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Finite extensions of mappings from a smooth variety

by Marek Karaś (Kraków)

Abstract. Let V, W be algebraic subsets of $\mathbf{k}^n$, $\mathbf{k}^m$ respectively, with $n \le m$. It is known that any finite polynomial mapping $f : V \to W$ can be extended to a finite polynomial mapping $F : \mathbf{k}^n \to \mathbf{k}^m$. The main goal of this paper is to estimate from above the geometric degree of a finite extension $F : \mathbf{k}^n \to \mathbf{k}^n$ of a dominating mapping $f : V \to W$, where V and W are smooth algebraic sets.

1. Introduction. Let $\mathbf{k}$ be any algebraically closed field of characteristic zero and V, W be algebraic subsets of $\mathbf{k}^n$, $\mathbf{k}^m$, respectively, with $n \le m$. It is known (under the assumption that $\mathbf{k}$ is an infinite field) that any finite mapping $f : V \to W$ can be extended to a finite mapping $F : \mathbf{k}^n \to \mathbf{k}^m$ (see [6]). There is the following natural problem: to estimate from above the number $\operatorname{gdeg} F$ (geometric degree of F) of points in the "generic fiber" of the best extension F (with $\operatorname{gdeg} F$ minimal). The author ([3]–[5]) solved this problem in a few cases.

If $n = m$ and $\operatorname{gdeg} F = 1$, then $F : \mathbf{k}^n \to \mathbf{k}^n$ is an isomorphism. Thus the answer to the above question is very important.

In this paper we consider a finite mapping from a smooth algebraic set such that the image is also a smooth algebraic set. We will show

Theorem 4.4. *Let $V, W \subset \mathbb{C}^n$ be smooth algebraic sets and let $f : V \to W$ be a finite dominating mapping that is dominating on every irreducible component. If $\dim V = \dim W = k$ and $4k+2 \le n$, then there exists a finite mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ such that $F|_V = f$ and*

$$\operatorname{gdeg} F \le (\operatorname{gdeg} f)^{2k+1}.$$

This is a generalization of the main Theorem of [2] which was the answer to the problem of extension of embeddings into affine space. The problem was set by S. S. Abhyankar [1].

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2. Preliminaries. The coordinate ring of the algebraic set V will be denoted by $\mathbf{k}[V]$. For any polynomial mapping $f = (f_1, \dots, f_n) : V \to W$ the homomorphism of the coordinate rings $\mathbf{k}[W] \ni P \mapsto P(f_1, \dots, f_n) \in \mathbf{k}[V]$ will be denoted by f^*.

DEFINITION 2.1. A polynomial mapping $f : V \to W$ is called *finite* if $\mathbf{k}[V]$ is an integral extension of the ring $f^*(\mathbf{k}[W]) = \mathbf{k}[f_1, \dots, f_m]$.

Finite mappings have the following properties:

THEOREM 2.2. (1) *If f is finite, then there exists a number $d \in \mathbb{N}$ such that $\# f^{-1}(y) \le d$ for all $y \in W$.*

(2) *If $\mathbf{k} = \mathbb{C}$, then f is finite iff f is proper (in the natural topology).*

(3) *If f, g are finite, then $g \circ f$ is finite.*

(4) *If f, g are polynomial mappings and $g \circ f$ is finite, then f is finite. If, moreover, f is dominating, then g is finite.*

(5) *If f is finite and dominating, then f is surjective.*

(6) *If $f : V \to W$ is finite and Z is an algebraic subset of V, then $f|_Z$ is finite.*

(7) *If $f : V \to W$ is finite and Z is an algebraic subset of V, then* $\dim f(Z) = \dim Z$.

(8) *If $f_i : V_i \to W_i$ are finite for $i = 1, 2$, and $f_1|_{V_1 \cap V_2} = f_2|_{V_1 \cap V_2}$, then $f_1 \cup f_2$ is finite.*

Proof. Assertions (1), (3), (4), (6) and (8) are easy consequences of the definition. For the proof of (2) see e.g. [8]. The proof of (5) can be found e.g. in [9] (Thm. I.5.5), and (7) is a consequence of the definition of dimension. ∎

If $f : V \to W$ is dominating then $f^* : \mathbf{k}[W] \to \mathbf{k}[V]$ is a monomorphism. In this case we will identify $\mathbf{k}[W]$ with $f^*(\mathbf{k}[W]) \subset \mathbf{k}[V]$. If V and W are irreducible, then $\mathbf{k}[W]$ and $\mathbf{k}[V]$ are integral domains and therefore f^* can be extended to a monomorphism $f^* : \mathbf{k}(W) \to \mathbf{k}(V)$ of fields. In the same way as before we will identify $\mathbf{k}(W)$ with $f^*(\mathbf{k}(W)) \subset \mathbf{k}(V)$.

We have the following

THEOREM 2.3 (see e.g. [7], Thm. 3.17). *Let $V \subset \mathbf{k}^n$ and $W \subset \mathbf{k}^m$ be irreducible algebraic sets of the same dimension. If $f : V \to W$ is dominating then there exists an open and dense subset $U \subset W$ such that*

$$\# f^{-1}(y) = [\mathbf{k}(V) : \mathbf{k}(W)] \quad \textit{for } y \in U.$$

In [7] Theorem 2.3 is stated for $\mathbf{k} = \mathbb{C}$ but the proof given there works for an algebraically closed field of characteristic zero.

From Theorem 2.3 and the theorem about the dimension of the fibers (see e.g. [9], Thm. I.6.7) we have

LEMMA 2.4. *Let $V \subset \mathbf{k}^n$ and $W \subset \mathbf{k}^m$ be algebraic sets. If W is irreducible and $f : V \to W$ dominating then there is an open and dense set $U \subset W$ such that the function $U \ni y \mapsto \# f^{-1}(y) \in \mathbb{N} \cup \{\infty\}$ is constant.*

P r o o f. Let $V = V_1 \cup \ldots \cup V_r$ be the decomposition of V into irreducible components. We have $\overline{f(V)} = \overline{f(V_1)} \cup \ldots \cup \overline{f(V_r)}$ and, since W is irreducible, $\overline{f(V_i)} = W$ for some $i \in \{1, \ldots, r\}$.

Let W^* be the union of those $\overline{f(V_i)}$ for which $\overline{f(V_i)} \neq W$. Since W is irreducible, $W \setminus W^*$ is an open and dense subset of W for which $f^{-1}(W \setminus W^*) \subset \bigcup_{i \in I} V_i$, where $I := \{i \in \{1, \ldots, r\} \mid \overline{f(V_i)} = W\}$. We may assume that $I = \{1, \ldots, k\}$ for some $k \leq r$. By Theorem 2.3 for all $i = 1, \ldots, k$ there exist $U_i \subset W$ open and dense in W such that

$$\#(f|_{V_i})^{-1}(y) = [\mathbf{k}(V_i) : \mathbf{k}(W)] \quad \text{for } y \in U_i.$$

For the set $\widetilde{U} := U_1 \cap \ldots \cap U_k$, which is also open and dense, we infer that

$$\#(f|_{V_i})^{-1} = [\mathbf{k}(V_i) : \mathbf{k}(W)] \quad \text{for } y \in \widetilde{U},\ i = 1, \ldots, k.$$

For $i \in \{1, \ldots, k\}$ set $V_i^* := V_i \cap \bigcup_{j \neq i} V_j$. Since $V_i^* \neq V_i$ and V_i is irreducible, we see that $\dim V_i^* < \dim V_i$ and $\overline{f(V_i^*)} \neq W$.

By irreducibility of W we get $\overline{f(V_1^*)} \cup \ldots \cup \overline{f(V_k^*)} \neq W$, and consequently $U^* := W \setminus (\overline{f(V_1^*)} \cup \ldots \cup \overline{f(V_k^*)})$ is open and dense in W.

Now we have

$$\#(f|_{V_1 \cup \ldots \cup V_k})^{-1}(y) = \sum_{i=1}^{k} [\mathbf{k}(V_i) : \mathbf{k}(V)] \quad \text{for } y \in \widetilde{U} \cap U^*,$$

where $\widetilde{U} \cap U^*$ is open and dense in W. Finally $U := \widetilde{U} \cap U^* \cap (W \setminus W^*)$ is an open and dense subset of W such that

$$\# f^{-1}(y) = \sum_{i=1}^{k} [\mathbf{k}(V_i) : \mathbf{k}(V)] \quad \text{for } y \in U. \ \blacksquare$$

Now we can state

DEFINITION 2.5. Let $f : V \to W$ be a dominating polynomial mapping. If W is an irreducible set then the constant number of points in the fibers of f on the set U (see Lemma 2.4) is called the *geometric degree* of the mapping f and denoted by $\operatorname{gdeg} f$. If the set W is reducible then

$$\operatorname{gdeg} f := \max\{\operatorname{gdeg}(f|_{f^{-1}(Z)}) \mid Z \subset W \text{ is an irreducible component of } W\}.$$

For any polynomial mapping we can define its geometric degree by putting $W = \overline{f(V)}$. In particular we can define the geometric degree for any finite mapping.

In the general case there may exists $y \in W$ with $\# f^{-1}(y) > \operatorname{gdeg} f$. This is illustrated by

EXAMPLE 2.6. Let $F : \mathbb{C} \ni t \mapsto (t^2 - 1, t(t^2 - 1)) \in \mathbb{C}^2$. Then $F(\mathbb{C}) = W$, where $W = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^2 + x^3\}$ and $\operatorname{gdeg} F = 1$, because F is a parametrization of the rational curve W, but $F^{-1}((0, 0)) = \{-1, 1\}$. If we take $f = F|_{\{-1,1\}}$, then we obtain a finite mapping $f : \{-1, 1\} \to \mathbb{C}^2$ with extension F and $\operatorname{gdeg} f = 2 > \operatorname{gdeg} F$.

The following theorem gives a condition under which it is impossible to find a point $y \in W$ with $\# f^{-1}(y) > \operatorname{gdeg} f$.

THEOREM 2.7 (see e.g. [9], Thm. II 5.6). *If $f : V \to W$ is finite and dominating, V and W are irreducible and W is normal then*

$$\# f^{-1}(y) \le \operatorname{gdeg} f \quad \textit{for } y \in W.$$

3. Extensions of projections. In [4] we proved

THEOREM 3.1 ([4], Thm. 3.10). *Let $V \subset \mathbf{k}^k \times \mathbf{k}^n$ be an irreducible algebraic set, and $\pi : V \to 0 \times \mathbf{k}^n$ the natural projection. If π is finite and $\pi(V)$ is normal then there exists a finite mapping $\Pi : \mathbf{k}^k \times \mathbf{k}^n \to \mathbf{k}^k \times \mathbf{k}^n$ such that $\Pi|_V = \pi$ and*

$$\operatorname{gdeg} \Pi \le (\operatorname{gdeg} \pi)^k.$$

Here, we prove a slight generalization (Theorem 3.6). A reducible set is meant to be normal if each of its points is normal.

First we define an auxiliary notion.

DEFINITION 3.2. A polynomial mapping $f : V \to W$ is called *dominating on the irreducible component $V' \subset V$* if $\overline{f(V')}$ is an irreducible component of W.

EXAMPLE 3.3. Let $V = \{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$, $W = \mathbb{C}$ and $f : V \ni (x, y) \mapsto x \in W$. The mapping f is dominating and dominating on the component $\{(x, y) \in \mathbb{C}^2 \mid y = 0\}$ but it is not dominating on the component $\{(x, y) \in \mathbb{C}^2 \mid x = 0\}$.

EXAMPLE 3.4. Let $V = W = \{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$ and $f : v \ni (x, y) \mapsto (x + y, 0) \in W$. The mapping f is dominating on every component of V but it is not dominating.

REMARK 3.5. If $f : V \to W$ is finite and dominating, then it is dominating on every component of maximal dimension.

THEOREM 3.6. *Let $V \subset \mathbf{k}^k \times \mathbf{k}^n$ be an algebraic set, and let $\pi : V \to 0 \times \mathbf{k}^n$ be the natural projection. If $\pi : V \to \pi(V)$ is finite and dominating* *on every component, and* $\pi(V)$ *is normal, then there exists a finite mapping* $\Pi : \mathbf{k}^k \times \mathbf{k}^n \to \mathbf{k}^k \times \mathbf{k}^n$ *such that* $\Pi|_V = \pi$ *and*

$$\operatorname{gdeg} \Pi \leq (\operatorname{gdeg} \pi)^k.$$

Proof. Let $V = V_1 \cup \ldots \cup V_r$ and $W = W_1 \cup \ldots \cup W_s$ be decompositions into irreducible components. Because W is normal, $W_i \cap W_j = \emptyset$ for $i \neq j$. We have

$$\pi^{-1}(W_1) = \bigcup_{i \in I_1} V_i, \quad \ldots, \quad \pi^{-1}(W_s) = \bigcup_{i \in I_s} V_i$$

where $I_1 \cup \ldots \cup I_s = \{1, \ldots, r\}$ and $I_i \cap I_j = \emptyset$ for $i \neq j$. For any $l \in \{1, \ldots, s\}$ consider the mapping

$$\pi_l = \pi|_{\pi^{-1}(W_l)} : \pi^{-1}(W_l) \to W_l.$$

By Definition 2.5 we have

$$\operatorname{gdeg} \pi_l \leq \operatorname{gdeg} \pi.$$

The mapping f is dominating on every component, so

$$\operatorname{gdeg} \pi_l = \sum_{j \in I_l} [\mathbf{k}(V_j) : \mathbf{k}(W_l)]$$

(see the proof of Lemma 2.4). For any $j \in I_l$ and $i \in \{1, \ldots, k\}$ let $H_{i,j,l} \in \mathbf{k}(W_l)[T]$ be the minimal monic polynomial for $x_i|_{V_j}$ (restriction of x_i to the set V_j) over the field $\mathbf{k}(W_l)$, where $x_1, \ldots, x_k$ and $y_1, \ldots, y_n$ are coordinates in $\mathbf{k}^k$ and $\mathbf{k}^n$ respectively. We have $\deg H_{i,j,l} \leq [\mathbf{k}(V_j) : \mathbf{k}(W_l)]$. The element $x_i|_{V_J}$ is integral over the ring $\mathbf{k}[W_l]$. The ring $\mathbf{k}[W_l]$ is normal, so $H_{i,j,l} \in \mathbf{k}[W_l][T]$. Now we put

$$H_{i,l} = \prod_{j \in I_l} H_{i,j,l}.$$

The polynomial $H_{i,l}$ is monic and

$$H_{i,l}(x_i) = 0 \quad \text{on } \bigcup_{j \in I_l} V_j.$$

Also

$$\deg H_{i,l} = \sum_{j \in I_l} \deg H_{i,j,l} \leq \sum_{j \in I_l} [\mathbf{k}(V_j) : \mathbf{k}(W_l)] = \operatorname{gdeg} \pi_l \leq \operatorname{gdeg} \pi.$$

Multiplying $H_{i,l}$, if necessary, by some power of T, we can assume that $\deg H_{i,l} = \operatorname{gdeg} \pi$. Thus we can write

$$H_{i,l} = T^d + a_{i,l,d-1} T^{d-1} + \ldots + a_{i,l,0}$$

where $d = \operatorname{gdeg} \pi$ and $a_{i,l,j} \in \mathbf{k}[W_l]$. The functions $a_{i,j} = a_{i,1,j} \cup \ldots \cup a_{i,s,j}$ are regular on $W_1 \cup \ldots \cup W_s = W$. Thus

$$H_i = T^d + a_{i,d-1} T^{d-1} + \ldots + a_{i,0}$$

is a monic polynomial in $\mathbf{k}[W][T]$ such that

$$H_i(x_i) = 0 \quad \text{on } V.$$

Let G_i be a polynomial of $\mathbf{k}[y_1,\dots,y_n][T]$ obtained from H_i by replacing $y_i|_W$ with y_i and

$$\Pi:\mathbf{k}^k\times\mathbf{k}^n\ni(x,y)\mapsto(G_1(x_1,y),\dots,G_k(x_k,y),y)\in\mathbf{k}^k\times\mathbf{k}^n.$$

Because $H_i(x_i|_V)=0$, we have $\Pi|_V=\pi$. Furthermore Π is finite. Indeed, $\mathbf{k}[x_1,\dots,x_k,y_1,\dots,y_n]$ is an integral extension of $\mathbf{k}[G_1(x_1,y),\dots$ $\dots,G_k(x_k,y),y_1,\dots,y_n]$ because

$$G_i(T,y)-G_i(x_i,y)\in\mathbf{k}[G_1(x_1,y),\dots,G_k(x_k,y),y_1,\dots,y_n][T]$$

is a monic polynomial which vanishes at x_i.

Finally, for each $(x',y')\in\mathbf{k}^k\times\mathbf{k}^n$ we have

$$\begin{aligned}\#\{(x,y)&\in\mathbf{k}^k\times\mathbf{k}^n:\Pi(x,y)=(x',y')\}\\&=\#\{(x,y)\in\mathbf{k}^k\times\mathbf{k}^n:y=y',\ G_i(x_i,y)=x_i'\text{ for }i=1,\dots,k\}\\&\le\deg G_1\cdot\ldots\cdot\deg G_k\le(\operatorname{gdeg}\pi)^k.\end{aligned}$$

Thus Π is a finite extension of π such that $\operatorname{gdeg}\Pi\le(\operatorname{gdeg}\pi)^k$. ∎

An example showing that the normality of $\pi(V)$ is necessary in Theorems 3.1 and 3.6 can be found in [4].

4. Proof of the main result. Let us recall some facts about embeddings.

Definition 4.1. A polynomial mapping $f:V\to\mathbf{k}^n$ is called an *embedding* if f is an isomorphism onto its image $f(V)=\overline{f(V)}$.

We have the following well known lemma (see e.g. [2]):

Lemma 4.2. *If $X\subset\mathbb{C}^n$ is a closed algebraic smooth set*, $\dim X=k$ *and* $n>2k+1$, *then we can change coordinates in such a way that the projection*

$$\phi:X\ni(x,y)\mapsto(0,y)\in0\times\mathbb{C}^{2k+1}$$

is an embedding.

We also have

Theorem 4.3 ([2], Thm. 1.2). *Let $X\subset\mathbb{C}^n$ be a closed algebraic set which is smooth and not necessarily irreducible of* (*not necessarily pure*) *dimension k. Let $\phi:X\to\mathbb{C}^n$ be an embedding. If $n\ge4k+2$ then there exists an isomorphism $\Phi:\mathbb{C}^n\to\mathbb{C}^n$ such that $\Phi|_X=\phi$.*

Now we are in a position to prove the main result.

Theorem 4.4. *Let $V,W\subset\mathbb{C}^n$ be smooth algebraic sets*, *and let $f:V\to W$ be a finite dominating mapping that is dominating on every irreducible component. If* $\dim V=\dim W=k$ *and* $4k+2\le n$, *then there exists a finite mapping $F:\mathbb{C}^n\to\mathbb{C}^n$ such that $F|_V=f$ and*

$$\operatorname{gdeg}F\le(\operatorname{gdeg}f)^{2k+1}.$$

P r o o f. By Lemma 4.2 we can assume that the projections

$$\phi_1 : V \to 0 \times \mathbb{C}^{2k+1} \quad \text{and} \quad \phi_2 : W \to 0 \times \mathbb{C}^{n-2k-1}$$

are embeddings. Put

$$\widetilde{V} = \phi_1(V), \quad \widetilde{W} = \phi_2(W) \quad \text{and} \quad \widetilde{f} = \phi_2 \circ f \circ \phi_1^{-1} : \widetilde{V} \to \widetilde{W}.$$

The mapping $\widetilde{f}$ is finite with $\operatorname{gdeg} \widetilde{f} = \operatorname{gdeg} f$.

Because $\widetilde{V} \subset 0 \times \mathbb{C}^{2k+1}$, $\widetilde{W} \subset 0 \times \mathbb{C}^{n-2k-1}$, we can consider the sets $\widetilde{V}$ and $\widetilde{W}$ as subsets of $\mathbb{C}^{2k+1}$ and $\mathbb{C}^{n-2k-1}$, respectively. Consider the isomorphism

$$\psi : \widetilde{V} \ni x \mapsto (x, \widetilde{f}(x)) \in \mathbb{C}^{2k+1} \times \mathbb{C}^{n-2k-1},$$

the set $\widehat{V} = \psi(\widetilde{V})$, and the projection

$$\pi : \widehat{V} \ni (x, y) \mapsto (0, y) \in 0 \times \mathbb{C}^{n-2k-1}.$$

We have $\widetilde{f} = \pi \circ \psi$, and ψ is an isomorphism, so π is finite and $\operatorname{gdeg} \pi = \operatorname{gdeg} \widetilde{f} = \operatorname{gdeg} f$. The set $\pi(\widehat{V}) = \widetilde{W}$ is smooth, because W is smooth, and by Theorem 3.6 there exists a finite mapping $\Pi : \mathbb{C}^{2k+1} \times \mathbb{C}^{n-2k-1} \to \mathbb{C}^{2k+1} \times \mathbb{C}^{n-2k-1}$ such that $\Pi|_{\widehat{V}} = \pi$ and

$$\operatorname{gdeg} \Pi \le (\operatorname{gdeg} \pi)^{2k+1}.$$

By Theorem 4.3 applied to $\psi : \widetilde{V} \to \widehat{V}$, $\phi_1 : V \to \widetilde{V}$ and $\phi_2 : W \to \widetilde{W}$ there exist isomorphisms $\Psi, \Phi_1, \Phi_2 : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\Psi|_{\widetilde{V}} = \psi, \qquad \Phi_1|_V = \phi_1, \qquad \Phi_2|_W = \phi_2.$$

Putting $F = \Phi_2^{-1} \circ \Pi \circ \Psi \circ \Phi_1$ we have a finite extension of f such that $\operatorname{gdeg} F \le (\operatorname{gdeg} f)^{2k+1}$. ■

Note that Theorem 4.4 is a generalization of Theorem 4.3. Indeed, if $f : V \to \mathbb{C}^n$ is an embedding and V is smooth then $f(V)$ is also smooth, f is a finite dominating mapping that is dominating on every component and $\operatorname{gdeg} f = 1$. By Theorem 4.4 there exists a finite mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ such that $F|_V = f$ and $\operatorname{gdeg} F = 1$. Now F is a birational mapping, and by Zariski's Main Theorem, it is an isomorphism.

Note, also, that if V is a pure dimensional set, then the assumption that f is dominating on every irreducible component is not necessary (see Remark 3.5).

By Theorem 4.4 we have

COROLLARY 4.5. *Under the assumptions of Theorem* 4.4 *there are infinitely many finite mappings* $F : \mathbb{C}^n \to \mathbb{C}^n$ *such that* $F|_V = f$ *and*

$$\operatorname{gdeg} F \le (\operatorname{gdeg} f)^{2k+1}.$$

P r o o f. Let $x \in \mathbb{C}^n \setminus V$. For any point $y \in \mathbb{C}^n \setminus W$ we define a mapping $f_y : V \cup \{x\} \to W \cup \{y\}$ such that $f_y|_V = f$ and $f_y(x) = y$. By Theorem 4.4

there exists a finite mapping $F_y : \mathbb{C}^n \to \mathbb{C}^n$ such that $F_y|_{V\cup\{x\}} = f_y$ and

$$\operatorname{gdeg} F_y \le (\operatorname{gdeg} f_y)^{2k+1} = (\operatorname{gdeg} f)^{2k+1}.$$

Obviously $F_y|_V = f$ and $F_y \neq F_{y'}$ for $y \neq y'$. ■

By Theorem 4.4 we also have

COROLLARY 4.6. *Let $V, W \subset \mathbb{C}^n$, where $n > 1$, be finite sets. For any mapping $f : V \to W$ there exists a finite mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ such that $F|_V = f$ and* $\operatorname{gdeg} F = \operatorname{gdeg} f$.

Because Lemma 4.2 and Theorem 4.3 are true for any algebraically closed field, we have

REMARK 4.7. Theorem 4.4 holds for any algebraically closed field of characteristic zero.

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Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: karas@im.uj.edu.pl

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