ANNALES
POLONICI MATHEMATICI
LXXV.2 (2000)

Existence and multiplicity results for nonlinear eigenvalue problems with discontinuities

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Abstract. We study eigenvalue problems with discontinuous terms. In particular we consider two problems: a nonlinear problem and a semilinear problem for elliptic equations. In order to study the existence of solutions we replace these two problems with their multivalued approximations and, for the first problem, we estabilish an existence result while for the second problem we prove the existence of multiple nontrivial solutions. The approach used is variational.

1. Introduction. The aim of this paper is to study nonlinear and semilinear eigenvalue problems with discontinuous terms. So let $Z \subset \mathbb{R}^N$ be a bounded domain with a C^1 boundary Γ. We start with the following eigenvalue problem:

$$(1)\quad \begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) - \lambda|x(z)|^{p-2}x(z) = f(z,x(z)) & \text{a.e. on } Z,\\ x|_\Gamma = 0, \quad 2 \le p < \infty. \end{cases}$$

Here $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that for all $z \in Z$, $f(z,\cdot)$ is locally bounded but not necessarily continuous and this implies that the problem (1) may not have any solutions. So in order to develop a reasonable existence theory, we need to pass to a multivalued extension of problem (1) by, roughly speaking, filling in the gaps at the discontinuity points of $f(z,\cdot)$. More precisely, we introduce the following two functions:

$$f_0(z,x) = \liminf_{x' \to x} f(z,x'), \qquad f_1(z,x) = \limsup_{x' \to x} f(z,x').$$

Note that when the one-sided limits $f(z,x^-)$ and $f(z,x^+)$ exist then $f_0(z,x) = \min\{f(z,x^-), f(z,x^+)\}$ and $f_1(z,x) = \max\{f(z,x^-), f(z,x^+)\}$.

2000 *Mathematics Subject Classification*: 35R70, 35J20, 58E05.
Key words and phrases: locally Lipschitz functional, subdifferential, discontinuous term, multivalued problem, eigenvalues, p-Laplacian.

Instead of (1) we consider the following multivalued version:

$$(2)\qquad \begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) - \lambda|x(z)|^{p-2}x(z) \\ \qquad\qquad\qquad \in [f_0(z,x(z)), f_1(z,x(z))] \quad \text{a.e. on } Z, \\ x|_\Gamma = 0, \quad 2 \le p < \infty. \end{cases}$$

Using a variational approach (for nonsmooth, locally Lipschitz energy functionals) we show that problem (2) has a solution.

We will also study the semilinear problem

$$(3)\qquad \begin{cases} -\Delta x(z) - \lambda x(z) = f(z,x(z)) \quad \text{a.e. on } Z, \\ x|_\Gamma = 0. \end{cases}$$

In this case $f(z,\cdot)$ need not be continuous either. So after introducing the functions $f_0(z,x)$ and $f_1(z,x)$, we replace (3) with its multivalued counterpart

$$(4)\qquad \begin{cases} -\Delta x(z) - \lambda x(z) \in [f_0(z,x(z)), f_1(z,x(z))] \quad \text{a.e. on } Z, \\ x|_\Gamma = 0. \end{cases}$$

For problem (4), using a result for locally Lipschitz functionals due to D. Goeleven, D. Motreanu and P. D. Panagiotopoulos [9], we prove the existence of multiple nontrivial solutions.

Similar eigenvalue problems were studied by K. J. Brown and H. Budin [3], P. Hess [10], P. Rabinowitz [14], D. G. De Figueiredo [8], C. Lefter and D. Motreanu [12], M. Ramos [15]. All these works deal with semilinear equations (i.e. $p = 2$) and with exception of [12], assume $f(\cdot,\cdot)$ to be continuous. The approaches vary although all use some aspects of the variational method. More specifically P. Rabinowitz [14], D. G. De Figueiredo [8], C. Lefter and D. Motreanu [12], M. Ramos [15] follow the variational approach. P. Hess [10] has a proof which combines variational and topological degree arguments, while K. J. Brown and H. Budin [3] employ a combination of variational and monotone iteration methods.

In this paper our approach is solely variational. We use the critical point theory for nonsmooth local energy functionals (see K. C. Chang [5]).

2. Mathematical preliminaries. Let X be a reflexive Banach space and X^* its topological dual. A function $\Phi : X \to \mathbb{R}$ is said to be *locally Lipschitz* if for every $x \in X$, there exists a neighbourhood U of x and a constant $k > 0$ depending on U such that $|\Phi(z) - \Phi(y)| \le k\|z-y\|$ for all $z, y \in U$. For this kind of functions the *generalized directional derivative* $\Phi^0(x;h)$ at $x \in X$ in the direction $h \in X$ is defined by

$$\Phi^0(x;h) = \limsup_{x' \to x,\ \lambda \downarrow 0} \frac{\Phi(x' + \lambda h) - \Phi(x')}{\lambda}.$$

It is known that the function $h \mapsto \Phi^0(x;h)$ is sublinear, continuous and it is the support function of the nonempty convex w^*-compact set

$$\partial\Phi(x) = \{x^* \in X : (x^*, h) \le \Phi^0(x;h) \text{ for all } h \in X\}.$$

The set $\partial\Phi(x)$ is called the *generalized* or *Clarke subdifferential* of Φ at x. If $\Phi, \Psi : X \to \mathbb{R}$ are locally Lipschitz functions, then $\partial(\Phi+\Psi)(x) \subseteq \partial\Phi(x) + \partial\Psi(x)$, while for any $\lambda \in \mathbb{R}$ we have $\partial(\lambda\Phi(x)) = \lambda\partial\Phi(x)$. Moreover, if $\Phi : X \to \mathbb{R}$ is convex, then this subdifferential coincides with the subdifferential in the sense of convex analysis. If $\Phi : X \to \mathbb{R}$ is strictly differentiable then $\partial\Phi(x) = \{\Phi'(x)\}$. A point $x \in X$ is a *critical point* of Φ if $0 \in \partial\Phi(x)$, while a *critical value* is the value assumed by Φ at a critical point. For more details we refer to the monograph of Clarke [6].

The compactness conditions for locally Lipschitz functionals $\Phi : X \to \mathbb{R}$ that we consider are the following:

• Φ satisfies the (PS)-*condition* if any sequence $\{x_n\}_{n\ge1} \subset X$ such that $\{\Phi(x_n)\}_{n\ge1}$ is bounded and $m(x_n) \to 0$ as $n \to \infty$ has a convergent subsequence (where $m(x_n) = \min_{x^*\in\partial\Phi(x_n)} \|x^*\|_{X^*}$; the existence of such an element follows from the fact that $\partial\Phi(x_n)$ is weakly compact and the norm functional on X^* is weakly semicontinuous);

• Φ satisfies the C-(PS)-*condition* if any sequence $\{x_n\}_{n\ge1} \subset X$ such that $\{\Phi(x_n)\}_{n\ge1}$ is bounded and $(1 + \|x_n\|)m(x_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Remark 1. The (PS)-condition is a generalization of the well known Palais–Smale condition proposed by Chang [5] in his nonsmooth critical point theory in order to obtain various minimax principles concerning the existence and characterization of critical points for locally Lipschitz functionals. The C-(PS)-condition is a weaker form of the (PS)-condition and it is the nonsmooth version of the condition introduced by Cerami in [4]. In a recent paper N. C. Kourogenis and N. S. Papageorgiou [11], using the C-(PS)-condition, derive minimax principles. For the convenience of the reader, we recall the following version of the Mountain Pass Theorem (cf. [11], Theorem 6).

Theorem 1. *If X is a reflexive Banach space*, $\Phi : X \to \mathbb{R}$ *is a locally Lipschitz functional satisfying the C-(PS)-condition and there are* $\varrho > 0$, $\alpha, \beta \in \mathbb{R}$ *and* $x_1 \in X$ *such that* $\|x_1\| > \varrho$ *and* $\max[\Phi(0), \Phi(x_1)] \le \alpha < \beta \le \inf_{\|x\|=\varrho} \Phi(x)$ *then Φ has a critical value $c \ge \beta$ characterized by*

$$c = \inf_{\gamma\in\Gamma} \max_{0\le t\le1} \Phi(\gamma(t))$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = x_1\}$.

Moreover we recall the abstract multiplicity result due to Goeleven–Motreanu–Panagiotopoulos (cf. [9], Theorem 2.1) that we will use to prove the existence of multiple nontrivial solutions for the problem (4).

THEOREM 2. *Let X be a reflexive Banach space. Suppose that $\Phi : X \to \mathbb{R}$ is an even, locally Lipschitz functional satisfying the (PS)-condition and the conditions*:

(i) $\Phi(0) = 0$,

(ii) *there exists a subspace X_1 of X of finite codimension and numbers $\beta, \varrho > 0$ such that $\Phi(x) \geq \beta$ for all $x \in X_1$ such that $\|x\| = \varrho$*,

(iii) *there is a finite-dimensional subspace X_2 of X, with $\dim X_2 > \operatorname{codim} X_1$, such that $\Phi(u) \to -\infty$ as $\|u\| \to \infty$, $u \in X_2$.*

Then Φ has at least $\dim X_2 - \operatorname{codim} X_1$ pairs of nontrivial critical points.

Finally, we recall the following definitions for an operator $A : X \to X^*$:

• A is said to be *monotone* if $\langle Ax_1 - Ax_2, x_1 - x_2\rangle \geq 0$ for all $x_1, x_2 \in X$;

• A is said to be *pseudomonotone* if for any sequence $\{x_n\}_{n\geq 1} \subset X$ such that $x_n \to x$ weakly in X and $\limsup_{n\to\infty}\langle Ax_n, x_n - x\rangle \leq 0$ it follows that $\langle Ax, x - w\rangle \leq \liminf_{n\to\infty}\langle Ax_n, x_n - w\rangle$ for all $w \in X$;

• A is said to be *demicontinuous* if for any sequence $\{x_n\}_{n\geq 1} \subset X$ such that $x_n \to x$ in X it follows that $Ax_n \to Ax$ weakly in X.

3. Existence result. We start with the quasilinear problem. Let λ_1 be the first eigenvalue of the p-Laplacian $-\Delta_p x = -\operatorname{div}(\|Dx\|^{p-2}Dx)$ on Z with zero Dirichlet boundary condition, which is the least real number λ for which the problem

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = \lambda|x(z)|^{p-2}x(z) & \text{a.e. on } Z,\\ x|_\Gamma = 0, \end{cases}$$

has a nontrivial solution. It is known (cf. [13]) that this first eigenvalue λ_1 is positive, isolated and simple (i.e. the associated eigenfunctions are constant multiples of each other). Moreover we have the following variational characterization of λ_1 via the Rayleigh quotient:

$$\lambda_1 = \min[\|Dx\|_p^p/\|x\|_p^p : x \in W_0^{1,p}(Z),\ x \geq 0],$$

where $\|\cdot\|_p$ denotes the norm in the space $L^p(Z)$.

One can prove that the corresponding eigenfunction u_1 is almost everywhere nonzero on Z. Let

$$\lambda_2 = \inf\{\lambda > 0 : \lambda \text{ is an eigenvalue of } -\Delta_p \text{ on } W_0^{1,p}(Z) \text{ and } \lambda > \lambda_1\}.$$

Note that since λ_1 is isolated, $\lambda_2 > \lambda_1$. Moreover, if $p = 2$, then λ_2 is the second eigenvalue of the Laplacian.

In what follows let $F(z,x)=\int_0^x f(z,r)\,dr$, $x\in\mathbb{R}$, be the *potential function* corresponding to f and $\Psi : W_0^{1,p}(Z)\to\mathbb{R}$ be the *energy functional* defined as $\Psi(x)=\int_Z F(z,x(z))\,dz$.

We introduce the following hypotheses on the function f:

$\boldsymbol{H}(f)_1$: $f : Z\times\mathbb{R}\to\mathbb{R}$ is a Borel measurable function such that for all $z\in Z$, $f(z,\cdot)$ is locally bounded and

(i) f_0, f_1 are both N-measurable (i.e. for every measurable function $x : Z\to\mathbb{R}$ the functions $z\mapsto f_i(z,x(z))$, $i=0,1$, are measurable);

(ii) there are $a\in L^\infty(Z,\mathbb{R}^+)$ and $c_1>0$ such that

$$|f(z,x)|\le a(z)+c_1|x|^{p-1}\qquad \text{a.e. on } Z,\ \forall x\in\mathbb{R};$$

(iii) there are $c_2>0$ and μ such that $1\le\mu\le p$ and

$$\limsup_{|x|\to\infty}\frac{f_i(z,x)x-pF(z,x)}{|x|^\mu}\le -c_2\qquad \text{uniformly for a.e. } z\in Z,\ i=0,1;$$

(iv) there are $c_3\in\mathbb{R}$ and $\overline{q}>p$ such that $\overline{q}-p<\mu$ if $1\le N\le p$, while $N(\overline{q}-p)/p<\mu$ and $\overline{q}<Np/(N-p)$ if $N>p$, with the property that

$$\limsup_{|x|\to\infty}\frac{F(z,x)}{|x|^{\overline{q}}}\le c_3\qquad \text{uniformly for a.e. } z\in Z;$$

$$\text{(v)}\qquad \liminf_{|x|\to\infty}\frac{pF(z,x)}{|x|^p}\ge 0\qquad \text{uniformly for a.e. } z\in Z;$$

$$\text{(vi)}\qquad \limsup_{|x|\to 0}\frac{pF(z,x)}{|x|^p}\le -\lambda_2\qquad \text{for a.e. } z\in Z.$$

REMARK 2. Condition $\boldsymbol{H}(f)_1$ (iii) was first introduced by D. G. Costa and C. A. Magalhães [7]. Note that this condition follows from the well known Ambrosetti–Rabinowitz condition (see [1], when $p=2$) which says: there are $\theta>2$ and $\xi>0$ such that $0<\theta F(z,x)<xf(z,x)$ for a.e. $z\in Z$ and all x such that $|x|\ge\xi$.

We now present an example of a function f that satisfies hypotheses $\boldsymbol{H}(f)_1$. For simplicity we drop the z-dependence and we assume $p>2$. Fix $M>1$ and $\eta,\xi>0$ such that $p\eta+\lambda_2-1\le\xi$. Let $k\in L^1_{\mathrm{loc}}(\mathbb{R})$ be the function defined by

$$k(x)=\begin{cases} -\xi|x|^{p-2}x, & |x|\le 1,\\ x^4, & 1<|x|\le M,\\ x, & |x|>M.\end{cases}$$

We define $f : Z\times\mathbb{R}\to\mathbb{R}$ by

$$f(z,x)=p\eta|x|^{p-2}x-\frac{|x|^{p-2}x}{|x|^p+1}+k(x).$$

Then the corresponding potential function is

$$F(z,x) = \eta|x|^p - \frac{1}{p}\log(|x|^p+1) + \theta(x),$$

where

$$\theta(x) = \int\limits_0^x k(r)\,dr = \begin{cases} x^2/2 - \xi/p + M^5/5 - M^2/2 - 1/5, & x < -M, \\ -\xi/p + x^5/5 + 1/5, & -M < x < -1, \\ -(\xi/p)|x|^p, & |x| \le 1, \\ -\xi/p + x^5/5 - 1/5, & 1 < x < M, \\ x^2/2 - \xi/p + M^5/5 - M^2/2 + 1/5, & x > M. \end{cases}$$

Evidently hypothesis $\boldsymbol{H}(f)_1$(ii) is satisfied. Also for $i = 1, 2$ we have

$$\frac{f_i(z,x)x - pF(z,x)}{|x|^2} = -\frac{|x|^p}{|x|^{p+2}+x^2} + 1 - \frac{\log(|x|^p+1)}{x^2} - \frac{p\theta(x)}{x^2} \to 1 - \frac{p}{2} < 0$$

as $|x| \to \infty$. So $\boldsymbol{H}(f)_1$(iii) is satisfied with $\mu = 2 < p$. Moreover for any $\overline{q} > p$ we have $F(z,x)/|x|^{\overline{q}} \to 0$ as $|x| \to \infty$ and so $\boldsymbol{H}(f)_1$(iv) holds. Now we deduce that

$$\frac{pF(z,x)}{|x|^p} = p\eta - \frac{\log(|x|^p+1)}{|x|^p} + \frac{p\theta(x)}{|x|^p} \to p\eta \quad \text{ as } |x| \to \infty,$$

while

$$\frac{pF(z,x)}{|x|^p} = p\eta - \frac{\log(|x|^p+1)}{|x|^p} + \frac{p\theta(x)}{|x|^p} \to p\eta - 1 - \xi \quad \text{ as } |x| \to 0.$$

So recalling the choice of η and ξ we see that $\boldsymbol{H}(f)_1$(v) and $\boldsymbol{H}(f)_1$(vi) are satisfied.

Now we start to study the problem (2). We introduce, for every $\lambda > 0$, the functional $R_\lambda : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$R_\lambda(x) = \frac{1}{p}\|Dx\|_p^p - \frac{\lambda}{p}\|x\|_p^p - \int\limits_Z F(z,x(z))\,dz = \frac{1}{p}\|Dx\|_p^p - \frac{\lambda}{p}\|x\|_p^p - \Psi(x).$$

It is well known (cf. [5]) that Ψ is locally Lipschitz. Hence so is R_λ.

Proposition 3. *If hypotheses $\boldsymbol{H}(f)_1$ hold, then R_λ satisfies the C-(PS)-condition.*

Proof. Let $\{x_n\}_{n\ge1} \subset W_0^{1,p}(Z)$ be a sequence such that $\{R_\lambda(x_n)\}_{n\ge1}$ is bounded and $(1+\|x_n\|)m(x_n) \to 0$ as $n \to \infty$. Let $x_n^* \in W^{-1,q}(Z)$ $(1/p+1/q=1)$, $x_n^* \in \partial R_\lambda(x_n)$, $n \ge 1$, be such that $\|x_n^*\| = m(x_n)$. We have

$$x_n^* = A(x_n) - \lambda J(x_n) - u_n^*,$$

where $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is defined by

$$\langle A(x), y\rangle = \int\limits_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))\,dz, \quad \forall x, y \in W_0^{1,p}(Z)$$

(here $\langle\cdot,\cdot\rangle$ denotes the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$), and $J : W_0^{1,p}(Z) \to L^q(Z) \subset W^{-1,q}(Z)$ is given by

$$J(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot)$$

and $u_n^* \in \partial\Psi(x_n)$. From [5] we know that $f_0(z, x_n(z)) \le u_n^* \le f_1(z, x_n(z))$ a.e. on Z. Moreover let $M > 0$ be such that $|R_\lambda(x_n)| \le M$ for all $n \ge 1$; we have

$$-\|Dx_n\|_p^p + \lambda\|x_n\|_p^p + \int_Z pF(z, x_n(z))\,dz \le pM, \quad \forall n \ge 1. \tag{5}$$

Also since $(1 + \|x_n\|)\|x_n^*\| \to 0$ as $n \to \infty$, there exists $\overline{n} \in \mathbb{N}$ such that

$$\langle A(x_n), x_n\rangle - \lambda(J(x_n), x_n)_{p,q} - \int_Z u_n^*(z)x_n(z)\,dz < 1, \quad \forall n \ge \overline{n}$$

(here $(\cdot,\cdot)_{p,q}$ denotes the duality brackets for the pair $(L^p(Z), L^q(Z))$). So we have

$$\|Dx_n\|_p^p - \lambda\|x_n\|_p^p - \int_Z u_n^*(z)x_n(z)\,dz < 1, \quad \forall n \ge \overline{n}. \tag{6}$$

Adding (5) and (6) above we obtain

$$\int_Z (pF(z, x_n(z)) - u_n^*(z)x_n(z))\,dz \le pM + 1, \quad \forall n \ge \overline{n}. \tag{7}$$

From $\boldsymbol{H}(f)_1$(iii) we find that there exist $\widehat{c}_2 > 0$ and $M_1 > 1$ such that for almost all $z \in Z$, all $|x| \ge M_1$ and all $u^* \in [f_0(z,x), f_1(z,x)]$, we have

$$\frac{u^*x - pF(z,x)}{|x|^\mu} \le -\widehat{c}_2. \tag{8}$$

Also using the growth hypothesis $\boldsymbol{H}(f)_1$(ii), we see that there exists $\beta_1 > 0$ such that

$$u^*x - pF(z,x) \le \beta_1 \quad \text{for almost all } z \in Z \text{ and all } |x| < M_1,$$

hence, from (8), setting $\beta_2 = \beta_1 + \widehat{c}_2|M_1|^\mu$, we obtain

$$u^*x - pF(z,x) \le -\widehat{c}_2|x|^\mu + \beta_2 \quad \text{for almost all } z \in Z \text{ and all } x \in \mathbb{R}.$$

Using this in (7), we obtain

$$pM + 1 \ge \int_Z (\widehat{c}_2|x_n(z)|^\mu - \beta_2)\,dz, \quad \forall n \ge \overline{n},$$

which implies that there exists $\beta_3 > 0$ such that

$$\|x_n\|_\mu \le \beta_3, \quad \forall n \ge 1, \tag{9}$$

that is, the sequence $\{x_n\}_{n\ge1}$ is bounded in $L^\mu(Z)$.

Now let $0 < \theta < 1$ be such that

$$\frac{1}{q} = \frac{1-\theta}{\mu} + \frac{\theta}{p^*}$$

where $p^* = Np/(N-p)$ if $p < N$, otherwise any p^* such that $1 \le \mu \le p < \overline{q} < p^*$. Using the interpolation inequality (cf. [2], p. 57) and (9) we have

$$\|x_n\|_{\overline{q}} \le \|x_n\|_{\mu}^{1-\theta} \|x_n\|_{p^*}^{\theta} \le \beta_3^{1-\theta} \|x_n\|_{p^*}^{\theta}, \quad \forall n \ge 1,$$

and so, from the Sobolev embedding theorem, there exists $\beta_4 > 0$ with the property

$$\|x_n\|_{\overline{q}} \le \beta_4 \|x_n\|_{1,p}^{\theta}, \quad \forall n \ge 1 \tag{10}$$

($\|\cdot\|_{1,p}$ denotes the norm in $W_0^{1,p}(Z)$). By $\boldsymbol{H}(f)_1$(ii), (iv), for fixed $\widehat{c}_3 > c_3$, there exists $c_4 > 0$ such that

$$F(z,x) \le \widehat{c}_3 |x|^{\overline{q}} + c_4 \quad \text{for almost all } z \in Z \text{ and all } x \in \mathbb{R}. \tag{11}$$

From the choice of the sequence $\{x_n\}_{n\ge1}$, we find that $R_\lambda(x_n) \le M$ for all $n \ge 1$, and so, using (11) we obtain

$$\frac{1}{p}\|Dx_n\|_p^p \le \frac{\lambda}{p}\|x_n\|_p^p + \widehat{c}_3 \|x_n\|_{\overline{q}}^{\overline{q}} + \beta_5, \quad \forall n \ge 1,$$

where $\beta_5 = c_4|Z| + M$; hence, as $p < \overline{q}$ it follows that there exists $\beta_6 > 0$ such that

$$\frac{1}{p}\|Dx_n\|_p^p \le \frac{\lambda}{p}\beta_6 \|x_n\|_{\overline{q}}^p + \widehat{c}_3 \|x_n\|_{\overline{q}}^{\overline{q}} + \beta_5, \quad \forall n \ge 1.$$

Now Young's inequality and (10) show that there exist two positive numbers depending on λ: $\beta_7(\lambda)$ and $\beta_8(\lambda)$, with the property

$$\frac{1}{p}\|Dx_n\|_p^p \le \beta_7(\lambda)\|x_n\|_{1,p}^{\theta\overline{q}} + \beta_8(\lambda), \quad \forall n \ge 1.$$

From our assumptions we can assume that $\theta\overline{q} < p$, in fact if $p < N$, then $(N/p)(\overline{q} - p) < \mu$ is equivalent to $\theta\overline{q} < p$, while if $p \ge N$, then we can always choose p^* large and then θ in the interpolation inequality small so that $\theta\overline{q} < p$. Therefore, recalling that from Poincaré's inequality (cf. [2], p. 174), $\|Dx\|_p$ is an equivalent norm in $W_0^{1,p}(Z)$, it follows that $\{x_n\}_{n\ge1}$ is bounded in $W_0^{1,p}(Z)$. Thus, by passing to a subsequence if necessary, we may assume that

$$x_n \to x \quad \text{weakly in } W_0^{1,p}(Z), \qquad x_n \to x \quad \text{in } L^p(Z),$$

and so $x_n(z) \to x(z)$ a.e. on Z and there exists $\eta \in L^p(Z)$ such that $|x_n(z)| \le \eta(z)$ a.e. on Z for all $n \ge 1$. Hence we have

$$\begin{aligned} \lambda(J(x_n), x_n - x)_{p,q} &\to 0 \quad \text{as } n \to \infty, \\ (u_n^*, x_n - x)_{p,q} &\to 0 \quad \text{as } n \to \infty, \\ \langle x_n^*, x_n - x\rangle &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Since $\langle x_n^*, x_n - x\rangle = \langle A(x_n), x_n - x\rangle - \lambda(J(x_n), x_n - x)_{p,q} - (u_n^*, x_n - x)_{p,q}$ for all $n \geq 1$, we infer that

$$\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle = \lim_{n\to\infty}\langle A(x_n), x_n - x\rangle = 0. \tag{12}$$

It is simple to see that $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is monotone and demicontinuous and so (cf. [16], p. 596) it is pseudomonotone; therefore from (12) we obtain

$$\lim_{n\to\infty}\langle A(x_n), x_n\rangle = \langle A(x), x\rangle,$$

which implies that

$$\|Dx_n\|_p \to \|Dx\|_p \quad \text{as } n \to \infty.$$

But we also have $Dx_n \to Dx$ weakly in $L^p(Z)$, thus, since $L^p(Z)$ is a uniformly convex space, we deduce that $x_n \to x$ in $W_0^{1,p}(Z)$ as $n \to \infty$. So R_λ satisfies the C-(PS)-condition. ∎

Using the previous proposition and the above-mentioned mountain pass theorem, we can prove the existence of nontrivial solutions for the problem (2).

THEOREM 4. *If hypotheses $\boldsymbol{H}(f)_1$ hold, then for every $\lambda \in (\lambda_1, \lambda_2)$ the problem* (2) *has a nontrivial solution.*

Proof. From $\boldsymbol{H}(f)_1$(v), given $\varepsilon > 0$ we can find $M_2 > 0$ such that

$$F(x,z) \geq -\frac{\varepsilon}{p}|x|^p \quad \text{for a.e. } z \in Z \text{ and all } x \text{ such that } |x| \geq M_2;$$

also from $\boldsymbol{H}(f)_1$(ii), there exists $\gamma > 0$ with

$$F(x,z) \geq -\gamma \quad \text{for a.e. } z \in Z \text{ and all } x \text{ such that } 0 \leq |x| < M_2.$$

Thus we deduce that

$$F(z,x) \geq -\frac{\varepsilon}{p}|x|^p - \gamma \quad \text{for a.e. } z \in Z \text{ and all } x \in \mathbb{R}.$$

Hence, for all $u_1 \in W_0^{1,p}(Z), u_1 \neq 0$, and all $t \in \mathbb{R}$, it follows that

$$\begin{aligned} R_\lambda(tu_1) &= \frac{|t|^p}{p}\|Du_1\|_p^p - \lambda\frac{|t|^p}{p}\|u_1\|_p^p - \int_Z F(z, tu_1(z))\,dz \\ &\leq \frac{|t|^p}{p}\left(1 - \frac{\lambda}{\lambda_1}\right)\|Du_1\|_p^p + \frac{\varepsilon|t|^p}{\lambda_1 p}\|Du_1\|_p^p + \gamma_1 \\ &= \frac{|t|^p}{p}\left(1 - \frac{\lambda-\varepsilon}{\lambda_1}\right)\|Du_1\|_p^p + \gamma_1, \end{aligned}$$

where $\gamma_1 = \gamma|Z|$. Take $\varepsilon > 0$ so that $\lambda_1 < \lambda - \varepsilon$. Then

$$R_\lambda(tu_1) \to -\infty \quad \text{as } |t| \to \infty. \tag{13}$$

Next using $\boldsymbol{H}(f)_1$(vi), given $\varepsilon > 0$ we can find $0 < \delta < 1$ such that

$$F(z,x) \le \frac{1}{p}(-\lambda_2 + \varepsilon)|x|^p \quad \text{for a.e. } z \in Z \text{ and all } x \text{ such that } |x| < \delta.$$

From (11) we know that

$$F(z,x) \le \widehat{c}_3 |x|^{\overline{q}} + c_4 \quad \text{for a.e. } z \in Z \text{ and all } x \in \mathbb{R},$$

so we have

$$F(z,x) \le \left(\widehat{c}_3 + \frac{c_4}{\delta^{\overline{q}}}\right)|x|^{\overline{q}} \quad \text{for a.e. } z \in Z \text{ and all } x \text{ such that } \delta \le |x| \le 1,$$

while

$$F(z,x) \le (\widehat{c}_3 + c_4)|x|^{\overline{q}} \quad \text{for a.e. } z \in Z \text{ and all } x \text{ such that } |x| \ge 1.$$

Thus finally we can find $\eta > 0$ such that

$$F(z,x) \le \frac{1}{p}(-\lambda_2 + \varepsilon)|x|^p + \eta |x|^{\overline{q}} \quad \text{for a.e. } z \in Z \text{ and all } x \in \mathbb{R}.$$

Therefore, taking $\varepsilon > 0$ such that $\lambda < \lambda_2 - \varepsilon$, for all $x \in W_0^{1,p}(Z)$ we obtain

$$\begin{aligned} R_\lambda(x) &= \frac{1}{p}\|Dx\|_p^p - \frac{\lambda}{p}\|x\|_p^p - \int_Z F(z,x(z))\,dz \\ &\ge \frac{1}{p}\|Dx\|_p^p - \frac{\lambda}{p}\|x\|_p^p + \frac{1}{p}(\lambda_2 - \varepsilon)\|x\|_p^p - \eta\|x\|_{\overline{q}}^{\overline{q}} \\ &\ge \frac{1}{p}\|Dx\|_p^p - \eta_1\|Dx\|_p^{\overline{q}} \\ &= \left(\frac{1}{p} - \eta_1\|Dx\|_p^{\overline{q}-p}\right)\|Dx\|_p^p, \end{aligned}$$

where η_1 is a positive constant found by using the Rayleigh quotient and the Sobolev embedding theorem because $p < \overline{q} < p^*$. Since $\|Dx\|_p$ is an equivalent norm in $W_0^{1,p}(Z)$, from this last inequality we deduce that there are $\varrho > 0$ and $\alpha, \beta \in \mathbb{R}$ such that

$$0 \le \alpha < \beta \le \inf_{\|x\|=\varrho} R_\lambda(x).$$

Now we observe that $R_\lambda(0) = 0$ and from (13) it is possible to find $x_1 \in W_0^{1,p}(Z)$ with $\|x_1\| > \varrho$ and $R_\lambda(x_1) \le 0$, therefore

$$\max[R_\lambda(0), R_\lambda(x_1)] \le \alpha < \beta \le \inf_{\|x\|=\varrho} R_\lambda(x).$$

So, from Proposition 3, we can apply Theorem 1 to find $\overline{x} \in W_0^{1,p}(Z)$ such that $0 \in \partial R_\lambda(\overline{x})$ and $R_\lambda(\overline{x}) \ge \beta > 0$. Hence $\overline{x} \ne 0$.

Also, since $0 \in \partial R_\lambda(\overline{x})$, there exists $u^* \in \partial\Psi(\overline{x})$ such that $0 = A(\overline{x}) - \lambda J(\overline{x}) - u^*$, hence

$$\langle A(\overline{x}), \varphi\rangle = \lambda(J(\overline{x}), \varphi)_{p,q} + (u^*, \varphi)_{p,q} \quad \text{for all } \varphi \in C_0^\infty(Z).$$

Thus

$$\int_Z \|D\overline{x}(z)\|^{p-2}(D\overline{x}(z), D\varphi(z))_N\,dz = \int_Z (\lambda|\overline{x}(z)|^{p-2}\overline{x}(z)\varphi(z) + u^*(z)\varphi(z))\,dz \quad \text{for all } \varphi \in C_0^\infty(Z).$$

From the definition of the distributional derivative, we infer that

$$-\operatorname{div}(\|D\overline{x}(z)\|^{p-2}D\overline{x}(z)) = \lambda|\overline{x}(z)|^{p-2}\overline{x}(z) + u^*(z) \quad \text{a.e. on } Z.$$

Since $u^*(z) \in \partial\Psi(\overline{x})(z) \subset [f_0(z, \overline{x}(z)), f_1(z, \overline{x}(z))]$ a.e. on Z (cf. [5]), we deduce that $\overline{x} \in W_0^{1,p}(Z)$ is a nontrivial solution of problem (2). ∎

4. Semilinear problem. Now we pass to the semilinear eigenvalue problem with discontinuous right hand side

$$\begin{cases} -\Delta x(z) - \lambda x(z) = f(z, x(z)) & \text{a.e. on } Z, \\ x|_\Gamma = 0. \end{cases} \tag{3}$$

Again the discontinuity of $f(z,\cdot)$ in (3) forces us to pass to the multivalued variant of (3). Namely we pass to the elliptic eigenvalue inclusion

$$\begin{cases} -\Delta x(z) - \lambda x(z) \in [f_0(z, x(z)), f_1(z, x(z))] & \text{a.e. on } Z, \\ x|_\Gamma = 0. \end{cases} \tag{4}$$

Also in this case we denote by $F(z,x)$ the function $F(z,x) = \int_0^x f(z,r)\,dr$, $x \in \mathbb{R}$, which is the potential function corresponding to f.

In order to prove the existence of multiple nontrivial solutions we make the following hypotheses on f:

$\boldsymbol{H}(f)_2$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that for all $z \in Z$, $f(z,\cdot)$ is locally bounded and

(i) f_0, f_1 are both N-measurable;

(ii) there are $a_1 \in L^\infty(Z, \mathbb{R}^+)$ and $c > 0$ such that

$$|f(z,x)| \le a_1(z) + c|x|^s \quad \text{a.e. on } Z,\ \forall x \in \mathbb{R},$$

where $0 \le s \le (N+2)/(N-2)$ and $N > 2$;

(iii) there are $\theta > 2$ and $\xi > 0$ such that

$$0 < \theta F(z,x) < xf(z,x) \quad \text{for a.e. } z \in Z \text{ and all } x \text{ such that } |x| \ge \xi;$$

(iv) $$\liminf_{|x|\to\infty} \frac{2F(z,x)}{x^2} \ge 0 \quad \text{uniformly for a.e. } z \in Z.$$

REMARK 3. Condition $\boldsymbol{H}(f)_2$(iii) forces the s in $\boldsymbol{H}(f)_2$(ii) to satisfy $1 < s \leq (N+2)/(N-2)$ (see [14], p. 9).

Again we introduce, for every $\lambda > 0$, the locally Lipschitz functional $R_\lambda : H_0^1(Z) \to \mathbb{R}$ defined by

$$R_\lambda(x) = \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda}{2}\|x\|_2^2 - \Psi(x),$$

where $\Psi : H_0^1(Z) \to \mathbb{R}$ is defined by $\Psi(x) = \int_Z F(z, x(z))\,dz$.

PROPOSITION 5. *If hypotheses $\boldsymbol{H}(f)_2$ hold, then R_λ satisfies the (PS)-condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subset H_0^1(Z)$ be a sequence with the property that there exists $M_3 > 0$ such that $|R_\lambda(x_n)| \leq M_3$ for all $n \geq 1$ and $m(x_n) \to 0$ as $n \to \infty$. As before let $x_n^* \in H^{-1}(Z)$, $x_n^* \in \partial R_\lambda(x_n)$, $n \geq 1$, be such that $\|x_n^*\| = m(x_n)$. We know that

$$x_n^* = A(x_n) - \lambda x_n - u_n^*, \quad n \geq 1,$$

where $A : H_0^1(Z) \to H^{-1}(Z)$ is defined by $\langle A(x), y\rangle = \int_Z (Dx(z), Dy(z))\,dz$ for $x, y \in H_0^1(Z)$ (here $\langle \cdot, \cdot \rangle$ denotes the duality brackets for the pair $(H_0^1(Z), H^{-1}(Z))$) and $u_n^* \in \partial\Psi(x_n)$. From [5] we know that $f_0(z, x_n(z)) \leq u_n^*(z) \leq f_1(z, x_n(z))$ a.e. on Z. It is simple to see that A is linear and monotone and so (cf. [16], p. 596) it is pseudomonotone.

Let $\eta \in (1/\theta, 1/2)$ and observe that there exists $\overline{n} \geq 1$ such that

$$-\eta\langle x_n^*, x_n\rangle \leq \eta\|x_n\|_{1,2}, \qquad \forall n \geq \overline{n},$$

which implies that

$$-\eta\langle A(x_n), x_n\rangle + \eta\lambda\|x_n\|_2^2 + \eta(u_n^*, x_n)_2 \leq \eta\|x_n\|_{1,2}, \qquad \forall n \geq \overline{n},$$

where $(\cdot, \cdot)_2$ denotes the inner product in the Hilbert space $L^2(Z)$. So from the definition of R_λ and since $2 < \theta$, we deduce that there exists $\widehat{\beta}_1$ such that

$$\begin{aligned}(14)\quad M_3 + \eta\|x_n\|_{1,2} \geq (1/2 - \eta)\|Dx_n\|_2^2 &- \lambda(1/2 - \eta)\widehat{\beta}_1\|x_n\|_\theta^2 \\ &- \int_Z (F(z, x_n(z)) - \eta u_n^*(z)x_n(z))\,dz, \qquad \forall n \geq \overline{n}.\end{aligned}$$

Using $\boldsymbol{H}(f)_2$(iii), we have

$$0 < \theta F(z, x) \leq x f_i(z, x) \quad \text{for a.e. } z \in Z \text{ and all } x \text{ with } |x| > \xi,\ i = 0, 1,$$

and so, for all $n \geq 1$, we obtain

$$(15)\quad 0 < \theta F(z, x_n(z)) \leq u_n^*(z)x_n(z) \quad \text{for a.e. } z \in Z \text{ with } |x_n(z)| > \xi.$$

Now, passing to integrals, we have

$$\int_Z (\eta u_n^*(z)x_n(z) - F(z,x_n(z)))\,dz = \int_A (\eta u_n^*(z)x_n(z) - F(z,x_n(z)))\,dz + \int_B \eta u_n^*(z)x_n(z)\,dz - \int_B F(z,x_n(z))\,dz, \qquad \forall n \ge 1,$$

where $A = \{z \in Z : |x_n(z)| > \xi\}$ and $B = \{z \in Z : |x_n(z)| \le \xi\}$. Note that, from $\boldsymbol{H}(f)_2$(ii), $|\eta u_n^*(z)x_n(z)| \le \eta(a_1(z) + c|x_n(z)|^s)|x_n(z)| \le \widehat{\beta}_2$ a.e. on B, where $\widehat{\beta}_2 = \eta(\|a_1\|_\infty + c\xi^s)\xi$. Hence, using (15) we obtain

$$\int_Z (\eta u_n^*(z)x_n(z) - F(z,x_n(z)))\,dz \ge \int_A (\eta\theta - 1)F(z,x_n(z))\,dz - \int_B F(z,x_n(z))\,dz - \widehat{\beta}_3, \qquad \forall n \ge 1, \tag{16}$$

where $\widehat{\beta}_3 = \widehat{\beta}_2|Z|$.

Hypothesis $\boldsymbol{H}(f)_2$(iii) implies (cf. [14], p. 9) that there exist $\widetilde{c}_3, \widetilde{c}_4 > 0$ such that $F(z,x) \ge \widetilde{c}_3|x|^\theta - \widetilde{c}_4$ a.e. on Z and for all $x \in \mathbb{R}$. So we have

$$\begin{aligned}\int_A (\eta\theta - 1)F(z,x_n(z))\,dz &\ge (\eta\theta - 1)\int_Z (\widetilde{c}_3|x_n(z)|^\theta - \widetilde{c}_4)\,dz - (\eta\theta - 1)\int_B (\widetilde{c}_3|x_n(z)|^\theta - \widetilde{c}_4)\,dz \\ &\ge (\eta\theta - 1)(\widetilde{c}_3\|x_n\|_\theta^\theta - \widetilde{c}_4|Z|) - (\eta\theta - 1)\widetilde{c}_3\xi^\theta|Z| \\ &\quad + (\eta\theta - 1)\widetilde{c}_4|B|, \qquad \forall n \ge 1.\end{aligned} \tag{17}$$

Also, by $\boldsymbol{H}(f)_2$(ii) there exists $\widehat{\beta}_4 > 0$ such that

$$\int_B F(z,x_n(z))\,dz \le \widehat{\beta}_4, \qquad \forall n \ge 1. \tag{18}$$

Using (17) and (18) in (16) we find that there exists $\widehat{\beta}_5 > 0$ such that

$$\int_Z (\eta u_n^*(z)x_n(z) - F(z,x_n(z)))\,dz \ge (\eta\theta - 1)\widetilde{c}_3\|x_n\|_\theta^\theta - \widehat{\beta}_5, \qquad \forall n \ge 1. \tag{19}$$

Now, using (19) in (14), we have

$$\begin{aligned}M_3 + \eta\|x_n\|_{1,2} \ge (1/2 - \eta)\|Dx_n\|_2^2 &- \lambda(1/2 - \eta)\widehat{\beta}_1\|x_n\|_\theta^2 \\ &+ (\eta\theta - 1)\widetilde{c}_3\|x_n\|_\theta^\theta - \widehat{\beta}_5, \qquad \forall n \ge \overline{n}.\end{aligned}$$

Since by hypothesis $2 < \theta$, from Young's inequality, for fixed $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|x_n\|_\theta^2 \le \delta(\varepsilon) + \varepsilon\|x_n\|_\theta^\theta$ for all $n \ge 1$; so we deduce that

$$M_3 + \eta\|x_n\|_{1,2} \geq (1/2-\eta)\|Dx_n\|_2^2 - \lambda(1/2-\eta)\widehat{\beta}_1\varepsilon\|x_n\|_\theta^\theta + (\eta\theta-1)\widetilde{c}_3\|x_n\|_\theta^\theta - \widehat{\beta}_6(\varepsilon), \quad \forall n \geq \overline{n},$$

where $\widehat{\beta}_6(\varepsilon) = \widehat{\beta}_5 + \lambda(1/2-\eta)\widehat{\beta}_1\delta(\varepsilon)$. Choose $\varepsilon > 0$ so that $(\eta\theta-1)\widetilde{c}_3 - \lambda(1/2-\eta)\widehat{\beta}_1\varepsilon > 0$. Then

$$M_3 + \eta\|x_n\|_{1,2} \geq (1/2-\eta)\|Dx_n\|_2^2 - \widehat{\beta}_6(\varepsilon), \quad \forall n \geq \overline{n}.$$

Since $\eta < 1/2$ and using Poincaré's inequality, we infer that $\{x_n\}_{n\geq 1}$ is bounded in $H_0^1(Z)$; thus we may assume that $x_n \to x$ weakly in $H_0^1(Z)$ and $x_n \to x$ in $L^2(Z)$ as $n \to \infty$. Since by hypothesis $s < (N+2)/(N-2)$, we have $s+1 < p^* = 2N/(N-2)$ and so $H_0^1(Z)$ is compactly embedded in $L^{s+1}(Z)$. Hence $x_n \to x$ in $L^{s+1}(Z)$ as $n \to \infty$.

Moreover from Theorem 2.1 of [5] we find that $u_n^* \in \partial\Psi(x_n) \subset L^\nu(Z)$ for all $n \geq 1$ where $1/\nu + 1/(s+1) = 1$, and since $f_0(z, x_n(z)) \leq u_n^*(z) \leq f_1(z, x_n(z))$ a.e. on Z and for all $n \geq 1$, from $\boldsymbol{H}(f)_2$(ii) we deduce that $\{u_n^*\}_{n\geq 1}$ is bounded in $L^\nu(Z)$.

Now, from our assumptions, we know that

$$x_n^* = A(x_n) - \lambda x_n - u_n^*, \quad n \geq 1,$$

and so

$$\langle A(x_n), x_n - x\rangle = \langle x_n^*, x_n - x\rangle + \lambda(x_n, x_n - x)_2 + (u_n^*, x_n - x)_{\nu,s+1}, \quad n \geq 1.$$

Therefore

$$\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle = 0,$$

and so from the pseudomonotonicity of $A : H_0^1(Z) \to H^{-1}(Z)$ we deduce

$$\|Dx_n\|_2 \to \|Dx\|_2 \quad \text{as } n \to \infty.$$

But we also have $Dx_n \to Dx$ weakly in $L^2(Z)$, thus, since $L^2(Z)$ is a uniformly convex space, we deduce that $x_n \to x$ in $H_0^1(Z)$ as $n \to \infty$. So R_λ satisfies the (PS)-condition. ∎

In what follows we denote by $(\lambda_n)_{n\geq 1}$ the increasing sequence of positive numbers which are distinct eigenvalues of $-\Delta$ with the Dirichlet boundary condition (cf. [2], p. 192). Using the previous proposition we can prove our multiplicity result:

Theorem 6. *If hypotheses $\boldsymbol{H}(f)_2$ hold, if for a.e. $z \in Z$, $f(z,\cdot)$ is odd and if there exists $k \geq 1$ such that*

$$\limsup_{x\to 0}\frac{2F(z,x)}{x^2} \leq -\lambda_{k+1} \quad \textit{uniformly for a.e. } z \in Z,$$

then for every $\lambda \in (\lambda_{k+1}, \lambda_k)$ the problem (4) *has k pairs of nontrivial solutions.*

P r o o f. From $\boldsymbol{H}(f)_2$(iv), given $\varepsilon > 0$, we can find $M_4 > 0$ such that

$$F(z,x) \geq -\frac{\varepsilon}{2}x^2 \quad \text{for a.e. } z \in Z \text{ and all } x \text{ such that } |x| > M_4.$$

Also from the growth hypothesis $\boldsymbol{H}(f)_2$(ii) we know that there exists $\widehat{\beta}_7$ such that

$$|F(z,x)| \leq \widehat{\beta}_7 \quad \text{for a.e. } z \in Z \text{ and all } x \text{ such that } |x| \leq M_4.$$

Hence we infer that

$$F(z,x) \geq -\frac{\varepsilon}{2}x^2 - \widehat{\beta}_7 \quad \text{for a.e. } z \in Z \text{ and all } x \in \mathbb{R}. \tag{20}$$

Let $V = \operatorname{span}\{u_m\}_{m=1}^k$, where $u_1, \ldots, u_k$ are the eigenfunctions corresponding to the first k eigenvalues of the Laplacian. Recall that for all $x \in V$ we have $\|Dx\|_2^2 \leq \lambda_k \|x\|_2^2$, so, using (20), we obtain

$$\begin{aligned} R_\lambda(x) &= \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda}{2}\|x\|_2^2 - \int_Z F(z,x(z))\,dz \\ &\leq \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda}{2}\|x\|_2^2 + \frac{\varepsilon}{2}\|x\|_2^2 + \widehat{\beta}_8 \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda-\varepsilon}{\lambda_k}\right)\|Dx\|_2^2 + \widehat{\beta}_8, \end{aligned}$$

where $\widehat{\beta}_8 = \widehat{\beta}_7|Z|$. Take $\varepsilon > 0$ so that $\lambda_k < \lambda - \varepsilon$. Then we see that

$$R_\lambda(x) \to -\infty \quad \text{as } \|x\|_{1,2} \to \infty. \tag{21}$$

Next using our assumption, given $\varepsilon > 0$ we can find $0 < \delta < 1$ such that

$$F(z,x) \leq (-\lambda_{k+1} + \varepsilon)\frac{x^2}{2} \quad \text{for a.e. } z \in Z \text{ and all } x \text{ such that } |x| < \delta.$$

Also from $\boldsymbol{H}(f)_2$(ii) we have

$$F(z,x) \leq \gamma_\delta |x|^s \quad \text{for a.e. } z \in Z \text{ and all } x \text{ such that } \delta \leq |x| \leq 1,$$

where $\gamma_\delta = (\|a_1\|_\infty/\delta^s + c)$. Moreover, there exists $\widetilde{\gamma} > 0$ such that for almost all $z \in Z$ and all x such that $|x| \geq 1$ we have

$$F(z,x) \leq \widetilde{\gamma}|x|^s \leq \widetilde{\gamma}|x|^r,$$

with $2 < r \leq 2N/(N-2)$ and $r \geq s+1$. Therefore we can find $\widehat{\gamma} > 0$ large enough such that

$$F(z,x) \leq (-\lambda_{k+1} + \varepsilon)\,\frac{x^2}{2} + \widehat{\gamma}|x|^r \quad \text{for a.e. } z \in Z \text{ and all } x \in \mathbb{R}.$$

Hence, for all $x \in H_0^1(Z)$, taking $\varepsilon > 0$ such that $\lambda + \varepsilon < \lambda_{k+1}$, it follows that

$$\begin{aligned} R_\lambda(x) &= \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda}{2}\|x\|_2^2 - \int_Z F(z,x(z))\,dz \\ &\geq \frac{1}{2}\|Dx\|_2^2 - \frac{\lambda}{2}\|x\|_2^2 + \frac{\lambda_{k+1}-\varepsilon}{2}\|x\|_2^2 - \widehat{\gamma}\|x\|_r^r \\ &\geq \frac{1}{2}\|Dx\|_2^2 - \gamma^*\|Dx\|_2^r \\ &= \left(\frac{1}{2} - \gamma^*\|Dx\|_2^{r-2}\right)\|Dx\|_2^2, \end{aligned}$$

where γ^* is a positive constant found by using Poincaré's inequality.

Since $\|Dx\|_2$ is an equivalent norm in $H_0^1(Z)$, from this last inequality we deduce that there are $\varrho, \beta > 0$ such that

$$R_\lambda(x) \geq \beta \quad \text{for all } x \in H_0^1(Z) \text{ such that } \|x\|_{1,2} = \varrho.$$

Now, since R_λ is even and $R_\lambda(0) = 0$, by Proposition 5 and (21), we can apply Theorem 2 with $X_1 = H_0^1(Z)$ and $X_2 = V$ to deduce the existence of k pairs of nontrivial critical points for R_λ. As in the proof of Theorem 4 we can verify that these pairs of nontrivial critical points are nontrivial solutions of problem (4). ∎

We conclude by presenting an example of a function f that satisfies hypotheses $\boldsymbol{H}(f)_2$ and the conditions of our Theorem 6. Again we drop the z-dependence and we suppose for simplicity that $N = 3$. Let $f : Z \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(z,x) = \begin{cases} -\sqrt[3]{x}, & |x| \leq 1, \\ 6x^5, & |x| > 1. \end{cases}$$

Evidently $f(z,\cdot)$ is odd and

$$F(z,x) = \begin{cases} -(3/4)\sqrt[3]{x^4}, & |x| \leq 1, \\ x^6 - 7/4, & |x| > 1. \end{cases}$$

Note that if $s = 5 = (N+2)/(N-2)$, then f satisfies the growth hypothesis $\boldsymbol{H}(f)_2$(ii). Also, assuming $\theta = 6$ and $\xi = 2$, for $|x| > \xi$ we obtain

$$0 < \theta F(z,x) = 6(x^6 - 7/4) \leq 6x^6 = xf(z,x).$$

So $\boldsymbol{H}(f)_2$(iii) is satisfied . Moreover if $|x| > 1$ then

$$\frac{2F(z,x)}{x^2} = 2x^4 - \frac{7}{2x^2} \to +\infty \quad \text{as } |x| \to \infty;$$

while if $|x| < 1$ then

$$\frac{2F(z,x)}{x^2} = -\frac{3}{2}x^{-2/3} \to -\infty \quad \text{as } |x| \to 0.$$

Therefore the function f satisfies hypotheses $\boldsymbol{H}(f)_2$ and the conditions of Theorem 6.

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Reçu par la Rédaction le 18.10.1999
Révisé le 17.4.2000