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On the Cartan–Norden theorem for affine Kähler immersions

by Maria Robaszewska (Kraków)

Abstract. In [O2] the Cartan–Norden theorem for real affine immersions was proved without the non-degeneracy assumption. A similar reasoning applies to the case of affine Kähler immersions with an anti-complex shape operator, which allows us to weaken the assumptions of the theorem given in [NP]. We need only require the immersion to have a non-vanishing type number everywhere on M.

1. Introduction. Let M be an n-dimensional complex manifold with a complex linear torsion-free connection ∇. A holomorphic immersion $f : M \to \mathbb{C}^{n+1}$ is called an *affine holomorphic immersion* if there exists a complex transversal bundle $\mathcal{N}$ such that ∇ is induced by its local sections in the following way. For $x \in M$ let ξ be a local section of $\mathcal{N}$ in the neighbourhood U of x (we require all local sections of $\mathcal{N}$ considered to be nowhere vanishing). Let X, Y be vector fields on U. Then we have the decomposition

$$D_X f_* Y = f_* \nabla_X Y + h(X,Y)\xi - h(JX,Y)J\xi,$$

known as the *Gauss formula*. We denote by J the complex structure on M as well as that on $\mathbb{C}^{n+1}$ and we write D for the standard connection on $\mathbb{C}^{n+1}$. The symmetric bilinear form h is called the *affine fundamental form*. The Gauss formula may also be written in the complex version

$$D_X f_* Y = f_* \nabla_X Y + h^c(X,Y)\xi$$

with $\mathbb{C}$-bilinear $h^c(X,Y) = h(X,Y) - ih(JX,Y)$. The (complex) rank of h^c is called the *type number* of f and denoted by tf.

The *shape operator* S and the *transversal forms* μ, ν are defined on U by the Weingarten formula:

$$D_X \xi = -f_* SX + \mu(X)\xi + \nu(X)J\xi.$$

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A connection ∇ whose curvature tensor satisfies the condition

$$R(JX,JY)=R(X,Y)\quad \text{for all vector fields } X,Y \tag{1}$$

is called an *affine Kähler connection* [NPP] and in this case an affine holomorphic immersion $f:(M,\nabla)\to(\mathbb{C}^{n+1},D)$ is said to be an *affine Kähler immersion*. The Cartan–Norden theorem deals with Kähler connections, which are known to have the property (1).

We recall the fundamental equations for an affine holomorphic immersion:

$$\begin{aligned}R(X,Y)Z&=h(Y,Z)SX-h(X,Z)SY-h(JY,Z)JSX+h(JX,Z)JSY\\&=h^c(Y,Z)SX-h^c(X,Z)SY \qquad (\textit{Gauss}),\end{aligned}$$

$$\begin{aligned}&(\nabla_X h)(Y,Z)+\mu(X)h(Y,Z)+\nu(X)h(JY,Z)\\&\quad=(\nabla_Y h)(X,Z)+\mu(Y)h(X,Z)+\nu(Y)h(JX,Z) \qquad (\textit{Codazzi I}),\end{aligned}$$

$$\begin{aligned}&(\nabla_X S)Y-\mu(X)SY-\nu(X)JSY\\&\quad=(\nabla_Y S)X-\mu(Y)SX-\nu(Y)JSX \qquad (\textit{Codazzi II}),\end{aligned}$$

$$h(X,SY)-h(Y,SX)=2d\mu(X,Y) \qquad (\textit{Ricci I}),$$

$$h(SX,JY)-h(SY,JX)=2d\nu(X,Y) \qquad (\textit{Ricci II}).$$

2. The Cartan–Norden theorem for affine holomorphic immersions with anti-complex shape operator. The theorem to be proved is the following:

THEOREM. *Let (M,g) be a connected, n-dimensional, pseudokählerian manifold with the Levi-Civita connection ∇ of g. Let $f:(M,\nabla)\to\mathbb{C}^{n+1}$ be an affine Kähler immersion such that the corresponding shape operator S is anti-complex.*

1) *If $tf>0$ on M, then either ∇ is flat or $R_x\neq 0$ for every $x\in M$.*

2) *If $R_x\neq 0$ for every $x\in M$, then there exists a Hermitian product G on $\mathbb{C}^{n+1}$ such that $f:(M,g)\to(\mathbb{C}^{n+1},G)$ is an isometric immersion.*

REMARK. The assumption that S is anti-complex is needed only if $tf=1$ everywhere on M, because if ∇ is affine Kähler and $tf>1$ at some point of M, then the shape operator must be anti-complex [O1].

Proof. We will first prove the following lemma:

LEMMA. *Let W be a connected component of the set $\{x\in M: R_x\neq 0\}$. Under the hypotheses of the theorem, there exists a Hermitian product G on $\mathbb{C}^{n+1}$ such that $f|_W:(W,g)\to(\mathbb{C}^{n+1},G)$ is an isometric immersion. Moreover, if ξ is a transversal vector field inducing the connection ∇ on an open set U and h, S are the affine fundamental form and the shape operator* *associated with* ξ *respectively*, *then*

$$h(X,Y) = \frac{1}{G(\xi,\xi)}\ g(SX,Y)$$

for all vector fields X, Y *on* $U \cap W$.

Proof. We have divided the proof into a sequence of steps, similar to those in [O2]. In Steps 1–6 we shall prove that there exists a non-vanishing function β on $U \cap W$ such that

$$h(X,Y) = \beta g(SX,Y)$$

on $U \cap W$, for all vector fields X, Y on $U \cap W$.

STEP 1. *If* $h_x \neq 0$, *then* $\operatorname{im} R_x = \operatorname{im} S_x$.

It follows from the Gauss equation that the complex subspace

$$\operatorname{im} R_x = \operatorname{span}\{R(X,Y)Z : X,Y,Z \in T_xM\}$$

is included in $\operatorname{im} S_x$. Let now $X \in T_xM$. If $X \in \ker h^c$, take Y, Z such that $h^c(Y,Z) \neq 0$. Then

$$SX = \frac{R(X,Y)Z}{h^c(Y,Z)} \in \operatorname{im} R_x.$$

If $X \notin \ker h^c$, take Y such that $h^c(X,Y) \neq 0$. Then

$$R(X,JX)JY = h^c(JX,JY)SX - h^c(X,JY)SJX = -2h^c(X,Y)SX,$$

so

$$SX = \frac{-R(X,JX)JY}{2h^c(X,Y)} \in \operatorname{im} R_x.$$

STEP 2. *If* $S_x \neq 0$, *then* $\ker h_x = \ker R_x$.

We only have to prove that $\ker h_x \supset \ker R_x$, where

$$\ker h_x = \{X \in T_xM : \forall Y \in T_xM,\ h(X,Y) = 0\} = \ker h^c_x,$$
$$\ker R_x = \{X \in T_xM : \forall Y,Z \in T_xM,\ R(Y,Z)X = 0\}.$$

Let $Z \in \ker R_x$ and let $Y \in T_xM$.

If $SY = 0$, then for arbitrary $X \in T_xM$,

$$0 = R(X,Y)Z = h^c(Y,Z)SX.$$

We can choose X such that $SX \neq 0$, so $h^c(Y,Z) = 0$.

If $SY \neq 0$, then the equality

$$0 = R(Y,JY)JZ = h^c(JY,JZ)SY - h^c(Y,JZ)SJY = -2h^c(Y,Z)SY$$

implies that $h^c(Y,Z) = 0$.

STEP 3. *If* $R_x \neq 0$, *then* $\ker h_x = \ker S_x = \ker R_x$.

From the Gauss equation we obtain $S_x \neq 0$ and $h_x \neq 0$, therefore $\ker h_x = \ker R_x$ and $\operatorname{im} R_x = \operatorname{im} S_x$. It remains to prove that $\ker h_x = \ker S_x$.

We first show that $\dim\ker S_x = \dim\ker h_x$.

If ∇ is the Levi-Civita connection for a Kählerian metric g, then for all X, Y, Z, W in T_xM we have $g(R(X,Y)Z,W) = -g(R(X,Y)W,Z)$, which implies

$$(\operatorname{im} R_x)^{\perp g} = \ker R_x$$

because of the non-degeneracy of g. For the non-degenerate g we have

$$\begin{aligned} 2n &= \dim_{\mathbb{R}}(\operatorname{im} R_x) + \dim_{\mathbb{R}}(\operatorname{im} R_x)^{\perp g} \\ &= \dim_{\mathbb{R}}(\operatorname{im} R_x) + \dim_{\mathbb{R}}(\ker R_x) \\ &= \dim_{\mathbb{R}}(\operatorname{im} S_x) + \dim_{\mathbb{R}}(\ker h_x). \end{aligned}$$

On the other hand we have

$$2n = \dim_{\mathbb{R}}(\operatorname{im} S_x) + \dim_{\mathbb{R}}(\ker S_x).$$

It now remains to prove the inclusion $\ker S_x \supset \ker h_x$.

Let $X \in \ker h_x$. The assertion of Step 2 implies that $X \in \ker R_x$. Recall that if ∇ is metrizable and torsion-free, then

$$\ker R_x = \{Y \in T_xM : \forall X \in T_xM,\ R(X,Y) = 0\}.$$

Consequently, for all Y, Z we have $R(X,Y)Z = 0$. Using the Gauss equation we obtain

$$0 = h^c(Y,Z)SX - h^c(X,Z)SY = h^c(Y,Z)SX$$

for all Y, Z. Since $h_x \neq 0$, we can find Y, Z such that $h^c(Y,Z) \neq 0$, therefore SX must equal 0.

Step 4. *If $R_x \neq 0$, then $SX = 0$ implies that $g(SZ,X) = 0$ for all $Z \in T_xM$.*

Let $X \in \ker S_x = (\operatorname{im} R_x)^{\perp g}$. Then for all $Y, Z \in T_xM$ we have

$$g(R(JZ,Z)Y,X) = 0 \quad \text{and} \quad g(R(JZ,Z)JY,X) = 0.$$

Using the Gauss equation we obtain

$$\begin{aligned} &h(Z,Y)g(SJZ,X) - h(Z,JY)g(SZ,X) = 0, \\ &h(Z,JY)g(SJZ,X) + h(Z,Y)g(SZ,X) = 0 \end{aligned} \tag{2}$$

for all $Y, Z \in T_xM$. Let $Z \in T_xM$. If $Z \in \ker h$, then, by Step 3, $Z \in \ker S$, so $g(SZ,X) = 0$. If $Z \notin \ker h$, then there exists $Y \in T_xM$ such that $h(Z,Y) \neq 0$. If we now consider (2), we obtain $g(SZ,X) = g(SJZ,X) = 0$.

Step 5. *If $R_x \neq 0$, then $h(X,Z) = 0$ implies $g(SX,Z) = 0$.*

Let $h(X,Z) = 0$. There are three possibilities:

(i) $h(JZ,X) \neq 0$,
(ii) $h(JZ,X) = 0$, but $Z \notin \ker h$,
(iii) $Z \in \ker h$.

If (i) holds, then from the equalities $g(R(X,JX)Z,Z)=0$, $h(X,Z)=0$ we obtain $h(JX,Z)g(SX,Z)=0$. Hence $g(SX,Z)=0$.

If (ii) holds, then we take Y such that $h(Y,Z)\neq 0$. From

$$g(R(X,Y)Z,Z)=0 \quad \text{and} \quad g(R(X,JY)Z,Z)=0$$

we obtain

$$h(Y,Z)g(SX,Z)+h(JY,Z)g(SJX,Z)=0,$$
$$h(JY,Z)g(SX,Z)-h(Y,Z)g(SJX,Z)=0,$$

which together with $h(Y,Z)\neq 0$ yields $g(SX,Z)=g(SJX,Z)=0$.

If (iii) holds, then, by Step 3, $Z\in\ker S$. Hence $g(SX,Z)=0$ by Step 4.

STEP 6. *There exists a non-vanishing function β on $U\cap W$ such that*

$$h(X,Y)=\beta g(SX,Y) \tag{3}$$

on $U\cap W$, for all vector fields X, Y on $U\cap W$.

Let $x\in U\cap W$. The forms $h_x(\cdot,\cdot)$ and $g_x(S_x\cdot,\cdot)$ are bilinear and for all $X,Y\in T_xM$ if $h_x(X,Y)=0$, then $g_x(S_xX,Y)=0$. If that is the case, one can find $\lambda_x\in\mathbb{R}$ such that $g_x(S_x\cdot,\cdot)=\lambda_x h_x(\cdot,\cdot)$. Moreover, $\lambda_x\neq 0$ because of the non-degeneracy of g and the fact that $S_x\neq 0$. For $x\in U\cap W$ we now define $\beta_x=1/\lambda_x$. As a function of x, β is smooth, because for every $x\in U\cap W$ we can find vector fields X,Y such that $g(SX,Y)\neq 0$ on a neighbourhood $U'\subset U\cap W$ of x, and then write

$$\beta=h(X,Y)/g(SX,Y).$$

STEP 7. *If $h(X,Y)=\beta g(SX,Y)$ on some open subset of $\{x\in M: R_x\neq 0\}$, then β satisfies the following equation*:

$$d\beta+2\beta\mu=0.$$

The proof is the same as in the case of anti-holomorphic transversal bundle in [NP], the difference is that now we do not have the condition $\nu(X)=\mu(JX)$.

From the first Codazzi equation we have, for all X,Y,Z,

$$\begin{aligned}
0 &= (\nabla_X h)(Y,Z)+\mu(X)h(Y,Z)+\nu(X)h(JY,Z)\\
&\quad -(\nabla_Y h)(X,Z)-\mu(Y)h(X,Z)-\nu(Y)h(JX,Z)\\
&= X(h(Y,Z))-h(\nabla_X Y,Z)-h(Y,\nabla_X Z)\\
&\quad +\mu(X)h(Y,Z)+\nu(X)h(JY,Z)\\
&\quad -Y(h(X,Z))+h(\nabla_Y X,Z)+h(X,\nabla_Y Z)\\
&\quad -\mu(Y)h(X,Z)-\nu(Y)h(JX,Z)\\
&= X(\beta g(SY,Z))-\beta g(S(\nabla_X Y),Z)-\beta g(SY,\nabla_X Z)\\
&\quad +\mu(X)\beta g(SY,Z)+\nu(X)\beta g(SJY,Z)
\end{aligned}$$

$$
\begin{aligned}
&- Y(\beta g(SX,Z)) + \beta g(S(\nabla_Y X),Z) + \beta g(SX,\nabla_Y Z)\\
&- \mu(Y)\beta g(SX,Z) - \nu(Y)\beta g(SJX,Z)\\
={}& X(\beta)g(SY,Z) + \beta g((\nabla_X S)(Y),Z)\\
&- \mu(X)\beta g(SY,Z) - \nu(X)\beta g(JSY,Z) + 2\mu(X)\beta g(SY,Z)\\
&- Y(\beta)g(SX,Z) - \beta g((\nabla_Y S)(X),Z)\\
&+ \mu(Y)\beta g(SX,Z) + \nu(Y)\beta g(JSX,Z) - 2\mu(Y)\beta g(SX,Z)\\
={}& (X(\beta)+2\mu(X)\beta)g(SY,Z) - (Y(\beta)+2\mu(Y)\beta)g(SX,Z)\\
&+ \beta g((\nabla_X S)(Y) - \mu(X)SY - \nu(X)JSY, Z)\\
&- \beta g((\nabla_Y S)(X) - \mu(Y)SX - \nu(Y)JSX, Z)\\
={}& (X(\beta)+2\mu(X)\beta)g(SY,Z) - (Y(\beta)+2\mu(Y)\beta)g(SX,Z),
\end{aligned}
$$

in which the last equality follows from the second Codazzi equation. We have

$$g((d\beta + 2\beta\mu)(X)SY - (d\beta + 2\beta\mu)(Y)SX, Z) = 0$$

for all X, Y, Z and from the non-degeneracy of g it follows that

$$(d\beta + 2\beta\mu)(X)SY - (d\beta + 2\beta\mu)(Y)SX = 0 \tag{4}$$

for all X, Y. We claim that

$$(d\beta + 2\beta\mu)(X) = 0 \tag{5}$$

for all X.

Let $X \in T_xM$. If $SX = 0$, then $(d\beta+2\beta\mu)(X)SY = 0$ for all $Y \in T_xM$. Since $S_x \neq 0$, we can find Y such that $SY \neq 0$. Hence (5) holds. If $SX \neq 0$, then we write (4) with $Y = JX$. From the linear independence of SX and $SJX = -JSX$ we conclude that (5) holds.

Using the function β we now define a Hermitian product G_x^ξ on $\mathbb{C}^{n+1}$:

$$
\begin{aligned}
G_x^\xi(f_*X, f_*Y) &= g(X,Y),\\
G_x^\xi(\xi_x,\xi_x) &= G_x^\xi(J\xi_x, J\xi_x) = 1/\beta_x,\\
G_x^\xi(f_*X,\xi_x) &= G_x^\xi(f_*X, J\xi_x) = G_x^\xi(\xi_x, J\xi_x) = 0
\end{aligned}
$$

for each $x \in U \cap W$.

It is easy to show that if we have two local sections of the transversal bundle $\mathcal{N}$, ξ and $\widetilde{\xi} = \phi\xi + \psi J\xi$, both defined on some open set U, then on $U \cap W$ we have $G^\xi = G^{\widetilde{\xi}}$. Indeed, let $x \in U \cap W$. Compare the values of G_x^ξ and $G_x^{\widetilde{\xi}}$, taking all the vectors f_*X with $X \in T_xM$ and $\widetilde{\xi}_x$ as generators of $\mathbb{C}^{n+1}$:

$$
\begin{aligned}
G_x^\xi(f_*X, f_*Y) &= g(X,Y) = G_x^{\widetilde{\xi}}(f_*X, f_*Y),\\
G_x^\xi(f_*X, \widetilde{\xi}_x) &= G_x^\xi(f_*X, \phi(x)\xi_x + \psi(x)J\xi_x) = 0 = G_x^{\widetilde{\xi}}(f_*X, \widetilde{\xi}_x),
\end{aligned}
$$

$$G^{\xi}_x(f_*X, J\widetilde{\xi}_x) = -G^{\xi}_x(f_*JX, \widetilde{\xi}_x) = 0 = G^{\widetilde{\xi}}_x(f_*X, J\widetilde{\xi}_x),$$

$$G^{\xi}_x(\widetilde{\xi}_x, \widetilde{\xi}_x) = G^{\xi}_x(\phi(x)\xi_x + \psi(x)J\xi_x, \phi(x)\xi_x + \psi(x)J\xi_x) = \frac{(\phi^2+\psi^2)(x)}{\beta_x},$$

whereas

$$G^{\widetilde{\xi}}_x(\widetilde{\xi}_x, \widetilde{\xi}_x) = 1/\widetilde{\beta}_x.$$

It remains to show that

$$\widetilde{\beta} = \frac{\beta}{\phi^2+\psi^2}. \tag{6}$$

Recall that we have

$$\widetilde{h}(X,Y) = \frac{\phi h(X,Y) - \psi h(JX,Y)}{\phi^2+\psi^2}, \qquad \widetilde{S}X = \phi SX + \psi JSX.$$

If we put $h(X,Y) = \beta g(SX,Y)$ into the first equation and make use of the anti-complexity of S, we obtain

$$\widetilde{h}(X,Y) = \frac{\beta}{\phi^2+\psi^2} g(\phi SX + \psi JSX, Y).$$

From the other equation it follows that

$$\widetilde{h}(X,Y) = \frac{\beta}{\phi^2+\psi^2} g(\widetilde{S}X, Y).$$

Comparing this with the equation $(\widetilde{3})$ defining $\widetilde{\beta}$ we have (6).

We can now define for $x \in W$ a Hermitian product G_x on $\mathbb{C}^{n+1}$: $G_x := G^{\xi}_x$, where ξ is a local section of the transversal bundle $\mathcal{N}$ whose domain contains x.

Step 8. *G is parallel relative to D.*

Let $X \in TM$. For $Y, Z \in \mathcal{X}(M)$ we have

$$\begin{aligned}
(D_XG)(f_*Y, f_*Z) &= X(G(f_*Y, f_*Z)) - G(D_Xf_*Y, f_*Z) - G(f_*Y, D_Xf_*Z) \\
&= X(G(f_*Y, f_*Z)) \\
&\quad - G(f_*\nabla_XY + h(X,Y)\xi - h(JX,Y)J\xi, f_*Z) \\
&\quad - G(f_*Y, f_*\nabla_XZ + h(X,Z)\xi - h(JX,Z)J\xi) \\
&= X(g(Y,Z)) - g(\nabla_XY, Z) - g(Y, \nabla_XZ) \\
&= (\nabla_Xg)(Y,Z) = 0,
\end{aligned}$$

$$\begin{aligned}
(D_XG)(f_*Y, \xi) &= X(G(f_*Y, \xi)) - G(D_Xf_*Y, \xi) - G(f_*Y, D_X\xi) \\
&= -G(f_*\nabla_XY + h(X,Y)\xi - h(JX,Y)J\xi, \xi) \\
&\quad - G(f_*Y, -f_*SX + \mu(X)\xi + \nu(X)J\xi) \\
&= -h(X,Y)\frac{1}{\beta} + g(Y, SX) = 0,
\end{aligned}$$

$$\begin{aligned}(D_X G)(\xi,\xi) &= X(G(\xi,\xi)) - 2G(D_X\xi,\xi)\\ &= X(G(\xi,\xi)) - 2G(-f_*SX + \mu(X)\xi + \nu(X)J\xi,\xi)\\ &= X\left(\frac{1}{\beta}\right) - 2\mu(X)\frac{1}{\beta}\\ &= -\frac{1}{\beta^2}(X(\beta) + 2\beta\mu(X)) = 0.\end{aligned}$$

Since $D_X G$ is hermitian as G is, we conclude that $D_X G = 0$. As W is assumed to be connected, the map $x \mapsto G_x$ is constant on W. The proof of the lemma is now complete. ∎

In Steps 9 and 10 we shall prove 1) of the theorem.

STEP 9. *If $h_x \neq 0$, then $R_x = 0$ implies $S_x = 0$.*

Let $e_1, \ldots, e_n$ be a complex basis of $T_x M$ such that $h^c(e_i, e_i) = 1$ for $i \in \{1, \ldots, r\}$, $h^c(e_i, e_i) = 0$ for $i \in \{r+1, \ldots, n\}$ and $h^c(e_i, e_j) = 0$ for $i \neq j$. We have $r > 0$ since $h_x \neq 0$.

Using the Gauss equation we obtain $0 = R(e_j, e_1)e_1 = Se_j$ for $j \neq 0$ and $0 = R(e_1, Je_1)Je_1 = -2Se_1$.

STEP 10. *The connected components of the set $\{x \in M : R_x \neq 0\}$ are closed in M.*

Let W be a non-empty connected component of $\{x \in M : R_x \neq 0\}$ in M. According to the lemma there is a Hermitian product G on $\mathbb{C}^{n+1}$ such that $f|_W : (W, g) \to (\mathbb{C}^{n+1}, G)$ is an isometric immersion. Let $y \in \overline{W}$. Let U be a neighbourhood of y with a local section ξ of the transversal bundle $\mathcal{N}$ and the corresponding $\mathcal{C}^\infty$ objects h, S on U. On $U \cap W$ we then have a smooth function $\beta : U \cap W \to \mathbb{R}\backslash\{0\}$ such that $\beta = 1/G(\xi,\xi)$ on $U \cap W$ and $h(X,Y) = \beta g(SX, Y)$ for vector fields X, Y on $U \cap W$.

Define

$$\alpha : U \ni x \mapsto G(\xi_x, \xi_x) \in \mathbb{R}.$$

We claim that

(i) $\alpha \neq 0$ on $\overline{W} \cap U$,

(ii) if X, Y are vector fields on U, then

$$\alpha h(X,Y) = g(SX, Y) \quad \text{on } \overline{W} \cap U.$$

Let $z \in \overline{W} \cap U$. Let V be an open neighbourhood of z such that $\overline{V} \subset U$; $\overline{W} \cap V \subset \overline{W \cap V}$, since V is open; $\overline{W \cap V} \subset \overline{W} \cap \overline{V} \subset \overline{W} \cap U \subset U$.

For arbitrary vector fields X, Y we define on U the continuous functions $\psi_1^{XY} := \alpha h(X,Y) - g(SX,Y)$ and $\psi_2^X := G(f_*X, \xi)$. Since ψ_1^{XY} and ψ_2^X vanish on $W \cap V$, we also have $\psi_1^{XY}(\overline{W \cap V}) = \{0\}$ and $\psi_2^X(\overline{W \cap V}) = \{0\}$.

In particular, they vanish at the point z. Since z was an arbitrary point from $\overline{W} \cap U$, we have (i) and (ii).

We aim to show that $y \in W$. Since $h_y \neq 0$, we can find vectors $X_y, Y_y \in T_yM$ such that $h(X_y, Y_y) \neq 0$ and then from the equality

$$\alpha_y h(X_y, Y_y) = g(S_y X_y, Y_y)$$

we conclude that $S_y \neq 0$. Step 9 now implies $R_y \neq 0$. If W_1 denotes the connected component of $\{x \in M : R_x \neq 0\}$ containing y, then obviously $W_1 = W$, so $y \in W$.

From Step 10 and the assumed connectedness of M it follows that if ∇ is not flat, then $R_x \neq 0$ for every $x \in M$. The second assertion of the theorem follows immediately from the lemma upon taking $W = M$. ■

References

[NPP] K. Nomizu, U. Pinkall and F. Podesta, *On the geometry of affine Kähler immersions*, Nagoya Math. J. 120 (1990), 205–222.

[NP] K. Nomizu and F. Podesta, *On the Cartan–Norden theorem for affine Kähler immersions*, Nagoya Math. J. 121 (1991), 127–135.

[O1] B. Opozda, *On some properties of the curvature and Ricci tensors in complex affine geometry*, Geom. Dedicata 55 (1995), 141–163.

[O2] —, *On the Cartan–Norden theorem*, Math. Z. 226 (1997), 309–316.

Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: robaszew@im.uj.edu.pl

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