

Quaternionic-Kähler geometry and almost Kähler \mathcal{A} -manifolds

by WŁODZIMIERZ JELONEK (Kraków)

Abstract. The aim of this paper is to give an easy explicit description of 3-K-contact structures on certain $\mathrm{SO}(3)$ -principal fibre bundles over quaternionic-Kähler manifolds.

0. Introduction. It has been known since 1975 [K] that any quaternionic-Kähler manifold (M, g_0) of positive scalar curvature admits a natural $\mathrm{SO}(3)$ -principal fibre bundle $p : P \rightarrow M$ such that (P, g) is a 3-Sasakian manifold and p is a Riemannian submersion. However for a long time the analogous construction for quaternionic-Kähler manifolds of negative scalar curvature was not given. Recently S. Tanno [T] proved that a natural $\mathrm{SO}(3)$ -principal bundle also in the case of negative scalar curvature admits a structure similar to 3-Sasakian structure, called by him the nS-structure. Alexandrov, Grantcharov and Ivanov [A-G-I] study the almost complex structures on the Salamon twistor bundle over quaternionic-Kähler manifolds. In particular they prove by generalizing results of Davidov and Muskarov [D-M] that in the case of negative scalar curvature the twistor bundle $(Z(M), g)$ admits an almost Kähler non-Kähler structure J and the Ricci tensor of $Z(M)$ is Hermitian with respect to J . They also prove that $(Z(M), g, J)$ satisfies the symmetry condition G_2 of A. Gray.

In this paper we prove the existence of a quasi 3-K-contact structure on the canonical $\mathrm{SO}(3)$ -principal bundle over a quaternionic-Kähler manifold. We use a method different from that of S. Tanno (who gave a more detailed description of this structure). We construct explicitly a homomorphism $F : Q \rightarrow P^-$ of principal fibre bundles where Q is the natural $\mathrm{Sp}(n)\mathrm{Sp}(1)$ -subbundle of the bundle $\mathrm{SO}(M)$ of oriented orthonormal frames of M .

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The class of \mathcal{A} -manifolds (for definition see [G]) is an important class of Riemannian manifolds which has appeared in a natural way during the investigation of spaces with volume preserving local geodesic symmetries (see [B], [D-N]). In our paper [J-1] we have explicitly constructed a family of locally non-homogeneous \mathcal{A} -manifolds parameterized by a real number $c > 0$ such that each of these manifolds has a Ricci tensor with two constant eigenvalues. These manifolds are $SO(3)$ -bundles over self-dual Einstein manifolds. In the present paper we consider a related problem and give examples of \mathcal{A} -manifolds with two constant eigenvalues on $SO(3)$ -bundles over quaternionic-Kähler manifolds (M, g) generalizing our results from [J-1], [J-3]. We also prove that the standard Riemannian structure $(Z(M), g_*)$ on the (generalized Salamon [S]) twistor bundle $Z(M)$ of a quaternionic-Kähler manifold M is a proper \mathcal{A} -structure and that $(Z(M), g_*)$ admits an almost Hodge structure with Hermitian Ricci tensor.

1. Preliminaries. We start with recalling some basic facts concerning the quaternionic-Kähler geometry (see [S], [Sw], [B]). We denote by $\mathfrak{X}(M)$ the Lie algebra of all local vector fields on M . If D is a vector bundle over M then $\Gamma(D)$ denotes the set of all local sections of D . We also write $\mathcal{A}^k(M) = \Gamma(\wedge^k TM^*)$.

Let (M, g) be an oriented Riemannian manifold, $\dim M = 4n$, $n > 1$, and $P = P(M, SO(4n))$ be the principal fibre bundle of oriented orthonormal frames $u : \mathbb{R}^{4n} \rightarrow TM$. We denote by ∇ the Levi-Civita connection of (M, g) . Let $\mathcal{G} \subset \text{End}(TM)$ be a 3-dimensional subbundle locally generated by three almost complex structures $\{I, J, K\}$ compatible with the metric g and satisfying the additional condition $I \circ J = -J \circ I = K$. We shall assume that the subbundle \mathcal{G} is parallel with respect to ∇ , i.e. $\nabla_X A \in \mathcal{G}$ for all $A \in \Gamma(\mathcal{G})$ and $X \in \mathfrak{X}(M)$. A Riemannian manifold with a bundle \mathcal{G} satisfying the above conditions is called a *quaternionic-Kähler manifold*. Every quaternionic-Kähler manifold is an Einstein manifold. If we regard the curvature tensor of (M, g) as a self-adjoint endomorphism of $\wedge^2 T^*M$ then

$$(R) \quad R|_{\mathcal{G}} = \alpha \text{id}|_{\mathcal{G}}$$

where α is a positive multiple of the scalar curvature τ of (M, g) (see [Sw], [G-L]). Note that $\alpha = \tau/(4(n+2))$ (see [K] and [G-L]).

The (sphere) subbundle $Z(M) = \{A \in \mathcal{G} : \|A\|^2 = 2n\}$ of \mathcal{G} is called the *twistor bundle* of the quaternionic-Kähler manifold M . It will be useful to consider $Z(M)$ as a sphere subbundle of $\wedge^2 TM$ (we identify the bundles $\wedge^2 T^*M$ and $\wedge^2 TM$ by means of g). With every $\omega \in Z(M) \subset \wedge^2 TM$ we associate an almost complex structure J_ω on $T_{p(\omega)}M$ by the formula

$$g(X, J_\omega Y) = g(X \wedge Y, \omega).$$

Let $\{1, i, j, k\}$ be the standard basis of the quaternion field \mathbb{H} . Let $V = \mathbb{H}^n$ and $\{I_0, J_0, K_0\}$ be the complex structures on V defined by $I_0x = -xi = (-x_1i, \dots, -x_ni)$, $J_0x = -xj = (-x_1j, \dots, -x_nj)$, $K_0x = -xk = (-x_1k, \dots, -x_nk)$ where $x = (x_1, \dots, x_n) \in \mathbb{H}^n$. On V we have the standard scalar product $\langle x, y \rangle = \sum x_i \bar{y}_i$. The group $\text{Sp}(n)$ is $\text{Sp}(n) = \{A \in \text{End}(\mathbb{H}^n) : \|Ax\| = \|x\| \text{ for } x \in V\}$. The group $\text{Sp}(n) \text{Sp}(1) = \text{Sp}(n) \times_{\mathbb{Z}_2} \text{Sp}(1)$ acts on V by the formula $(A, q)x = Axq^{-1}$.

We shall identify the space $\bigwedge^2 \mathbb{R}^{4n}$ with $\mathfrak{so}(4n)$. The scalar product on $\mathfrak{so}(4n)$ is given by $\langle A, B \rangle = -\frac{1}{2} \text{tr } A \circ B$. Note that $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \subset \mathfrak{so}(4n)$. Let $\pi : \mathfrak{so}(4n) \rightarrow \mathfrak{sp}(1)$ be the orthogonal projection with respect to this product. The components of a vector $X \in \mathfrak{g}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ are denoted by X^+, X^- respectively, i.e. $X = X^+ + X^-$ where $X^+ \in \mathfrak{sp}(n)$ and $X^- \in \mathfrak{sp}(1)$. Note that $X^- = \pi(X)$ if $X \in \mathfrak{g}$. The algebra $\mathfrak{sp}(1)$ is spanned by $\{I_0, J_0, K_0\}$. We have $\|I_0\|^2 = \|J_0\|^2 = \|K_0\|^2 = 2n$.

Let us define the 1-forms $\phi_{I_0}, \phi_{J_0}, \phi_{K_0}$ on $\mathfrak{so}(4n) = \bigwedge^2 \mathbb{R}^{4n}$ as follows: $\phi_I(X) = \langle I, X \rangle$ where $I \in \{I_0, J_0, K_0\}$. Note that for simple bivectors $X \wedge Y$ we have

$$\phi_I(X \wedge Y) = \langle I, X \wedge Y \rangle = g(IX, Y)$$

where g is the standard scalar product on $\mathbb{R}^{4n} = \mathbb{H}^n$. The projection π is given by the formula

$$(1.1) \quad \pi(A) = \frac{1}{2n} (\phi_{I_0}(A)I_0 + \phi_{J_0}(A)J_0 + \phi_{K_0}(A)K_0).$$

We shall use the following

DEFINITION 1. Two vectors $X, Y \in \mathbb{H}^n$ are called \mathbb{H} -dependent if

$$(1.2) \quad Y \in \text{span}_{\mathbb{R}}\{X, I_0X, J_0X, K_0X\}.$$

Note that if $Y \perp \text{span}_{\mathbb{R}}\{X, I_0X, J_0X, K_0X\}$ then

$$(1.3a) \quad \pi(X \wedge Y) = 0$$

and

$$(1.3b) \quad \|\pi(X \wedge Y)\|^2 = \frac{1}{2n} \|X\|^2 \|Y\|^2$$

if X, Y are \mathbb{H} -dependent and $X \perp Y$.

The bundle \mathcal{G} determines a reduction of the $\text{SO}(4n)$ -principal fibre bundle P to the $\text{Sp}(n) \text{Sp}(1)$ -principal fibre bundle $Q(M)$ of orthonormal frames u compatible with the almost quaternionic structure given by \mathcal{G} , i.e. such that $uI_0u^{-1} \in \mathcal{G}$, $uJ_0u^{-1} \in \mathcal{G}$, $uK_0u^{-1} \in \mathcal{G}$ where we identify \mathbb{R}^{4n} with \mathbb{H}^n in the standard way. The bundle \mathcal{G} has a natural scalar product defined by $\langle I, J \rangle = -\frac{1}{2} \text{tr } I \circ J$.

Let $u \in Q$. By α_u we mean an isomorphism

$$(1.4) \quad \alpha_u : \mathfrak{so}(4n) \rightarrow \bigwedge^2 T_{p(u)}M$$

defined by

$$\alpha_u(A) = \sum_{i < j} A_j^i e_i \wedge e_j$$

where $u = (e_1, \dots, e_{4n})$. The mapping α_u is an isometry. For fixed $u \in Q$ we can define the projection $\pi_u : \bigwedge^2 T_{p(u)}M \rightarrow \tilde{\mathcal{G}} = \alpha_u(\mathfrak{sp}(1))$ by $\pi_u(\gamma) = \alpha_u(\pi(\alpha_u^{-1}(\gamma)))$. Note that $\pi = \pi_u$ is just the orthogonal projection on $\tilde{\mathcal{G}}$ and does not depend on u and that $\tilde{\mathcal{G}}$ corresponds (under the isomorphism given by the metric g) to the bundle \mathcal{G} . If $\gamma \in \bigwedge^2 T_x M$ then we write $\gamma^- = \gamma_- = \pi(\gamma)$. Note that if $\gamma = \alpha_u(X)$ where $X \in \mathfrak{g} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$, then $\gamma^- = \alpha_u(X^-)$. Two vectors $X, Y \in T_x M$ are called \mathbb{H} -dependent if $Y \in \text{span}\{X, IX, JX, KX\}$ where a quaternionic structure $\{I, J, K\}$ is some local basis of \mathcal{G} . The relationships analogous to (1.3a) and (1.3b) hold for any two vectors $X, Y \in TM$.

We let P^- be the principal $\text{SO}(3)$ -bundle of oriented orthogonal bases $\{I, J, K\} \subset \mathcal{G}_x$, $x \in M$ of \mathcal{G} ($I \circ J = K = -J \circ I$). We have a homomorphism $F : Q \rightarrow P^-$ of principal fibre bundles defined by

$$(1.4) \quad F(u) = (uI_0u^{-1}, uJ_0u^{-1}, uK_0u^{-1}),$$

which is equivariant with respect to the homomorphism

$$(*) \quad \Phi : \text{Sp}(n) \text{Sp}(1) \rightarrow \text{SO}(3)$$

defined by $\Phi(A, q) = \phi_q \in \text{SO}(3) \subset \text{Gl}(\text{Im } \mathbb{H})$ where $\phi_q(I) = R_{q^{-1}} \circ I \circ R_q$ and R_q is right multiplication by q ($R_q(x) = xq$ for $x \in V$) for $I \in \text{Im } \mathbb{H} = \mathbb{R}\{I_0, J_0, K_0\}$. We denote by ϕ the related homomorphism of Lie algebras

$$\phi : \mathfrak{g} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \rightarrow \mathfrak{so}(3).$$

Note that $\phi = \phi_+ + \phi_-$ where $\phi_+ = \phi|_{\mathfrak{sp}(n)} = 0$ and $\phi_- = \phi|_{\mathfrak{sp}(1)} : \mathfrak{g} \rightarrow \mathfrak{so}(3)$. Consider the mappings of fibre bundles

$$(1.5) \quad \Phi_i : P^-(M) \rightarrow Z(M)$$

given by $\Phi_i(J_1, J_2, J_3) = J_i$. Note that $Z(M) = P^-(M)/\text{SO}(2)_i = P^-(M) \times_{\text{SO}(3)} (\text{SO}(3)/\text{SO}(2)_i)$ where $\text{SO}(2)_i = \{\phi_q \in \text{SO}(\text{Im } \mathbb{H}) : \phi_q J_i = J_i\}$ for $i \in \{1, 2, 3\}$ where $J_1 = I_0$, $J_2 = J_0$, $J_3 = K_0$, and that $P^-(M)$ is a principal S^1 -bundle over $Z(M)$.

Let (P, \bar{g}) , (M, g) be Riemannian spaces. A *Riemannian submersion* $p : P \rightarrow M$ (see [O'N]) is a submersion which preserves the lengths of horizontal vectors. We shall denote by V the distribution of vertical vectors ($X \in V$ if X is tangent to the fibre $p^{-1}(x)$ for some $x \in M$) and by H the horizontal distribution which is an orthogonal complement of V . Let \mathcal{H} and \mathcal{V} denote the projections of the tangent space TM of M onto the subspaces H , V , respectively. We denote by T and A the O'Neill tensors defined as follows

(see [O'N]):

$$\begin{aligned} T_X Y &= \mathcal{H}(\nabla_{\mathcal{V}X} \mathcal{V}Y) + \mathcal{V}(\nabla_{\mathcal{V}X} \mathcal{H}Y), \\ A_X Y &= \mathcal{V}(\nabla_{\mathcal{H}X} \mathcal{H}Y) + \mathcal{H}(\nabla_{\mathcal{H}X} \mathcal{V}Y). \end{aligned}$$

Finally, recall that a Riemannian manifold (M, g) is called an \mathcal{A} -manifold (see [G]) if its Ricci tensor ρ satisfies the condition $\nabla_X \rho(X, X) = 0$ for all $X \in TM$. An \mathcal{A} -manifold M is called *proper* if $\nabla \rho \neq 0$ and M is locally irreducible. An almost Hermitian manifold (M, g, J) is called an *almost Kähler manifold* if its Kähler form Ω ($\Omega(X, Y) = g(X, JY)$) is closed ($d\Omega = 0$). An almost Kähler manifold is called an *almost Hodge manifold* if its Kähler form defines a cohomology class which is a (real) multiple of an integral class ([J-2]).

2. Killing tensors with two constant eigenvalues. By a *Killing tensor* on a Riemannian space (M, g) with Levi-Civita connection ∇ we mean a $(1, 1)$ tensor S defined on M which satisfies the following conditions:

- (a) $g(SX, Y) = g(X, SY)$ for every $X, Y \in \mathfrak{X}(M)$,
- (b) $g(\nabla S(X, X), X) = 0$ for every $X \in \mathfrak{X}(M)$.

DEFINITION 2. A distribution $D \subset TM$ is called *geodesic* if $\nabla_X X \in \Gamma(D)$ for every section $X \in \Gamma(D)$.

PROPOSITION 1. Let (M, g) be a Riemannian manifold and let $TM = D_1 \oplus D_2$ where D_1, D_2 are orthogonal geodesic distributions on M . Define the tensor S on M by

$$(2.1) \quad SX = \begin{cases} \lambda X & \text{if } X \in D_1, \\ \mu X & \text{if } X \in D_2, \end{cases}$$

where λ, μ are some real numbers. Then S is a Killing tensor. If $\lambda \neq \mu$ and at least one of the distributions D_i is not integrable then $\nabla S \neq 0$.

PROOF. It suffices to show (see [J-1]) that $\nabla S(X, X) = 0$ if $X \in \Gamma(D_i)$ where $i \in \{1, 2\}$. Assume for example that $X \in \Gamma(D_1)$. Then from (2.1) we obtain

$$\nabla S(X, X) = -(S - \lambda \text{Id})(\nabla_X X) = 0$$

since D_1 is geodesic. It is known that $\nabla S = 0$ if and only if both distributions are integrable, which finishes the proof. ■

COROLLARY 1. Assume that $p : P \rightarrow M$ is a Riemannian submersion with totally geodesic fibres. Denote by V, H the distributions of vertical and horizontal vectors respectively. Then the tensor S defined by

$$SX = \begin{cases} \lambda X & \text{if } X \in V, \\ \mu X & \text{if } X \in H, \end{cases}$$

is a Killing tensor. If $\lambda \neq \mu$ and the bundle $p : P \rightarrow M$ is not trivial (i.e. the O'Neill tensor $A \neq 0$) then $\nabla S \neq 0$.

Proof. Note that if $p : P \rightarrow M$ is a Riemannian submersion then the distribution H is geodesic. Since the fibres $p^{-1}(x)$ are totally geodesic the distribution V is also totally geodesic. ■

3. The structure of \mathcal{A} -manifold on a principal $\mathrm{SO}(3)$ -bundle over a quaternionic-Kähler manifold. Let $P = P(M, G)$ be a principal fibre bundle over a Riemannian manifold (M, g) with a compact, connected, semisimple Lie structure group G . Let \mathfrak{g} be the Lie algebra of G . Suppose that $\omega \in \mathcal{A}^1(P) \otimes \mathfrak{g}$ is a connection form on P . Define a metric g_c on P by

$$(c) \quad g_c = -c^2 B(\omega(X), \omega(Y)) + p^*g(X, Y)$$

where $B(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$ is the Killing form of \mathfrak{g} . In [J-1] we have proved the following theorem:

PROPOSITION 2. *The group G is a group of isometries of (P, g_c) . The orbits of the action of G on P are totally geodesic submanifolds of P isometric to the group G with biinvariant metric given by $-c^2 B$.*

Recall that a Riemannian manifold (M, g) with curvature tensor R is said to have a *harmonic curvature* if $\delta R = 0$ (or equivalently if the Ricci endomorphism $S \in \mathrm{End}(TM)$ is a Codazzi tensor, which means that $\nabla S(X, Y) = \nabla S(Y, X)$ for all $X, Y \in \mathfrak{X}(M)$). Let now Q be the principal $\mathrm{Sp}(n) \mathrm{Sp}(1)$ -fibre bundle of orthonormal frames over (M, g) . Assume that $\omega \in \mathcal{A}^1(Q) \otimes (\mathfrak{sp}(n) \oplus \mathfrak{sp}(1))$ is a connection form associated with the Levi-Civita connection $\bar{\nabla}$ of the metric g . Define a metric g_c on Q by formula (c). We denote by $H = \{X \in TQ : \omega(X) = 0\}$ the horizontal and by V the vertical distribution of P . Note that every quaternionic-Kähler manifold is Einstein, hence it has harmonic curvature tensor. Thus we have (see [J-1]):

PROPOSITION 3. *The Ricci endomorphism of (Q, g_c) leaves invariant the horizontal and vertical distributions of Q .*

The homomorphism $F : Q \rightarrow P^-$ given by (1.4) and the Levi-Civita connection form ω on Q induce a connection form $\omega^- \in \mathcal{A}^1(P^-) \otimes \mathfrak{so}(3)$ on the bundle P^- such that $\phi_- \omega = F^* \omega^-$. Define the metric

$$(c^*) \quad g_{c^*}(X, Y) = -\frac{1}{2}c^2 \mathrm{tr}(\omega^-(X) \circ \omega^-(Y)) + p^*g(X, Y)$$

on P^- . Then (since (M, g) has harmonic curvature tensor and $F^* \Omega^- = \phi_- \Omega$ where $\Omega = D\omega$, $\Omega^- = D\omega^-$ are the curvature forms of the connections Γ , Γ^- on Q, P^- respectively) we have exactly as in [J-1]:

PROPOSITION 4. *The Ricci endomorphism of (P^-, g_{c^*}) leaves invariant the horizontal and vertical distributions of (P^-, g_{c^*}) .*

Now we are in a position to prove:

PROPOSITION 5. *Let a Riemannian manifold (M, g) , $\dim M = 4n$, be a quaternionic-Kähler space with non-zero scalar curvature τ_* . Then the principal fibre bundle $p : P^- \rightarrow M$ of oriented normed orthogonal bases of the vector bundle \mathcal{G} admits a one-parameter family P_c^- of proper \mathcal{A} -structures on P^- such that $p : P_c^- \rightarrow M$ is a Riemannian submersion with fibres homothetic to $\mathbb{R}P^3 = \text{SO}(3)$ with the standard metric of constant curvature.*

Proof. Let $F : Q \rightarrow P^-$ be the homomorphism of principal fibre bundles defined by (1.4). Define a metric g_{c^*} on P^- by (c^*) . Below we write g_c instead of g_{c^*} . The natural projection $p : P^- \rightarrow M$ is a Riemannian submersion of $P_c^- = (P^-, g_c)$ onto (M, g) . From Proposition 2 it follows that O'Neill's tensor $T = 0$ (the fibres of p are totally geodesic and homothetic to $\text{SO}(3) = \mathbb{R}P^3$ with constant sectional curvature $K = 1/(2nc^2)$). To compute the tensor A recall that if $X, Y \in T_{F(u)}P^-$ are horizontal vector fields then

$$(3.1) \quad \omega^-([X, Y])_{F(u)} = -\Omega^-(X, Y)_{F(u)} = -\phi_- \Omega(\bar{X}, \bar{Y})_u = -(\Omega(\bar{X}, \bar{Y})_u)^-$$

where \bar{X}, \bar{Y} are horizontal vector fields on Q such that $dF(\bar{X}) = X \circ F$. Hence $A_X Y$ is a vertical vector field such that $\omega^-(A_X Y) = -\frac{1}{2}\Omega^-(X, Y)$.

If V is a vertical vector field and $dF(\bar{V}) = V \circ F$ then

$$\begin{aligned} g_c(A_X V, Y)_{F(u)} &= -g_c(V, A_X Y)_{F(u)} = -\frac{1}{2}c^2 \text{tr}(\omega^-(V)_{F(u)} \circ \omega^-(A_X Y)_{F(u)}) \\ &= \frac{1}{4}c^2 \text{tr}(\omega^-(V)_{F(u)} \circ \Omega^-(X, Y)_{F(u)}) \\ &= \frac{1}{4}c^2 \text{tr}(\phi_- \omega(\bar{V})_u \circ \phi_- \Omega(\bar{X}, \bar{Y})_u). \end{aligned}$$

We can assume that X, Y are horizontal lifts of vector fields $X_*, Y_* \in \mathfrak{X}(M)$. Consequently, we obtain

$$\begin{aligned} g_c(A_X V, Y)_{F(u)} &= \frac{1}{2}c^2 R(X_* \wedge Y_*, \alpha_u(\omega^-(V))) \\ &= \frac{1}{2}c^2 g(X_* \wedge Y_*, R|_{\mathcal{G}}(\alpha_u \omega^-(V))) \end{aligned}$$

and thus, in view of (R),

$$(3.2) \quad g_c(A_X V, Y)_{F(u)} = \frac{1}{2}c^2 \alpha g(X_* \wedge Y_*, \alpha_u(\omega^-(V))).$$

Let $\{V_1, V_2, V_3\}$ be an orthonormal basis of $\mathfrak{sp}(1)$ and let $\{v_1, v_2, v_3\}$ be the Killing vector fields corresponding (under the homomorphism F) to V_i . Then $g_c(v_i, v_i) = c^2$ and $g(\alpha_u(V_i), \alpha_u(V_i)) = 1$. Hence

$$(3.3) \quad c^2 \|A_X Y\|^2 = \sum_{i=1}^3 g_c(A_X Y, v_i)^2 = \frac{1}{4}c^4 \sum_{i=1}^3 g(X_* \wedge Y_*, R|_{\mathcal{G}}(\alpha_u(V_i)))^2.$$

From (3.3) it follows that if $\|X\| = \|Y\| = 1$, X, Y are \mathbb{H} -dependent and $X \perp Y$ then (see (1.3a) and (1.3b))

$$\|A_X Y\|^2 = \frac{1}{4}c^2\alpha^2\|(X_* \wedge Y_*)_- \|^2 = \frac{1}{8n}c^2\alpha^2$$

and $\|A_X Y\| = 0$ if $X \perp \mathcal{G}Y = \mathbb{R}\{IY, JY, KY\}$. Let $\{Y_{1*}, \dots, Y_{4n*}\}$ be a local orthonormal frame on M such that any four vectors $Y_{4k+1}, Y_{4k+2}, Y_{4k+3}, Y_{4k+4}$ are \mathbb{H} -dependent for $k = 0, 1, \dots, n-1$. Let $\{Y_1, \dots, Y_{4n}\}$ be its horizontal lift. We can assume that $X = Y_1$. Note that if X is horizontal and $\|X\| = 1$ then for a vertical vector field V we have

$$\begin{aligned} (3.4) \quad \|A_X V\|^2 &= \sum_{i=1}^4 g_c(A_X V, Y_i)^2 \\ &= \frac{1}{4}c^4 \sum_{i=1}^4 g((X_* \wedge Y_{i*})_-, R_{|\mathcal{G}}(\alpha_u(\omega^-(V))))^2 \\ &= \frac{1}{8n}c^4 \|R_{|\mathcal{G}}(\alpha_u(\omega^-(V)))\|^2 \\ &= \frac{1}{8n}c^4\alpha^2 \|\alpha_u(\omega^-(V))\|^2 = \frac{1}{8n}c^2\alpha^2 \|V\|^2 \end{aligned}$$

in view of the relation $c^2\|\alpha_u(\omega(V))\|^2 = g_c(V, V)$. Hence we have (we assume that $\|X\| = \|Y\| = 1$, $X \perp Y$ and P_{XY} denotes the plane spanned by $\{X, Y\}$)

$$\begin{aligned} K(P_{v_i v_j}) &= \frac{1}{2nc^2} \quad \text{if } i \neq j, \quad K(P_{X v_i}) = \frac{1}{8n}c^2\alpha^2, \\ K(P_{XY}) &= K(P_{X_* Y_*}) - \frac{3}{8n}c^2\alpha^2 \quad \text{if } X, Y \text{ are } \mathbb{H}\text{-dependent,} \\ K(P_{XY}) &= K(P_{X_* Y_*}) \quad \text{if } X \perp \text{span}\{IY, JY, KY\} = \mathcal{G}Y. \end{aligned}$$

Consequently, if V is a vertical vector field such that $\|V\| = 1$ then

$$(3.5) \quad \varrho(V, V) = \frac{1}{nc^2} + \frac{1}{2}c^2\alpha^2$$

and if X is horizontal and $\|X\| = 1$ then

$$(3.6) \quad \varrho(X, X) = \frac{1}{4n}\tau_* - 3\frac{3}{8n}c^2\alpha^2 + \frac{3}{8n}c^2\alpha^2 = \frac{(n+2)\alpha}{n} - \frac{3}{4n}c^2\alpha^2.$$

From Proposition 4 we get

$$\varrho(\xi, X) = 0$$

for any vertical vector field ξ and any horizontal vector field X , i.e. the Ricci endomorphism S of (P_c^-, g_c) preserves the horizontal and vertical distributions. From (3.5) and (3.6) we deduce that S has exactly two constant

eigenvalues

$$\lambda = \frac{1}{nc^2} + \frac{1}{2}c^2\alpha^2 \quad \text{and} \quad \mu = \frac{(n+2)\alpha}{n} - \frac{3}{4n}c^2\alpha^2.$$

Note that $\lambda = \mu$ if and only if $c^2 = 2/\alpha$ or $c^2 = 2/((2n+3)\alpha)$ and for those values of c the manifold P_c^- is an Einstein space. Denote by H the horizontal distribution of the submersion $p_c : P_c^- \rightarrow M$. Since for $\lambda \neq \mu$ the distribution $D_0 = \text{span}\{v_1, v_2, v_3\}$ is an eigendistribution of the Ricci tensor which is totally geodesic and the leaves of the foliation D_0 are locally irreducible it follows that if P_c^- were locally reducible we would have $D_0^\perp = H = D_1 \oplus D_2$ where $D_0 \oplus D_1$ and D_2 would be parallel distributions. For a Killing vector field $\xi = v_1 \in D_0$ define $T = \nabla\xi$. Then T is skew-symmetric, $TD_0 \subset D_0$ and $TD_2 \subset D_1$, $TD_1 \subset D_1$. Thus we would have $TX = 0$ for every $X \in D_2$, contrary to (3.4). It follows that P_c^- is locally irreducible. For $c^2 \neq 2/\alpha$ and $c^2 \neq 2/((2n+3)\alpha)$ from Proposition 1 it follows that (P^-, g_c) is a proper \mathcal{A} -manifold. ■

Hence we get

COROLLARY 2. *Let (M, g) be a quaternionic-Kähler manifold with non-zero scalar curvature τ_* . For $c^2 = 2/\alpha$ or $c^2 = 2/((2n+3)\alpha)$ where $\alpha = \tau_*/(4(n+2))$ the manifold P_c^- is an Einstein space. For other c^2 the manifold (P_c^-, g_c) is a proper \mathcal{A} -manifold whose Ricci tensor has exactly two constant eigenvalues*

$$\lambda = \frac{1}{nc^2} + \frac{1}{2}c^2\alpha^2 \quad \text{and} \quad \mu = \frac{(n+2)\alpha}{n} - \frac{3}{4n}c^2\alpha^2.$$

If $c^2 < 4(n+2)/(3\alpha)$ then (P_c^-, g_c) has positive Ricci curvature.

4. Almost Kähler \mathcal{A} -structure on the twistor bundle $Z(M)$. Let (P, g) be a Riemannian manifold and ξ be a unit Killing vector field on P . Define a tensor field ϕ by $\phi(X) = \nabla_X\xi$ and a 1-form η by $\eta(X) := g(\xi, X)$. Then we call (P, g, ξ, ϕ, η) a *K-contact structure* if the following relation is satisfied:

$$(K) \quad \phi^2 = -\text{id} + \eta \otimes \xi.$$

Assume that (P, g) is a circle bundle over a manifold M and that the fundamental vector field of the action of the group S^1 is a constant multiple of a unit Killing vector ξ on P . If ξ gives a K-contact structure (ξ, ϕ, η) on P then there is a unique metric g_* and a unique almost complex structure J_* on M such that (M, g_*, J_*) is an almost Kähler manifold and

$$d\eta(X, Y) = g(X, \phi Y) = p^*\Omega(X, Y)$$

where Ω is the Kähler form of (M, g_*, J_*) (i.e. $\Omega(X, Y) = g(X, J_*Y)$). Assume that $c^2 = 1/(2n)$ (see (c*)) and let $g = g_{1/(2n)}$ be the fixed metric

on P^- . Hence

$$g(X, Y) = -\frac{1}{4n} \operatorname{tr}(\omega^-(X) \circ \omega^-(Y)) + p^*g_0(X, Y)$$

where g_0 is the metric on the quaternionic-Kähler manifold M . Then the vector fields (V_1, V_2, V_3) corresponding via F to $v_1 = I_0, v_2 = J_0, v_3 = K_0 \in \mathfrak{sp}(1) \subset \mathfrak{so}(4n)$ are unit Killing vector fields (in view of Proposition 2). We start with:

PROPOSITION 6. *Assume that (M, g_0) is a quaternionic-Kähler manifold with scalar curvature τ_0 such that $|\tau_0| = 16n(n+2)$. Then the fields V_1, V_2, V_3 define on (P^-, g) three K-contact structures. The tensors ϕ_i induce on $(Z(M), g_*)$ the almost Kähler structure J_* which does not depend on i .*

Proof. It is clear that V_i is a unit Killing field. Define the 1-form η_i by $\eta_i(X) = g(V_i, X)$. We only have to show that $\phi_i = \nabla V_i$ defines on $H_i = \ker \eta_i$ an almost complex structure. Note that $\phi_i(X) = A_X V_i$ if X is horizontal and that ϕ_i preserves the horizontal and vertical bundles. From (3.2) it follows that the restriction of ϕ_i to the horizontal distribution is an almost complex structure. In fact we have $\alpha = 4n\varepsilon$ and

$$(4.1) \quad g(A_X V_i, Y) = \frac{1}{2} c^2 \alpha g(X_* \wedge Y_*, \alpha_u(v_i)) = \varepsilon g(X_*, J_{\alpha_u(v_i)} Y_*)$$

where X, Y are horizontal with respect to the submersion $p: P^- \rightarrow M$, $\varepsilon = \operatorname{sgn} \tau_0$ and $p(X) = X_*, p(Y) = Y_*$. On the other hand the restriction to the vertical distribution also gives an almost complex structure (on the fibres, Φ_i (see (1.5)) is just the Hopf fibration $\mathbb{R}P^3 \rightarrow \mathbb{C}P^1$). The last statement of the proposition follows from the fact that J_* is one of the two natural almost complex structures J_+, J_- on $Z(M)$ whose definitions we recall below. The theorem follows from the well known properties of K-contact structures (see [J-2], [J-3]). ■

Below the following definition will be useful:

DEFINITION 3. Assume that a Lie group G ($G = \operatorname{SO}(3)$ or $G = \operatorname{SU}(2)$) acts by isometries on a Riemannian manifold (M, g) in such a way that the orbits $\operatorname{orb}_G(x)$ of the action are totally geodesic submanifolds isometric to G with its canonical metric of constant sectional curvature $K = 1$. Let $\{\xi_1, \xi_2, \xi_3\}$ be Killing vector fields corresponding to the generators of the Lie algebra $\mathfrak{so}(3)$ and denote by H the orthogonal complement of $\operatorname{span}\{\xi_1, \xi_2, \xi_3\}$. Assume also that every vector field ξ_i gives a K-contact structure. Hence

$$(K1) \quad [\xi_i, \xi_j] = 2\varepsilon_{ijk} \xi_k \quad \text{and} \quad \nabla_{\xi_i} \xi_j = \varepsilon_{ijk} \xi_k.$$

Let the fields ξ_1, ξ_2, ξ_3 satisfy the condition (for pairwise different i, j, k)

$$(K2) \quad J_i \circ J_j = \varepsilon \varepsilon_{ijk} J_k,$$

i.e. $(\varepsilon J_1, \varepsilon J_2, \varepsilon J_3)$ is an almost quaternionic-structure on H , where $\phi_i = \nabla \xi_i$ and $J_i = -\phi_i|_H$. Then we call $(M, \xi_i, \phi_i, \eta_i)$ a *3-K-contact structure* if $\varepsilon = 1$ and a *quasi 3-K-contact structure* if $\varepsilon = -1$.

REMARK. Note that if $\varepsilon = 1$ then the above definition is equivalent to the usual definition of a 3-K-contact structure (see [K], [Ku] and [J-3]). The notion of nS-structure introduced by S. Tanno is stronger than the notion of quasi 3-K-contact structure and the relation between these two notions is similar to that between the 3-K-contact structure and 3-Sasakian structure in the positive case. An nS-structure is a quasi 3-K-contact structure satisfying an additional condition similar to the condition of being Sasakian in the positive case.

PROPOSITION 7. *Let an oriented Riemannian 4n-manifold (M, g) be a quaternionic-Kähler manifold with scalar curvature τ_0 such that $|\tau_0| = 16n(n + 2)$. Then the fibre bundle $p : P^- \rightarrow M$ furnished with the metric g admits a 3-K-contact structure if $\tau_0 > 0$ and a quasi 3-K-contact structure if $\tau_0 < 0$.*

PROOF. We prove that the triple (V_1, V_2, V_3) gives the required structure on $P^-(M)$. Define $\Psi_i = \Phi_i \circ F : Q(M) \rightarrow Z(M)$ (see (1.5)). Then $\Psi_i(u) = \alpha_u(v_i)$. Note that $p : (P^-(M), g) \rightarrow (M, g_0)$ is a Riemannian submersion. Let $u \in Q$ and $v = F(u) \in P^-(M)$. Define $H_v = \{X \in TP^- : \eta_1(X) = \eta_2(X) = \eta_3(X) = 0\}$. Then $p : H_v \rightarrow T_{p(v)}M$ is an isometry and $J_i = -\phi_i|_{H_v}$ is given by $J_i = \varepsilon J_{\Psi_i(u)}$, where we identify H_v with $T_{p(v)}M$ via p and where ε is the sign of τ_0 . Note that $J_i \circ J_j = \varepsilon \varepsilon_{ijk} J_k$ (since $v_i v_j = \varepsilon_{ijk} v_k$). Thus the assumptions of Definition 3 are satisfied, which completes the proof. ■

REMARK. The case $\tau_0 > 0$ of Proposition 7 is well known. Note that if $\tau_0 > 0$ then our 3-K-contact structure is 3-Sasakian. The case $\tau_0 < 0$ was proved earlier in a different way by S. Tanno (see [T], Th. B). In fact Tanno proved more: he showed that our quasi 3-K-contact structure is in fact an nS-structure. He also described the Ricci tensor of (P^-, g) .

Let (M, g_0) be a quaternionic-Kähler manifold. We have two natural almost complex structures J_-, J_+ on the twistor bundle $\pi : Z(M) \rightarrow M$. Both structures preserve the vertical and horizontal distributions of $Z(M)$. They are given as follows (see [E-S]):

$$(4.2) \quad g_*(X, J_+Y) = g_0(\pi(X), J_\omega \pi(Y)) = g_*(X, J_-Y)$$

if X, Y are horizontal with respect to $\pi : Z(M) \rightarrow M$, hence on $H_\omega \approx T_{\pi(\omega)}M$ the almost complex structures $J_+ = J_-$ are just J_ω . Write $\xi = V_1, \Phi = \Phi_1$. If X is a vertical vector then we define

$$(4.3) \quad J_+X = -d\Phi(\nabla_{X^*}\xi), \quad J_-X = d\Phi(\nabla_{X^*}\xi)$$

where X^* is the horizontal lift of $X \in VZ(M)$ with respect to the Riemannian submersion $\Phi : P^- \rightarrow Z$. It is well known (see [B]) that the structure J_+ is integrable, while J_- is not. In what follows we also write $J_1 = J_+$ and $J_{-1} = J_-$.

In my paper [J-2] I have proved that a regular K-contact manifold P (i.e. P is a principal S^1 -bundle over M and $p : (P, g) \rightarrow (M, g_*)$ is a Riemannian submersion) is an \mathcal{A} -manifold if and only if (M, g_*) with the induced almost Kähler structure J_* is an almost Kähler \mathcal{A} -manifold whose Ricci tensor ϱ_* is Hermitian, i.e. $\varrho_*(J_*X, J_*Y) = \varrho_*(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$.

PROPOSITION 8. *Let (M, g_0) be a quaternionic-Kähler manifold with scalar curvature $\tau_0 = 16n(n+2)\varepsilon$ where $\varepsilon \in \{-1, 1\}$. Then the manifold $(Z(M), g_*, J_\varepsilon)$ is an almost Kähler \mathcal{A} -manifold. If $\tau_0 > 0$ then $(Z(M), g_*, J_+)$ is Kähler-Einstein and the 3-K-contact structure on $(P^-(M), g)$ is a 3-Sasakian structure. If $\tau_0 < 0$ then $(Z(M), g_*, J_-)$ is a strictly almost Hodge \mathcal{A} -manifold whose Ricci tensor has two constant eigenvalues $\lambda_* = 4n + 4$ and $\mu_* = -4n - 12$ of multiplicity 2 and $4n$ respectively. The Ricci tensor ϱ_* of $(Z(M), g_*)$ is Hermitian with respect to both structures J_+, J_- , i.e. $\varrho_*(J_+X, J_+Y) = \varrho_*(J_-X, J_-Y) = \varrho_*(X, Y)$. If M is compact then $Z(M)$ is a compact space.*

Proof. Note that (P^-, g) is an \mathcal{A} -manifold (see [J-1]). Let $\Phi = \Phi_1$ and $\xi = V_1$. Then the Killing field ξ defines on (P^-, g) a K-contact structure and $\Phi : P^- \rightarrow Z(M)$ is an S^1 -principal fibre bundle. Thus from Theorem 1 of [J-2] it follows that $(Z(M), g_*, J_*)$ is an almost Kähler \mathcal{A} -manifold with Hermitian Ricci tensor. From (4.1)–(4.3) it follows that the almost Kähler structure J_* induced on $Z(M)$ by $\phi = \nabla\xi$ coincides with $-J_+$ if $\tau_0 > 0$ and with J_- if $\tau_0 < 0$. Hence if $\tau_0 > 0$ then J_* is integrable. Consequently, $(Z(M), g_*, J_+)$ is a Kähler manifold. Thus every Killing vector field V_i gives a Sasakian structure on $(P^-(M), g)$ and the triple (V_1, V_2, V_3) defines a 3-Sasakian structure on $(P^-(M), g)$ (see [K]). It is also an Einstein space since from Corollary 2 it follows that for $c^2 = 1/(2n)$ and $\alpha = 4n$ the manifold (P^-, g) is Einstein. Thus $(Z(M), g_*)$ is also Einstein in this case (see also [B] and [S]).

If $\tau_0 < 0$ then the Ricci tensor S of (P^-, g_c) has two constant eigenvalues whose eigendistributions are just the vertical and horizontal distributions. If $c^2 = 1/(2n)$ and $\alpha = -4n$ then the eigenvalues are $\lambda = 4n+2$ and $\mu = -4n-14$. Hence it follows from [J-2] that the Ricci tensor S_* of $(Z(M), g_*)$ has exactly two eigenvalues $\lambda_* = \lambda+2 = 4n+4$ and $\mu_* = \mu+2 = -4n-12$ whose eigendistributions are the vertical and horizontal distributions respectively. Since J_+, J_- also preserve the vertical and horizontal distributions it follows that $S_* \circ J_\varepsilon = J_\varepsilon \circ S_*$, which means that the Ricci tensor of $(Z(M), g_*)$ is Hermitian with respect to both structures. If $\tau_0 < 0$ then $J_* = J_-$ is not

integrable and from the construction it is clear that $(Z(M), g, J_*)$ is a strictly almost Hodge \mathcal{A} -manifold. The scalar curvature τ_* of $(Z(M), g_*)$ is negative, $\tau_* = -8n(2n + 5) + 8$. ■

REMARK. Note that all the above results are proved by different methods in [A-G-I] (with the exception of the fact that $Z(M)$ is an almost Hodge \mathcal{A} -manifold). Let us recall the Goldberg conjecture: A compact Einstein almost Kähler manifold is Kähler, formulated by S. I. Goldberg [Go] almost thirty years ago. This conjecture was proved by K. Sekigawa [Se] under the assumption that the scalar curvature is non-negative. The case of negative scalar curvature is still open. In relation to this conjecture, examples of strictly almost Kähler Riemannian manifolds which are close to Einstein are of great interest (see [D-M]). Compact examples are of particular interest. Note that there are examples of compact quaternionic-Kähler manifolds in any dimension $4n$ (for example compact quotients of symmetric spaces dual to the symmetric spaces discussed by Wolf [W]). In that way we obtain examples of compact almost Hodge \mathcal{A} -manifolds with Hermitian Ricci tensor in every dimension $4n + 2$ (see also [J-2]). These examples indicate that the Goldberg conjecture may be false in general.

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Institute of Mathematics
Cracow University of Technology
Warszawska 24
31-155 Kraków, Poland
E-mail: wjelon@usk.pk.edu.pl

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