeed, one can easily verify that the set

$$Z = \Big\{(z, y) : \Big\|z + \int_0^t y(s)\, ds\Big\| < R',\ \forall t\Big\} \cap \{(z, y) : \|y\|_C < R''\} \cap \mathrm{Ker}(P),$$

where

$$R' > \frac{\|K_{P_L}\|S + TM}{1 - \|K_{P_L}\| \cdot \|L_2\| - TN} \quad \text{and} \quad R'' > M + NR',$$

satisfies Remark 4.2.

References

[1] R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina and B. N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators*, Birkhäuser, Basel, 1992.

[2] S. Armentrout and T. M. Price, *Decomposition into compact sets with UV properties*, Trans. Amer. Math. Soc. 141 (1969), 433–442.

[3] Yu. G. Borisovich, B. D. Gel'man, A. D. Myshkis and V. V. Obukhovskiĭ, *Topological methods in the fixed point theory of multivalued mappings*, Russian Math. Surveys 35 (1980), 65–143.

[4] Yu. G. Borisovich, V. G. Zvyagin and Yu. I. Sapronov, *Nonlinear Fredholm mappings and Leray–Schauder theory*, ibid. 32 (1977), 3–54.

[5] K. Gęba, *Algebraic topology methods in the theory of compact fields in Banach spaces*, Fund. Math. 46 (1964), 177–209.

[6] K. Gęba, I. Massabò and A. Vignoli, *Generalized topological degree and bifurcation*, in: Nonlinear Functional Analysis and its Applications (Maratea, 1985), Reidel, 1986, 55–73.
[7] S. Goldberg, *Unbounded Linear Operators. Theory and Applications*, McGraw-Hill, 1966.
[8] L. Górniewicz, *Homological methods in fixed-point theory of multi-valued maps*, Dissertationes Math. 129 (1976).
[9] L. Górniewicz and W. Kryszewski, *Bifurcation invariants for acyclic mappings*, Rep. Math. Phys. 31 (1992), 217–239.
[10] J. Izé, I. Massabò and A. Vignoli, *Degree theory for equivariant maps I*, Trans. Amer. Math. Soc. 315 (1989), 433–510.
[11] J. L. Kelley, *General Topology*, Van Nostrand, New York, 1955.
[12] W. Kryszewski, *Homotopy properties of set-valued mappings*, Wyd. Uniwersytetu Mikołaja Kopernika, Toruń, 1997.
[13] —, *Topological and approximation methods in the degree theory of set-valued maps*, Dissertationes Math. 336 (1994).
[14] —, *The fixed-point index for the class of compositions of acyclic set-valued maps on ANR's*, Bull. Sci. Math. 120 (1996), 129–151.
[15] —, *Remarks to the Vietoris Theorem*, Topol. Methods Nonlinear Anal. 8 (1996), 383–405.
[16] R. C. Lacher, *Cell-like mappings and their generalizations*, Bull. Amer. Math. Soc. 83 (1977), 336–552.
[17] J. Mawhin, *Topological Degree Methods in Nonlinear Boundary Value Problems*, CBMS Regional Conf. Ser. in Math. 40, Amer. Math. Soc., Providence, 1979.
[18] T. Pruszko, *Some applications of the topological degree theory to multi-valued boundary value problems*, Dissertationes Math. 229 (1984).

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