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On coefficient inequalities in the Carathéodory class of functions

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Abstract. Some inequalities are proved for coefficients of functions in the class $P(\alpha)$, where $\alpha \in [0,1)$, of functions with real part greater than α. In particular, new inequalities for coefficients in the Carathéodory class $P(0)$ are given.

1. Main results. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk. We denote by $P(\alpha)$, where $\alpha \in [0,1)$, the class of functions p regular in $\mathbb{D}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \qquad z \in \mathbb{D}, \tag{1.1}$$

such that $\operatorname{Re} p(z) > \alpha$ for $z \in \mathbb{D}$ (see [2, p. 105]). The well known class of Carathéodory functions having positive real part in $\mathbb{D}$, denoted by P, coincides with $P(0)$. The class $P(\alpha)$, although not explicitly defined, appeared first in [4], where Robertson defined functions convex of order α and starlike of order α.

Using the well known estimates $|p_n| \le 2$, $n \in \mathbb{N}$, [1; 2, p. 80] for the coefficients of $p \in P$ it is easy to prove the lemma below (see [2, p. 101]).

Lemma 1.1. *Fix $\alpha \in [0,1)$ and let q of the form*

$$q(z) = q_0 + \sum_{n=1}^{\infty} q_n z^n, \qquad z \in \mathbb{D}, \tag{1.2}$$

be regular in $\mathbb{D}$. If $\operatorname{Re} q(z) > \alpha$ for $z \in \mathbb{D}$, then

$$|q_n| \le 2(\operatorname{Re} q_0 - \alpha), \qquad n \in \mathbb{N}. \tag{1.3}$$

Estimates (1.3) *are sharp.*

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An extremal function for which equalities hold in (1.3) is

$$q(z)=\frac{q_0+(\overline{q}_0-2\alpha)z}{1-z},\qquad z\in\mathbb{D}.$$

REMARK 1.2. For $\alpha=0$ we have $|q_n|\le 2\operatorname{Re}q_0$, $n\in\mathbb{N}$, for the coefficients q_n of a function q such that $\operatorname{Re}q(z)>0$ for $z\in\mathbb{D}$.

An interesting generalization of Remark 1.2 can be found in [3].

As an immediate consequence of (1.3) we have the following estimates for the coefficients of $p\in P(\alpha)$ of the form (1.1) which can be found in [4, p. 386]:

$$|p_n|\le 2(1-\alpha),\qquad n\in\mathbb{N}. \tag{1.4}$$

Now we formulate two basic theorems of this paper.

THEOREM 1.3. *Fix* $\alpha\in[0,1)$ *and* $\xi\in\overline{\mathbb{D}}$. *If* $p\in P(\alpha)$, *then the function*

$$q(z)=q(\xi;z)=\frac{\xi-\overline{\xi}z[(1-2\alpha)z+\alpha\xi]}{z}+\frac{(z-\xi)(1-\overline{\xi}z)}{z}p(z), \tag{1.5}$$

$z\in\mathbb{D}$, *is regular in* $\mathbb{D}$ *and*

$$\operatorname{Re}q(\xi;z)\ge\alpha,\qquad z\in\mathbb{D}. \tag{1.6}$$

Equality holds in (1.6) *only if* $|\xi|=1$ *and*

$$p(z)=p(\alpha,\xi;z)=\frac{1+(1-2\alpha)\overline{\xi}z}{1-\overline{\xi}z},\qquad z\in\mathbb{D}. \tag{1.7}$$

Proof. Observe first that the function (1.5) has a removable singularity at $z=0$ since

$$q(\xi;0)=\lim_{z\to 0}q(\xi;z)=1+(1-\alpha)|\xi|^2-\xi p_1. \tag{1.8}$$

Assume first that p, and hence q, is regular on $\partial\mathbb{D}$. For $z=e^{i\theta}$, $\theta\in\mathbb{R}$, we have

$$\begin{aligned}q(\xi;e^{i\theta})&=2i\operatorname{Im}(\xi e^{-i\theta})-\alpha|\xi|^2\\&\quad+2\alpha\overline{\xi}e^{i\theta}+[1+|\xi|^2-2\operatorname{Re}(\xi e^{-i\theta})]p(e^{i\theta}).\end{aligned}$$

Since $p\in P(\alpha)$ we see that

$$\begin{aligned}\operatorname{Re}q(\xi;e^{i\theta})&=-\alpha|\xi|^2+2\alpha\operatorname{Re}(\overline{\xi}e^{i\theta})\\&\quad+[(1+|\xi|^2)-2\operatorname{Re}(\xi e^{-i\theta})]\operatorname{Re}p(e^{i\theta})\\&\ge\alpha+2\alpha(\operatorname{Re}(\overline{\xi}e^{i\theta})-\operatorname{Re}(\xi e^{-i\theta}))=\alpha.\end{aligned}$$

By the mimimum principle for harmonic functions the above inequality holds in $\overline{\mathbb{D}}$, i.e. $\operatorname{Re}q(\xi;z)\ge\alpha$ for $z\in\overline{\mathbb{D}}$.

If p is not regular on $\partial\mathbb{D}$, then we consider the functions $p_r(z)=p(rz)$, $z\in\mathbb{D}$, for $r\in(0,1)$. Replacing p by p_r on the right hand side of (1.5) we obtain the corresponding function $q_r(\xi;z)$, $z\in\mathbb{D}$. Repeating the above considerations we get the strict inequality $\operatorname{Re} q_r(\xi; z) > \alpha$ for $z \in \mathbb{D}$. Letting $r \to 1$ we see that $p_r \to p$ and $q_r \to q$. Consequently, $\operatorname{Re} q(\xi; z) \ge \alpha$ for $z \in \mathbb{D}$.

If $|\xi| < 1$, then from (1.5) we have $q(\xi; \xi) = 1 - (1-\alpha)|\xi|^2 > \alpha$. Hence $\operatorname{Re} q(\xi; z) > \alpha$ for $z \in \mathbb{D}$ and $|\xi| < 1$.

If $|\xi| = 1$, then from (1.5) we deduce that

$$\begin{aligned}\operatorname{Re} q(\xi; 0) - \alpha &= (1-\alpha)(1 + |\xi|^2) - \operatorname{Re}(\xi p_1)\\ &\ge (1-\alpha)(1+|\xi|^2) - |\xi p_1| \ge (1-\alpha)(1-|\xi|)^2 = 0\end{aligned}$$

for all $\alpha \in [0,1)$. Equality holds only if $\operatorname{Re}(\xi p_1) = |\xi p_1|$ and $|p_1| = 2(1-\alpha)$. Hence $p_1 = 2(1-\alpha)\overline{\xi}$, which holds only for the function p defined by (1.7). Then $q(\xi; z) = \alpha$ for each $\xi \in \partial\mathbb{D}$ and all $z \in \mathbb{D}$.

Now we prove the converse theorem for $\xi \in \mathbb{D}$.

THEOREM 1.4. *Fix $\alpha \in [0,1)$ and $\xi \in \mathbb{D}$. Assume that q is regular in $\mathbb{D}$, $\operatorname{Re} q(z) > \alpha$ for $z \in \mathbb{D}$ and*

$$q(\xi) = 1 - (1-\alpha)|\xi|^2. \tag{1.9}$$

Then the function

$$\begin{aligned}p(z) &= p(\xi; z)\\ &= \frac{z}{(z-\xi)(1-\overline{\xi}z)}\left(q(z) - \frac{\xi - \overline{\xi}z[(1-2\alpha)z + \alpha\xi]}{z}\right), \qquad z \in \mathbb{D},\end{aligned} \tag{1.10}$$

is regular in $\mathbb{D}$ and $p \in P(\alpha)$.

P r o o f. Simple calculations lead to

$$p(\xi; \xi) = \lim_{z \to \xi} p(\xi; z) = \frac{1 + (1-2\alpha)|\xi|^2 + \xi q'(\xi)}{1 - |\xi|^2}$$

so at $z = \xi$ the function p has a removable singularity. Moreover $p(\xi; 0) = 1$ for each $\xi \in \mathbb{D}$. Therefore p is regular in $\mathbb{D}$ and of the form (1.1) for each $\xi \in \mathbb{D}$.

Now we prove that $p \in P(\alpha)$. Assume first that q is regular on $\partial\mathbb{D}$. By (1.10), so is p, and for $z = e^{i\theta}$, $\theta \in \mathbb{R}$, we have

$$p(\xi; e^{i\theta}) = \frac{1}{1 + |\xi|^2 - 2\operatorname{Re}(\xi e^{-i\theta})}(q(e^{i\theta}) + \alpha|\xi|^2 - 2\alpha\overline{\xi}e^{i\theta} - 2i\operatorname{Im}(\overline{\xi}e^{i\theta})).$$

Since $\operatorname{Re} q(z) \ge \alpha$ for $z \in \partial\mathbb{D}$ we obtain

$$\begin{aligned}\operatorname{Re} p(\xi; e^{i\theta}) &= \frac{1}{1 + |\xi|^2 - 2\operatorname{Re}(\xi e^{-i\theta})}(\operatorname{Re} q(e^{i\theta}) + \alpha|\xi|^2 - 2\alpha \operatorname{Re}(\overline{\xi}e^{i\theta}))\\ &\ge \frac{1 + |\xi|^2 - 2\operatorname{Re}(\overline{\xi}e^{i\theta})}{1 + |\xi|^2 - 2\operatorname{Re}(\xi e^{-i\theta})}\,\alpha = \alpha.\end{aligned}$$

By the mimimum principle for harmonic functions the above inequality is true in $\overline{\mathbb{D}}$, i.e. $\operatorname{Re} p(\xi; z) \geq \alpha$ for $z \in \overline{\mathbb{D}}$.

If q is not regular on $\partial\mathbb{D}$, then arguing as in the part of Theorem 1.3 concerning p_r (setting $q_r(z) = q(rz)$, $z \in \mathbb{D}$, for $r \in (0,1)$, and using (1.10) in place of (1.5)) we obtain $\operatorname{Re} p(\xi; z) \geq \alpha$ for $z \in \mathbb{D}$.

Finally, recall that $p(\xi; 0) = 1$ for each $\xi \in \mathbb{D}$. This implies that $\operatorname{Re} p(\xi; z) > \alpha$ for $z \in \mathbb{D}$ and $\xi \in \mathbb{D}$. Therefore $p \in P(\alpha)$.

For $\alpha = 0$ we obtain from Theorems 1.3 and 1.4 the following results.

COROLLARY 1.5. *Fix* $\xi \in \overline{\mathbb{D}}$. *If* $p \in P$, *then the function*

$$q(z) = q(\xi; z) = \frac{\xi - \overline{\xi} z^2}{z} + \frac{(z - \xi)(1 - \overline{\xi} z)}{z} p(z), \qquad z \in \mathbb{D},$$

is regular in $\mathbb{D}$ *and* $\operatorname{Re} q(\xi; z) \geq 0$ *for* $z \in \mathbb{D}$. *Equality holds only if* $|\xi| = 1$ *and*

$$p(z) = p(0, \xi; z) = \frac{1 + \overline{\xi} z}{1 - \overline{\xi} z}, \qquad z \in \mathbb{D}.$$

COROLLARY 1.6. *Fix* $\xi \in \mathbb{D}$. *Assume that* q *is regular in* $\mathbb{D}$, $\operatorname{Re} q(z) > 0$ *for* $z \in \mathbb{D}$ *and* $q(\xi) = 1 - |\xi|^2$. *Then the function*

$$p(z) = p(\xi; z) = \frac{z}{(z - \xi)(1 - \overline{\xi} z)} \left(q(z) - \frac{\xi - \overline{\xi} z^2}{z} \right), \qquad z \in \mathbb{D},$$

is regular in $\mathbb{D}$ *and* $p \in P$.

For $\xi = 1$ Corollary 1.5 is due to Robertson [5].

2. Applications. In this section we apply Theorem 1.3 to obtain some inequalities for coefficients of functions in the class $P(\alpha)$. In the case when $\alpha = 0$ these results generalize the well known estimates for coefficients of functions in the Carathéodory class.

THEOREM 2.1. *Fix* $\alpha \in [0,1)$ *and* $\xi \in \overline{\mathbb{D}}$. *If* $p \in P(\alpha)$ *and* p *is of the form* (1.1), *then*

$$|\xi p_2 - (1 + |\xi|^2) p_1 + 2(1 - \alpha)\overline{\xi}| \leq 2((1 - \alpha)(1 + |\xi|^2) - \operatorname{Re}(\xi p_1)), \tag{2.1}$$

$$|\xi p_{n+1} - (1 + |\xi|^2) p_n + \overline{\xi} p_{n-1}| \leq 2((1 - \alpha)(1 + |\xi|^2) - \operatorname{Re}(\xi p_1)), \tag{2.2}$$

$$\big|\, ||\xi| p_{n+1} - p_n| - |\xi| \cdot ||\xi| p_n - p_{n-1}| \,\big| \leq 2((1 - \alpha)(1 + |\xi|^2) - |\xi| \operatorname{Re} p_1) \tag{2.3}$$

for $n = 2, 3, \ldots$ *Estimates* (2.1)–(2.3) *are sharp.*

Proof. By Theorem 1.3 the function q defined by (1.5) is regular in $\mathbb{D}$ and $\operatorname{Re} q(\xi; z) \geq \alpha$ for $z \in \mathbb{D}$. We can assume that q is of the form (1.2).

From (1.5) we have

$$zq(\xi;z) = [1+(1-\alpha)|\xi|^2 - \xi p_1]z + [-\xi p_2 + (1+|\xi|^2)p_1 - 2(1-\alpha)\overline{\xi}\,]z^2 + \dots$$
$$+ [-\xi p_n + (1+|\xi|^2)p_{n-1} - \overline{\xi}p_{n-2}]z^n + \dots$$

Consequently,

$$q_0 = 1+(1-\alpha)|\xi|^2 - \xi p_1, \qquad q_1 = -\xi p_2 + (1+|\xi|^2)p_1 - 2(1-\alpha)\overline{\xi}$$

and

$$q_n = -\xi p_n + (1+|\xi|^2)p_{n-1} - \overline{\xi}p_{n-2} \qquad \text{for } n = 2,3,\dots$$

Now using (1.3) and the formula for q_0 we obtain (2.1) and (2.2).

To prove (2.3) assume that $\xi = |\xi|e^{i\varphi}$, $\varphi \in [0,2\pi)$. Since $p \in P(\alpha)$, the function $p(e^{-i\varphi}z)$, $z \in \mathbb{D}$, also belongs to $P(\alpha)$, and applying (2.2) to it we have

$$\begin{aligned} \big|\,||\xi|p_{n+1} - p_n| - |\xi|\cdot||\xi|p_n - p_{n-1}|\,\big| &= \big|\,|\xi e^{-i\varphi}p_{n+1} - p_n| - |\overline{\xi}|\cdot|\xi e^{-i\varphi}p_n - p_{n-1}|\,\big| \\ &\le |\xi e^{-i\varphi}p_{n+1} - (1+|\xi|^2)p_n + \overline{\xi}e^{i\varphi}p_{n-1}| \\ &\le 2((1-\alpha)(1+|\xi|^2) - \operatorname{Re}(\xi e^{-i\varphi}p_1)) \\ &= 2((1-\alpha)(1+|\xi|^2) - |\xi|\operatorname{Re} p_1). \end{aligned}$$

The function

$$p(\alpha,1;z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)\sum_{n=1}^{\infty} z^n, \qquad z \in \mathbb{D}, \tag{2.4}$$

is in $P(\alpha)$ and gives equalities in (2.1)–(2.3).

For $\alpha = 0$ the above yields the following result.

Corollary 2.2. *Fix $\xi \in \overline{\mathbb{D}}$. If $p \in P$ and p is of the form* (1.1), *then*

$$|\xi p_2 - (1+|\xi|^2)p_1 + 2\overline{\xi}| \le 2(1+|\xi|^2 - \operatorname{Re}(\xi p_1)), \tag{2.5}$$
$$|\xi p_{n+1} - (1+|\xi|^2)p_n + \overline{\xi}p_{n-1}| \le 2(1+|\xi|^2 - \operatorname{Re}(\xi p_1)), \tag{2.6}$$
$$\big|\,||\xi|p_{n+1} - p_n| - |\xi|\cdot||\xi|p_n - p_{n-1}|\,\big| \le 2(1+|\xi|^2 - |\xi|\operatorname{Re} p_1) \tag{2.7}$$

for $n = 2,3,\dots$ Estimates (2.5)–(2.7) *are sharp.*

Corollary 2.3. *If $p \in P$ and p is of the form* (1.1), *then*

$$\begin{aligned} &|p_2 - 2p_1 + 2| \le 2\operatorname{Re}(2 - p_1), \\ &|p_{n+1} - 2p_n + p_{n-1}| \le 2(2 - \operatorname{Re} p_1), \end{aligned} \tag{2.8}$$

$$\begin{aligned} &|p_2 + 2p_1 + 2| \le 2(2 + \operatorname{Re} p_1), \\ &|p_{n+1} + 2p_n + p_{n-1}| \le 2(2 + \operatorname{Re} p_1), \end{aligned} \tag{2.9}$$

$$|p_2 + 2ip_1 - 2| \le 2(2 + \operatorname{Im} p_1),$$
(2.10)
$$|p_{n+1} + 2ip_n - p_{n-1}| \le 2(2 + \operatorname{Im} p_1),$$

$$|p_2 - 2ip_1 - 2| \le 2(2 - \operatorname{Im} p_1),$$
(2.11)
$$|p_{n+1} - 2ip_n - p_{n-1}| \le 2(2 - \operatorname{Im} p_1),$$

for $n = 2, 3, \ldots$ *All estimates are sharp.*

Proof. Estimates (2.8) follow from (2.5) and (2.6) by setting $\xi = 1$. Setting $\xi = -1$ in (2.5) and (2.6) we obtain (2.9). Analogously, setting $\xi = i$ and $\xi = -i$ in (2.5) and (2.6) we get (2.10) and (2.11), respectively. The function $p(z) = p(0, \xi; z)$, $z \in \mathbb{D}$, defined by (1.6), for suitable ξ as above, is extremal for the cases considered.

Taking $|\xi| = 1$ in (2.7) we have

Corollary 2.4. *If* $p \in P$ *and* p *is of the form* (1.1), *then*

$$\big|\,|p_{n+1} - p_n| - |p_n - p_{n-1}|\,\big| \le 2(2 - \operatorname{Re} p_1)$$

for $n = 2, 3, \ldots$ *The estimates are sharp.*

Setting $\xi = 1/n$ and $\xi = 1 - 1/n$, $n = 2, 3, \ldots$, in (2.6) we have respectively:

Corollary 2.5. *If* $p \in P$ *and* p *is of the form* (1.1), *then*

$$\left|p_{n+1} - \left(n + \frac{1}{n}\right)p_n + p_{n-1}\right| \le 2\left(n + \frac{1}{n} - \operatorname{Re} p_1\right)$$

for $n = 2, 3, \ldots$ *The estimates are sharp.*

Corollary 2.6. *If* $p \in P$ *and* p *is of the form* (1.1), *then*

$$\left|p_{n+1} - \left(\frac{n}{n-1} + \frac{n-1}{n}\right)p_n + p_{n-1}\right| \le 2\left(\frac{n}{n-1} + \frac{n-1}{n} - \operatorname{Re} p_1\right)$$

for $n = 2, 3, \ldots$ *The estimates are sharp.*

Theorem 2.7. *Fix* $\alpha \in [0, 1)$ *and* $\xi \in \overline{\mathbb{D}}$. *If* $p \in P(\alpha)$ *and* p *is of the form* (1.1), *then*

$$(2.12)\qquad |\xi p_{n+1} - p_n|$$
$$\le \begin{cases} 2\dfrac{1-|\xi|^n}{1-|\xi|}\,[(1-\alpha)(1+|\xi|^2) - \operatorname{Re}(\xi p_1)] + |2(1-\alpha) - \xi p_1|\cdot|\xi|^n, & |\xi| < 1, \\ (2n+1)|2(1-\alpha) - \xi p_1|, & |\xi| = 1, \end{cases}$$

$$\begin{aligned}(2.13)\quad & \big|\,|\xi p_{n+1}| - |p_n|\,\big| \\ & \le \begin{cases} 2\dfrac{1-|\xi|^n}{1-|\xi|}\,[(1-\alpha)(1+|\xi|^2)-|p_1|\operatorname{Re}\xi] + |2(1-\alpha)-\xi|p_1||\cdot|\xi|^n, & |\xi|<1, \\ (2n+1)|2(1-\alpha)-\xi|p_1||, & |\xi|=1, \end{cases}\end{aligned}$$

for $n = 2, 3, \ldots$ The estimates are sharp for each $\xi \in [0, 1]$.

P r o o f. By Theorem 1.3 the function q defined by (1.5) is regular in $\mathbb{D}$ and $\operatorname{Re} q(\xi; z) \ge \alpha$ for $z \in \mathbb{D}$. We can assume that q is of the form (1.2).

From (1.5) we have

$$\begin{aligned}(2.14)\quad \frac{q(z)}{1-\overline{\xi}z} &= \frac{\xi - \overline{\xi}z[(1-2\alpha)z + \alpha\xi]}{z(1-\overline{\xi}z)} + \frac{z-\xi}{z}\,p(z) \\ &= 1-2\alpha + \frac{\xi}{z} - [(1-\alpha)(1-|\xi|^2)-\alpha]\frac{1}{1-\overline{\xi}z} + \left(1-\frac{\xi}{z}\right)p(z) \\ &= [1 + (1-\alpha)|\xi|^2 - \xi p_1] \\ &\quad + [p_1 - \xi p_2 - ((1-\alpha)(1-|\xi|^2)-\alpha)\overline{\xi}\,]z + \ldots + \\ &\quad + [p_n - \xi p_{n+1} - ((1-\alpha)(1-|\xi|^2)-\alpha)\overline{\xi}^n\,]z^n + \ldots\end{aligned}$$

But

$$\begin{aligned}\frac{q(z)}{1-\overline{\xi}z} &= q_0 + [q_1 + q_0\overline{\xi}\,]z + \ldots + \\ &\quad + [q_n + q_{n-1}\overline{\xi} + \ldots + q_1\overline{\xi}^{n-1} + q_0\overline{\xi}^n]z^n + \ldots\end{aligned}$$

By the above and from (2.14) we have

$$\begin{aligned} q_0 &= 1 + (1-\alpha)|\xi|^2 - \xi p_1, \\ q_1 + q_0\overline{\xi} &= p_1 - \xi p_2 - [(1-\alpha)(1-|\xi|^2)-\alpha]\overline{\xi}\end{aligned}$$

and

$$q_n + q_{n-1}\overline{\xi} + \ldots + q_1\overline{\xi}^{n-1} + q_0\overline{\xi}^n = p_n - \xi p_{n+1} - [(1-\alpha)(1-|\xi|^2)-\alpha]\overline{\xi}^n$$

for all $n \in \mathbb{N}$. From estimates (1.3) it follows that

$$\begin{aligned}&|\xi p_{n+1} - p_n| \\ &\quad \le |q_n + q_{n-1}\overline{\xi} + \ldots + q_1\overline{\xi}^{n-1}| + |q_0 + (1-\alpha)(1-|\xi|^2)-\alpha|\cdot|\xi|^n \\ &\quad \le 2(\operatorname{Re} q_0 - \alpha)(1 + |\xi| + \ldots + |\xi|^{n-1}) + |2(1-\alpha) - \xi p_1|\cdot|\xi|^n,\end{aligned}$$

which gives estimates (2.12).

In order to prove (2.13) assume that $p_1 = |p_1|e^{i\psi}$, $\psi \in [0, 2\pi)$. Let $|\xi| = 1$. Since $p \in P(\alpha)$, the function $p(e^{-i\psi}z)$, $z \in \mathbb{D}$, also belongs to $P(\alpha)$, and applying (2.12) we have

$$\begin{aligned}\big|\,|\xi p_{n+1}| - |p_n|\,\big| &\le |\xi e^{-i\psi}p_{n+1} - p_n| \le (2n+1)|2(1-\alpha) - \xi e^{-i\psi}p_1| \\ &= |2(1-\alpha) - \xi|p_1|\,|.\end{aligned}$$

Analogously we prove (2.13) when $|\xi| < 1$.

If $\xi \in [0, 1)$, then equalities in (2.12) and (2.13) are achieved for the coefficients of the function (2.4).

For $\xi = 1$ the factor $2n + 1$ which appears on the right hand side of (2.12) and (2.13) cannot be replaced by a smaller one. To see this consider for each $\alpha \in [0, 1)$ and $\theta \in [0, 2\pi)$ the function

$$\begin{aligned} p_{\alpha,\theta}(z) &= \frac{1 - 2\alpha z \cos\theta - (1 - 2\alpha) z^2}{1 - 2z\cos\theta + z^2} \\ &= 1 + 2(1 - \alpha) \sum_{n=2}^{\infty} \cos(n\theta) z^n, \qquad z \in \mathbb{D}. \end{aligned}$$

Then for $\xi = 1$ we have

$$\begin{aligned} |p_{n+1} - p_n| &= 2(1 - \alpha) |\cos((n + 1)\theta) - \cos(n\theta)| \\ &= 4(1 - \alpha) \left| \frac{\sin((2n + 1)\theta/2)}{\sin(\theta/2)} \right| \sin^2(\theta/2) \\ &\le 4(1 - \alpha)(2n + 1) \sin^2(\theta/2) \end{aligned}$$

for all $\theta \in [0, 2\pi)$. Taking θ sufficiently small we see that the factor $2n + 1$ is the best possible.

For $\alpha = 0$ Theorem 2.7 has the following form.

Corollary 2.8. *Fix* $\xi \in \overline{\mathbb{D}}$. *If* $p \in P$ *and* p *is of the form* (1.1), *then*

(2.15) $$|\xi p_{n+1} - p_n| \le \begin{cases} 2\dfrac{1 - |\xi|^n}{1 - |\xi|} [1 + |\xi|^2 - \operatorname{Re}(\xi p_1)] + |2 - \xi p_1| \cdot |\xi|^n & \text{for } |\xi| < 1, \\ (2n + 1)|2 - \xi p_1| & \text{for } |\xi| = 1, \end{cases}$$

(2.16) $$\big| |\xi p_{n+1}| - |p_n| \big| \le \begin{cases} 2\dfrac{1 - |\xi|^n}{1 - |\xi|} [1 + |\xi|^2 - |p_1| \operatorname{Re}\xi] + |2 - \xi|p_1|| \cdot |\xi|^n & \text{for } |\xi| < 1, \\ (2n + 1)|2 - \xi|p_1|| & \text{for } |\xi| = 1, \end{cases}$$

for $n = 2, 3, \ldots$ *The estimates are sharp for* $\xi \in [0, 1]$.

For $\xi = 1$ the result of Corollary 2.8 was obtained by Robertson [5].

Setting $\xi = 1/n$, $n = 2, 3, \ldots$, we get from Corollary 2.8 the following results.

Corollary 2.9. *If* $p \in P$ *and* p *is of the form* (1.1), *then*

$$|p_{n+1} - n p_n| \le 2\frac{1 - (1/n)^n}{n - 1} (n^2 + 1 - n \operatorname{Re} p_1) + \left(\frac{1}{n}\right)^n |2n - p_1|$$

for $n = 2, 3, \ldots$ *The estimates are sharp.*

In particular, for $n = 2$ and $n = 3$ we have

$$|p_3 - 2p_2| \le \frac{15}{2} - 3\operatorname{Re} p_1 + \left|1 - \frac{p_1}{4}\right|,$$

$$|p_4 - 3p_3| \le \frac{1}{27}(260 - 78\operatorname{Re} p_1 + |6 - p_1|).$$

For $\xi = 1 - 1/(n+1)$, $n \in \mathbb{N}$, Corollary 2.8 yields

Corollary 2.10. *If $p \in P$ and p is of the form* (1.1), *then*

$$|np_{n+1} - (n+1)p_n| \le 2\left(1 - \left(\frac{n}{n+1}\right)^n\right)(2n^2 + 2n + 1 - n(n+1)\operatorname{Re} p_1) + \left(\frac{n}{n+1}\right)^n |2(n+1) - np_1|$$

for $n \in \mathbb{N}$. The estimates are sharp.

In particular, for $n = 1$ and $n = 2$ we have

$$|p_2 - 2p_1| \le 5 - 2\operatorname{Re} p_1 + \left|2 - \frac{p_1}{2}\right|,$$

$$|2p_3 - 3p_2| \le \frac{1}{27}(260 - 78\operatorname{Re} p_1 + |6 - p_1|).$$

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