On coefficient inequalities in the Carathéodory class of functions

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Abstract. Some inequalities are proved for coefficients of functions in the class $P(\alpha)$, where $\alpha \in [0, 1)$, of functions with real part greater than $\alpha$. In particular, new inequalities for coefficients in the Carathéodory class $P(0)$ are given.

1. Main results. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk. We denote by $P(\alpha)$, where $\alpha \in [0, 1)$, the class of functions $p$ regular in $D$ of the form

\begin{equation}
    p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in D,
\end{equation}

such that $\text{Re} p(z) > \alpha$ for $z \in D$ (see [2, p. 105]). The well known class of Carathéodory functions having positive real part in $D$, denoted by $P$, coincides with $P(0)$. The class $P(\alpha)$, although not explicitly defined, appeared first in [4], where Robertson defined functions convex of order $\alpha$ and starlike of order $\alpha$.

Using the well known estimates $|p_n| \leq 2, \ n \in \mathbb{N}$, [1; 2, p. 80] for the coefficients of $p \in P$ it is easy to prove the lemma below (see [2, p. 101]).

Lemma 1.1. Fix $\alpha \in [0, 1)$ and let $q$ of the form

\begin{equation}
    q(z) = q_0 + \sum_{n=1}^{\infty} q_n z^n, \quad z \in D,
\end{equation}

be regular in $D$. If $\text{Re} q(z) > \alpha$ for $z \in D$, then

\begin{equation}
    |q_n| \leq 2(\text{Re} q_0 - \alpha), \quad n \in \mathbb{N}.
\end{equation}

Estimates (1.3) are sharp.

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An extremal function for which equalities hold in (1.3) is
\[ q(z) = \frac{q_0 + (\overline{q}_0 - 2\alpha)z}{1 - z}, \quad z \in \mathbb{D}. \]

**Remark 1.2.** For \( \alpha = 0 \) we have \( |q_n| \leq 2 \, \text{Re} \, q_0, \ n \in \mathbb{N}, \) for the coefficients \( q_n \) of a function \( q \) such that \( \text{Re} \, q(z) > 0 \) for \( z \in \mathbb{D} \).

An interesting generalization of Remark 1.2 can be found in [3].

As an immediate consequence of (1.3) we have the following estimates for the coefficients of \( p \in P(\alpha) \) of the form (1.1) which can be found in [4, p. 386]:
\[ |p_n| \leq 2(1 - \alpha), \quad n \in \mathbb{N}. \quad (1.4) \]

Now we formulate two basic theorems of this paper.

**Theorem 1.3.** Fix \( \alpha \in [0, 1) \) and \( \xi \in \mathbb{D} \). If \( p \in P(\alpha) \), then the function
\[ q(z) = q(\xi; z) = \frac{\xi - \overline{\xi}z[(1 - 2\alpha)z + \alpha \xi]}{z} + \frac{(z - \xi)(1 - \overline{\xi}z)}{z} p(z), \]
\( z \in \mathbb{D}, \) is regular in \( \mathbb{D} \) and
\[ \text{Re} \, q(\xi; z) \geq \alpha, \quad z \in \mathbb{D}. \quad (1.6) \]
Equality holds in (1.6) only if \( |\xi| = 1 \) and
\[ p(z) = p(\alpha, \xi; z) = \frac{1 + (1 - 2\alpha)\overline{\xi}z}{1 - \overline{\xi}z}, \quad z \in \mathbb{D}. \quad (1.7) \]

**Proof.** Observe first that the function (1.5) has a removable singularity at \( z = 0 \) since
\[ q(\xi; 0) = \lim_{z \to 0} q(\xi; z) = 1 + (1 - \alpha)|\xi|^2 - \xi p_1. \quad (1.8) \]

Assume first that \( p \), and hence \( q \), is regular on \( \partial \mathbb{D} \). For \( z = e^{i\theta}, \ \theta \in \mathbb{R}, \) we have
\[ q(\xi; e^{i\theta}) = 2i \, \text{Im}(\xi e^{-i\theta}) - \alpha|\xi|^2 + 2\alpha \overline{\xi} e^{i\theta} + [1 + |\xi|^2 - 2 \, \text{Re}(\xi e^{-i\theta})] p(e^{i\theta}). \]
Since \( p \in P(\alpha) \) we see that
\[ \text{Re} \, q(\xi; e^{i\theta}) = - \alpha|\xi|^2 + 2\alpha \, \text{Re}(\overline{\xi} e^{i\theta}) + [(1 + |\xi|^2 - 2 \, \text{Re}(\xi e^{-i\theta})] \, \text{Re}(p(e^{i\theta}) \geq \alpha + 2\alpha(\text{Re}(\overline{\xi} e^{i\theta}) - \text{Re}(\xi e^{-i\theta})) = \alpha. \]

By the minimum principle for harmonic functions the above inequality holds in \( \mathbb{D} \), i.e. \( \text{Re} \, q(\xi; z) \geq \alpha \) for \( z \in \mathbb{D} \).

If \( p \) is not regular on \( \partial \mathbb{D} \), then we consider the functions \( p_r(z) = p(rz), \ z \in \mathbb{D}, \) for \( r \in (0, 1) \). Replacing \( p \) by \( p_r \) on the right hand side of (1.5) we obtain the corresponding function \( q_r(\xi; z), \ z \in \mathbb{D}. \) Repeating the above
considerations we get the strict inequality \( \text{Re} q_r(\xi; z) > \alpha \) for \( z \in \mathbb{D} \). Letting \( r \to 1 \) we see that \( p_r \to p \) and \( q_r \to q \). Consequently, \( \text{Re} q(\xi; z) \geq \alpha \) for \( z \in \mathbb{D} \).

If \( |\xi| < 1 \), then from (1.5) we have \( q(\xi; \xi) = 1 - (1 - \alpha)|\xi|^2 > \alpha \). Hence \( \text{Re} q(\xi; z) > \alpha \) for \( z \in \mathbb{D} \) and \( |\xi| < 1 \).

If \( |\xi| = 1 \), then from (1.5) we deduce that

\[
\text{Re} q(\xi; 0) - \alpha = (1 - \alpha)(1 + |\xi|^2) - \text{Re}(\xi p_1) \\
\geq (1 - \alpha)(1 + |\xi|^2) - |\xi p_1| \geq (1 - \alpha)(1 - |\xi|)^2 = 0
\]

for all \( \alpha \in [0, 1) \). Equality holds only if \( \text{Re}(\xi p_1) = |\xi p_1| \) and \( |p_1| = 2(1 - \alpha) \).

Hence \( p_1 = 2(1 - \alpha) \xi \), which holds only for the function \( p \) defined by (1.7).

Then \( q(\xi; z) = \alpha \) for each \( \xi \in \partial \mathbb{D} \) and all \( z \in \mathbb{D} \).

Now we prove the converse theorem for \( \xi \in \mathbb{D} \).

**Theorem 1.4.** Fix \( \alpha \in [0, 1) \) and \( \xi \in \mathbb{D} \). Assume that \( q \) is regular in \( \mathbb{D} \), \( \text{Re} q(z) > \alpha \) for \( z \in \mathbb{D} \) and

\[
q(\xi) = 1 - (1 - \alpha)|\xi|^2.
\]

Then the function

\[
p(z) = p(\xi; z) = \frac{z}{(z - \xi)(1 - \overline{\xi}z)} \left( q(z) - \frac{\xi - \overline{\xi}z[(1 - 2\alpha)z + \alpha \xi]}{z} \right), \quad z \in \mathbb{D},
\]

is regular in \( \mathbb{D} \) and \( p \in P(\alpha) \).

**Proof.** Simple calculations lead to

\[
p(\xi; \xi) = \lim_{z \to \xi} p(\xi; z) = \frac{1 + (1 - 2\alpha)|\xi|^2 + \xi q'(\xi)}{1 - |\xi|^2}
\]

so at \( z = \xi \) the function \( p \) has a removable singularity. Moreover \( p(\xi; 0) = 1 \) for each \( \xi \in \mathbb{D} \). Therefore \( p \) is regular in \( \mathbb{D} \) and of the form (1.1) for each \( \xi \in \mathbb{D} \).

Now we prove that \( p \in P(\alpha) \). Assume first that \( q \) is regular on \( \partial \mathbb{D} \).

By (1.10), so is \( p \), and for \( z = e^{i\theta} \), \( \theta \in \mathbb{R} \), we have

\[
p(\xi; e^{i\theta}) = \frac{1}{1 + |\xi|^2 - 2 \text{Re}(\xi e^{-i\theta})} (q(e^{i\theta}) + \alpha |\xi|^2 - 2\alpha \overline{\xi} e^{i\theta} - 2i \text{Im}(\overline{\xi} e^{i\theta})).
\]

Since \( \text{Re} q(z) \geq \alpha \) for \( z \in \partial \mathbb{D} \) we obtain

\[
\text{Re} p(\xi; e^{i\theta}) = \frac{1}{1 + |\xi|^2 - 2 \text{Re}(\xi e^{-i\theta})} \left( \text{Re} q(e^{i\theta}) + \alpha |\xi|^2 - 2\alpha \text{Re}(\overline{\xi} e^{i\theta}) \right) \\
\geq \frac{1 + |\xi|^2 - 2 \text{Re}(\overline{\xi} e^{i\theta})}{1 + |\xi|^2 - 2 \text{Re}(\xi e^{-i\theta})} \alpha = \alpha.
\]
By the minimum principle for harmonic functions the above inequality is true in \(\mathbb{D}\), i.e. \(\text{Re} p(\xi; z) \geq \alpha\) for \(z \in \mathbb{D}\).

If \(q\) is not regular on \(\partial \mathbb{D}\), then arguing as in the part of Theorem 1.3 concerning \(p_r\) (setting \(q_r(z) = q(rz)\), \(z \in \mathbb{D}\), for \(r \in (0, 1)\), and using (1.10) in place of (1.5)) we obtain \(\text{Re} p(\xi; z) \geq \alpha\) for \(z \in \mathbb{D}\).

Finally, recall that \(p(\xi; 0) = 1\) for each \(\xi \in \mathbb{D}\). This implies that \(\text{Re} p(\xi; z) > \alpha\) for \(z \in \mathbb{D}\) and \(\xi \in \mathbb{D}\). Therefore \(p \in P(\alpha)\).

For \(\alpha = 0\) we obtain from Theorems 1.3 and 1.4 the following results.

**Corollary 1.5.** Fix \(\xi \in \mathbb{D}\). If \(p \in P\), then the function

\[
q(z) = q(\xi; z) = \frac{\xi - \overline{\xi} z^2}{z} + \frac{(z - \xi)(1 - \overline{\xi} z)}{z} p(z), \quad z \in \mathbb{D},
\]

is regular in \(\mathbb{D}\) and \(\text{Re} q(\xi; z) \geq 0\) for \(z \in \mathbb{D}\). Equality holds only if \(|\xi| = 1\) and

\[
p(z) = p(0, \xi; z) = \frac{1 + \overline{\xi} z}{1 - \overline{\xi} z}, \quad z \in \mathbb{D}.
\]

**Corollary 1.6.** Fix \(\xi \in \mathbb{D}\). Assume that \(q\) is regular in \(\mathbb{D}\), \(\text{Re} q(z) > 0\) for \(z \in \mathbb{D}\) and \(q(\xi) = 1 - |\xi|^2\). Then the function

\[
p(z) = p(\xi; z) = \frac{z}{(z - \xi)(1 - \overline{\xi} z)} \left( q(z) - \frac{\xi - \overline{\xi} z^2}{z} \right), \quad z \in \mathbb{D},
\]

is regular in \(\mathbb{D}\) and \(p \in P\).

For \(\xi = 1\) Corollary 1.5 is due to Robertson [5].

**2. Applications.** In this section we apply Theorem 1.3 to obtain some inequalities for coefficients of functions in the class \(P(\alpha)\). In the case when \(\alpha = 0\) these results generalize the well known estimates for coefficients of functions in the Carathéodory class.

**Theorem 2.1.** Fix \(\alpha \in [0, 1)\) and \(\xi \in \mathbb{D}\). If \(p \in P(\alpha)\) and \(p\) is of the form (1.1), then

\[
(2.1) \quad |\xi p_2 - (1 + |\xi|^2)p_1 + 2(1 - \alpha)| \leq 2((1 - \alpha)(1 + |\xi|^2) - \text{Re}(\xi p_1)) ,
\]

\[
(2.2) \quad |\xi p_{n+1} - (1 + |\xi|^2)p_n + \overline{\xi} p_{n-1}| \leq 2((1 - \alpha)(1 + |\xi|^2) - \text{Re}(\xi p_1)) ,
\]

\[
(2.3) \quad ||\xi [p_{n+1} - p_n] - |\xi| \cdot ||\xi [p_n - p_{n-1}]| \leq 2((1 - \alpha)(1 + |\xi|^2) - |\xi| \text{Re} p_1) \quad \text{for} \quad n = 2, 3, \ldots
\]

Estimates (2.1)–(2.3) are sharp.

**Proof.** By Theorem 1.3 the function \(q\) defined by (1.5) is regular in \(\mathbb{D}\) and \(\text{Re} q(\xi; z) \geq \alpha\) for \(z \in \mathbb{D}\). We can assume that \(q\) is of the form (1.2).
From (1.5) we have

\[ zq(\xi; z) = [1 + (1 - \alpha)|\xi|^2 - \xi p_1]z + [-\xi p_2 + (1 + |\xi|^2)p_1 - 2(1 - \alpha)\xi]z^2 + \ldots \]
\[ + [-\xi p_n + (1 + |\xi|^2)p_{n-1} - \bar{\xi}p_{n-2}]z^n + \ldots \]

Consequently,

\[ q_0 = 1 + (1 - \alpha)|\xi|^2 - \xi p_1, \quad q_1 = -\xi p_2 + (1 + |\xi|^2)p_1 - 2(1 - \alpha)\xi \]

and

\[ q_n = -\xi p_n + (1 + |\xi|^2)p_{n-1} - \bar{\xi}p_{n-2} \quad \text{for } n = 2, 3, \ldots \]

Now using (1.3) and the formula for \( q_0 \) we obtain (2.1) and (2.2).

To prove (2.3) assume that \( \xi = |\xi|e^{i\varphi}, \varphi \in [0, 2\pi) \). Since \( p \in P(\alpha) \), the function \( p(e^{-i\varphi}z) \), \( z \in \mathbb{D} \), also belongs to \( P(\alpha) \), and applying (2.2) to it we have

\[ ||\xi||_{P_{n+1} - p_n} - |\xi| \cdot ||\xi||_{p_n - p_{n-1}} \cdot ||\xi||_{p_n - p_{n-1}} = 1 + 2(1 - \alpha)(1 + |\xi|^2) - |\xi|Re(p_1) \]

The function

\[ p(\alpha, 1; z) = \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha) \sum_{n=1}^{\infty} z^n, \quad z \in \mathbb{D}, \]

is in \( P(\alpha) \) and gives equalities in (2.1)–(2.3).

For \( \alpha = 0 \) the above yields the following result.

**Corollary 2.2.** Fix \( \xi \in \mathbb{D} \). If \( p \in P \) and \( p \) is of the form (1.1), then

\[ ||\xi||_{p_2 - (1 + |\xi|^2)p_1 + 2\xi} \leq 2(1 + |\xi|^2 - Re(\xi p_1)), \]

\[ ||\xi||_{p_{n+1} - (1 + |\xi|^2)p_n + \bar{\xi}p_{n-1}} \leq 2(1 + |\xi|^2 - Re(\xi p_1)), \]

\[ ||\xi||_{p_{n+1} - p_n} - |\xi| \cdot ||\xi||_{p_n - p_{n-1}} \cdot ||\xi||_{p_n - p_{n-1}} \leq 2(1 + |\xi|^2 - |\xi|Re(p_1)) \]

for \( n = 2, 3, \ldots \) Estimates (2.5)–(2.7) are sharp.

**Corollary 2.3.** If \( p \in P \) and \( p \) is of the form (1.1), then

\[ |p_2 - 2p_1 + 2| \leq 2Re(2 - p_1), \]
\[ |p_{n+1} - 2p_n + p_{n-1}| \leq 2(2 - Re(p_1)), \]
\[ |p_2 + 2p_1 + 2| \leq 2(2 + Re(p_1)), \]
\[ |p_{n+1} + 2p_n + p_{n-1}| \leq 2(2 + Re(p_1)), \]
\[(2.10)\] 
\[|p_2 + 2ip_1 - 2| \leq 2(2 + \text{Im} p_1),\]
\[|p_{n+1} + 2ip_n - p_{n-1}| \leq 2(2 + \text{Im} p_1),\]
\[(2.11)\] 
\[|p_2 - 2ip_1 - 2| \leq 2(2 - \text{Im} p_1),\]
\[|p_{n+1} - 2ip_n - p_{n-1}| \leq 2(2 - \text{Im} p_1),\]
for \(n = 2, 3, \ldots\) All estimates are sharp.

**Proof.** Estimates (2.8) follow from (2.5) and (2.6) by setting \(\xi = 1\).

Setting \(\xi = i\) and \(\xi = -i\) in (2.5) and (2.6) we obtain (2.9). Analogously, setting \(\xi = i\) and \(\xi = -i\) in (2.5) and (2.6) we get (2.10) and (2.11), respectively. The function \(p(z) = p(0, \xi; z), z \in \mathbb{D}\), defined by (1.6), for suitable \(\xi\) as above, is extremal for the cases considered.

Taking \(|\xi| = 1\) in (2.7) we have

**Corollary 2.4.** If \(p \in P\) and \(p\) is of the form (1.1), then

\[|p_{n+1} - p_n| - |p_n - p_{n-1}| \leq 2(2 - \text{Re} p_1)\]
for \(n = 2, 3, \ldots\) The estimates are sharp.

Setting \(\xi = 1/n\) and \(\xi = 1 - 1/n, n = 2, 3, \ldots\), in (2.6) we have respectively:

**Corollary 2.5.** If \(p \in P\) and \(p\) is of the form (1.1), then

\[|p_{n+1} - \left(\frac{n+1}{n}\right)p_n + p_{n-1}| \leq 2\left(\frac{n+1}{n} - \text{Re} p_1\right)\]
for \(n = 2, 3, \ldots\) The estimates are sharp.

**Corollary 2.6.** If \(p \in P\) and \(p\) is of the form (1.1), then

\[|p_{n+1} - \left(\frac{n}{n-1} + \frac{n-1}{n}\right)p_n + p_{n-1}| \leq 2\left(\frac{n}{n-1} + \frac{n-1}{n} - \text{Re} p_1\right)\]
for \(n = 2, 3, \ldots\) The estimates are sharp.

**Theorem 2.7.** Fix \(\alpha \in [0, 1)\) and \(\xi \in \overline{\mathbb{D}}\). If \(p \in P(\alpha)\) and \(p\) is of the form (1.1), then

\[(2.12)\] 
\[|\xi p_{n+1} - p_n| \leq \begin{cases} 2 \frac{1 - |\xi|^n}{1 - |\xi|} [(1 - \alpha)(1 + |\xi|^2) - \text{Re}(\xi p_1)] + |2(1 - \alpha) - \xi p_1| \cdot |\xi|^n, & |\xi| < 1, \\ (2n + 1)|2(1 - \alpha) - \xi p_1|, & |\xi| = 1, \end{cases}\]
By the above and from (2.14) we have

\[
q(z) = \frac{\xi - \bar{\xi}z[(1 - 2\alpha)z + \alpha \xi]}{z(1 - \bar{\xi}z)} + \frac{z - \xi}{z} p(z)
\]

\[
= 1 - 2\alpha + \frac{\xi}{z} - [(1 - \alpha)(1 - |\xi|^2) - \alpha] \frac{1}{1 - \bar{\xi}z} + \left(1 - \frac{\xi}{z}\right) p(z)
\]

\[
= \left[1 + (1 - \alpha)|\xi|^2 - \xi p_1\right] + \left[p_1 - \xi p_2 - ((1 - \alpha)(1 - |\xi|^2) - \alpha)\bar{\xi}\right] z + \ldots +
\]

\[
+ \left[p_n - \xi p_{n+1} - ((1 - \alpha)(1 - |\xi|^2) - \alpha)\bar{\xi}^n\right] z^n + \ldots
\]

But

\[
q(z) = q_0 + [q_1 + q_0 \bar{\xi}] z + \ldots +
\]

\[
+ [q_n + q_{n-1} \bar{\xi} + \ldots + q_1 \bar{\xi}^{n-1} + q_0 \bar{\xi}^n] z^n + \ldots
\]

By the above and from (2.14) we have

\[
q_0 = 1 + (1 - \alpha)|\xi|^2 - \xi p_1,
\]

\[
q_1 + q_0 \bar{\xi} = p_1 - \xi p_2 - [(1 - \alpha)(1 - |\xi|^2) - \alpha] \bar{\xi}
\]

and

\[
q_n + q_{n-1} \bar{\xi} + \ldots + q_1 \bar{\xi}^{n-1} + q_0 \bar{\xi}^n = p_n - \xi p_{n+1} - [(1 - \alpha)(1 - |\xi|^2) - \alpha] \bar{\xi}^n
\]

for all \(n \in \mathbb{N}\). From estimates (1.3) it follows that

\[
|\xi p_{n+1} - p_n|
\]

\[
\leq |q_n + q_{n-1} \bar{\xi} + \ldots + q_1 \bar{\xi}^{n-1}| + |q_0 + (1 - \alpha)(1 - |\xi|^2) - \alpha| \cdot |\xi|^n
\]

\[
\leq 2(\text{Re} q_0 - \alpha)(1 + |\xi| + \ldots + |\xi|^{n-1}) + 2(1 - \alpha) - \xi p_1 - |\xi|^n,
\]

which gives estimates (2.12).

In order to prove (2.13) assume that \(p_1 = |p_1|e^{i\psi}, \ \psi \in [0, 2\pi]\). Let \(|\xi| = 1\). Since \(p \in P(\alpha)\), the function \(p(e^{-i\psi}z)\), \(z \in \mathbb{D}\), also belongs to \(P(\alpha)\), and applying (2.12) we have

\[
|\xi p_{n+1} - p_n| \leq |\xi e^{-i\psi} p_{n+1} - p_n| \leq (2n + 1)|2(1 - \alpha) - \xi e^{-i\psi} p_1|
\]

\[
= |2(1 - \alpha) - \xi |p_1| |
\]
Analogously we prove (2.13) when $|\xi| < 1$.

If $\xi \in [0, 1)$, then equalities in (2.12) and (2.13) are achieved for the coefficients of the function (2.4).

For $\xi = 1$ the factor $2n + 1$ which appears on the right hand side of (2.12) and (2.13) cannot be replaced by a smaller one. To see this consider for each $\alpha \in [0, 1)$ and $\theta \in [0, 2\pi)$ the function

$$p_{\alpha, \theta}(z) = \frac{1 - 2\alpha z \cos \theta - (1 - 2\alpha)z^2}{1 - 2z \cos \theta + z^2} = 1 + 2(1 - \alpha) \sum_{n=2}^{\infty} \cos(n\theta)z^n, \quad z \in \mathbb{D}.$$  

Then for $\xi = 1$ we have

$$|p_{n+1}^\xi - p_n| = 2(1 - \alpha)|\cos((n + 1)\theta) - \cos(n\theta)|$$

$$= 4(1 - \alpha) \left| \frac{\sin((2n + 1)\theta/2)}{\sin(\theta/2)} \right| \sin^2(\theta/2)$$

$$\leq 4(1 - \alpha)(2n + 1)\sin^2(\theta/2)$$

for all $\theta \in [0, 2\pi)$. Taking $\theta$ sufficiently small we see that the factor $2n + 1$ is the best possible.

For $\alpha = 0$ Theorem 2.7 has the following form.

**Corollary 2.8.** Fix $\xi \in \mathbb{D}$. If $p \in P$ and $p$ is of the form (1.1), then

$$(2.15) \quad |\xi p_{n+1} - p_n|$$

$$\leq \begin{cases} 2 \frac{1 - |\xi|^n}{1 - |\xi|} \left[ 1 + |\xi|^2 - |\xi|p_1 \right] + |2 - \xi p_1| \cdot |\xi|^n & \text{for } |\xi| < 1, \\ (2n + 1)|2 - \xi p_1| & \text{for } |\xi| = 1, \end{cases}$$

$$(2.16) \quad \|\xi p_{n+1} - p_n\|$$

$$\leq \begin{cases} 2 \frac{1 - |\xi|^n}{1 - |\xi|} \left[ 1 + |\xi|^2 - |p_1| Re \xi \right] + |2 - \xi |p_1|| \cdot |\xi|^n & \text{for } |\xi| < 1, \\ (2n + 1)|2 - \xi |p_1|| & \text{for } |\xi| = 1, \end{cases}$$

for $n = 2, 3, \ldots$ The estimates are sharp for $\xi \in [0, 1]$.

For $\xi = 1$ the result of Corollary 2.8 was obtained by Robertson [5]. Setting $\xi = 1/n$, $n = 2, 3, \ldots$, we get from Corollary 2.8 the following results.

**Corollary 2.9.** If $p \in P$ and $p$ is of the form (1.1), then

$$|p_{n+1} - np_n| \leq 2 \frac{1 - (1/n)^n}{n - 1} (n^2 + 1 - n Re p_1) + \left( \frac{1}{n} \right)^n |2n - p_1|$$

for $n = 2, 3, \ldots$ The estimates are sharp.
In particular, for \( n = 2 \) and \( n = 3 \) we have
\[
\begin{align*}
|p_3 - 2p_2| &\leq \frac{15}{2} - 3 \text{Re} p_1 + \left|1 - \frac{p_1}{4}\right|, \\
|p_4 - 3p_3| &\leq \frac{1}{27}(260 - 78 \text{Re} p_1 + |6 - p_1|).
\end{align*}
\]

For \( \xi = 1 - 1/(n+1), \ n \in \mathbb{N} \), Corollary 2.8 yields

**Corollary 2.10.** If \( p \in P \) and \( p \) is of the form (1.1), then
\[
|np_{n+1} - (n+1)p_n| \leq 2 \left(1 - \left(\frac{n}{n+1}\right)^n\right)(2n^2 + 2n + 1 - n(n+1) \text{Re} p_1)
+ \left(\frac{n}{n+1}\right)^n |2(n+1) - np_1|
\]
for \( n \in \mathbb{N} \). The estimates are sharp.

In particular, for \( n = 1 \) and \( n = 2 \) we have
\[
\begin{align*}
|p_2 - 2p_1| &\leq 5 - 2 \text{Re} p_1 + \left|2 - \frac{p_1}{2}\right|, \\
|2p_3 - 3p_2| &\leq \frac{1}{27}(260 - 78 \text{Re} p_1 + |6 - p_1|).
\end{align*}
\]

**References**


