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Killing tensors and warped products

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Abstract. We present some examples of Killing tensors and give their geometric interpretation. We give new examples of non-compact complete and compact Riemannian manifolds whose Ricci tensor ϱ satisfies the condition $\nabla_X \varrho(X, X) = \frac{2}{n+2} X\tau g(X, X)$.

0. Introduction. Killing tensors are symmetric $(0, 2)$ tensors ϱ on a Riemannian manifold (M, g) satisfying the condition

$$\nabla_X \varrho(X, X) = 0 \tag{K}$$

for all $X \in \mathfrak{X}(M)$ or equivalently $\mathfrak{C}_{X,Y,Z} \nabla_X \varrho(Y, Z) = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$ where $\mathfrak{X}(M)$ denotes the space of all local vector fields on M, $\mathfrak{C}$ denotes the cyclic sum and ∇ denotes the Levi-Civita connection of (M, g). The condition (K) is a generalization of the condition $\nabla\varrho = 0$. Another generalization of this condition is $\nabla_X \varrho(Y, Z) = \nabla_Y \varrho(X, Z)$, which gives the class of Codazzi tensors. The Codazzi tensors are quite frequently used in Riemannian geometry. For example the second fundamental form of any hypersurface immersed in a Euclidean space is a Codazzi tensor. On the other hand it is difficult to find general examples of Killing tensors in the literature. It is only known that a Ricci tensor of any naturally reductive homogeneous space (and more generally of any D'Atri space) has this property.

The aim of the present paper is to show that Killing tensors appear quite naturally in Riemannian geometry. We prove that on every warped product $M_0 \times_{f_1} M_1 \times \ldots \times_{f_k} M_k$ there exists a Killing tensor $\Phi(X, Y) = g(SX, Y)$ such that the functions $\lambda_0 = \mu \in \mathbb{R}$ and $\lambda_i = \mu + C_i f_i^2$ for $i > 1$ are eigenfunctions of S for any $\mu \in \mathbb{R}$ and any real constants $C_i \in \mathbb{R} - \{0\}$. Conversely, let $\Phi(X, Y) = g(SX, Y)$ be a Killing tensor with an integrable almost product structure given by its eigendistributions and with eigenvalues $\mu, \lambda_1, \ldots, \lambda_k$ such that $\mu \in \mathbb{R}$ is constant and $\bigoplus_{j>0} D_j \subset \ker d\lambda_i$ (i.e. $\nabla\lambda_i \in$

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$\Gamma(D_0)$). If M is a simply connected, complete Riemannian manifold then $M = M_0 \times_{f_1} M_1 \times \ldots \times_{f_k} M_k$ where $TM_i = D_i = \ker(S - \lambda_i \operatorname{Id})$ and $f_i^2 = |\lambda_i - \mu|$.

A manifold M is called an $\mathcal{A}$-*manifold* (see [G]) if its Ricci tensor satisfies condition (K). The scalar curvature τ of an $\mathcal{A}$-manifold (M,g) is constant. All the known examples of $\mathcal{A}$-manifolds have Ricci tensors with constant eigenvalues. Besse [B] (p. 433) defines a class of manifolds whose Ricci tensor ϱ satisfies the condition $\nabla_X \varrho(X,X) = \frac{2}{n+2} X\tau g(X,X)$ for every $X \in TM$ (in Besse's notation $D\varrho \in C^\infty(Q \oplus A)$). A. Gray [G] also considered these manifolds and denoted this class by $\mathcal{A} \oplus \mathcal{C}^\perp$ (see [G], p. 265). It is remarked in [B] that very little is known about such manifolds if $\dim M > 2$. In the present paper we give a system of equations for the warping functions $f_1, f_2, \ldots, f_k \in C^\infty(M_0)$ and conditions on manifolds $(M_0, g_0), (M_1, g_1), \ldots, (M_k, g_k)$ under which the manifold $M = M_0 \times_{f_1} M_1 \times \ldots \times_{f_k} M_k$ is an $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold whose Ricci tensor has eigenfunctions $\lambda_0 = \mu + \frac{2}{n+2}\tau$, $\lambda_i = \mu + \frac{2}{n+2}\tau + C_i f_i^2$ where $\mu \in \mathbb{R}$ and $C_i \in \mathbb{R} - \{0\}$. We present very simple explicit examples of complete $\mathcal{A} \oplus \mathcal{C}^\perp$-manifolds (M,g) (with $M = \mathbb{R}^n$ for every $n > 2$) and many examples of compact $\mathcal{A} \oplus \mathcal{C}^\perp$-manifolds whose Ricci tensor has more than two eigenfunctions.

1. Killing tensors—preliminaries. Assume that M_i are smooth connected manifolds and g_i are smooth Riemannian metrics on M_i. All the manifolds, tensors and distributions considered in this paper are assumed to be smooth (of class C^∞). We also write $g(X,Y) = \langle X, Y\rangle$. Our present aim is to study in more detail the $(1,1)$ tensors S such that $\Phi(X,Y) = \langle SX, Y\rangle$ is a Killing tensor (which means that $\nabla_X \Phi(X,X) = 0$ for all $X \in TM$). We call such tensors $\mathcal{A}$-*tensors* or simply *Killing tensors* (and we write $S \in \mathcal{A}$). Hence S satisfies the following conditions:

$$\text{(a)} \qquad \langle SX, Y\rangle = \langle SY, X\rangle,$$

$$\text{(b)} \qquad \langle \nabla S(X,X), X\rangle = 0$$

for all $X, Y \in \mathfrak{X}(M)$. Define as in [D] the integer valued function $E_S(x) =$ (the number of distinct eigenvalues of S_x) and set $M_S = \{x \in M : E_S$ is constant in a neighborhood of $x\}$. Then M_S is an open submanifold of M and in every component U of M_S the eigenfunctions λ_i of S are smooth functions $\lambda_i|_U \in C^\infty(U)$. Set $D_i = \ker(S - \lambda_i \operatorname{Id})$. Then $D_i \perp D_j$ if $i \neq j$. We denote by $\Gamma(D_i)$ the set of all local sections of the vector bundle D_i. If $f \in C^\infty(M)$ then $\nabla f \in \mathfrak{X}(M)$ denotes the gradient of f, i.e. ∇f is the vector field on M such that $g(\nabla f, X) = df(X)$ for every $X \in TM$. We denote by H^f the Hessian of f which is defined by $H^f(X,Y) = XYf - df(\nabla_X Y)$ for every $X, Y \in TM$. The Hessian H^f is a symmetric $(0,2)$ tensor on M, which means that $H^f(X,Y)=H^f(Y,X)$. In what follows we consider each component U of M_S separately so we can assume that $M_S=M$. Note that

$$TM=\bigoplus_{i=1}^{k} D_i.$$

Denote by $p_i: TM\to D_i$ the orthogonal projection of TM on D_i.

Definition. A distribution $D\subset TM$ is called *umbilical* if there exists a vector field $\xi\in\mathfrak{X}(M)$ such that

$$\nabla_X X=p(\nabla_X X)+g(X,X)\xi$$

for every local section $X\in\Gamma(D)$ where p denotes the orthogonal projection $p: TM\to D$. If D is in addition integrable then we call D *totally umbilical*. The field ξ is called the *mean curvature normal* of the distribution D.

Proposition 1. *Assume that S is a Killing tensor. Then all the distributions $D_i=\ker(S-\lambda_i\,\mathrm{Id})$ are umbilical.*

Proof. Recall (see [J-1]) that if $S\in\mathcal{A}$ and $X\in\Gamma(D_i)$, $Y\in\Gamma(D_j)$ where $i\neq j$ then

$$\langle\nabla_X X,Y\rangle=\frac{1}{2}\frac{Y\lambda_i}{\lambda_j-\lambda_i}\|X\|^2. \tag{1.1}$$

Write $\nabla_X X=p_i(\nabla_X X)+h_i(X,X)$ where $h_i(X,X)\perp D_i$. From (1.1) it follows that for all $Y\in TM$ we have $\langle h_i(X,X),Y\rangle=\phi_i(Y)\langle X,X\rangle$, where ϕ_i is a one-form defined by

$$\phi_i(Y)=\frac{1}{2}\sum_{j\neq i}\frac{1}{\lambda_j-\lambda_i}d\lambda_i\circ p_j.$$

Hence $h_i(X,X)=\langle X,X\rangle\xi_i$ where the mean curvature normal field $\xi_i\in\mathfrak{X}(M)$ is defined as follows (note that $d\lambda_j\circ p_j=0$, see [J-1]):

$$\xi_i=-\frac{1}{2}\sum_{j\neq i}p_j(\nabla\ln|\lambda_i-\lambda_j|).\ \blacksquare \tag{1.2}$$

2. Killing tensors with integrable eigendistributions. In this section we investigate Killing tensors whose eigendistributions form an integrable almost product structure $(D_1,\ldots,D_k)$. This means that all the distributions $D_{i_1}\oplus\ldots\oplus D_{i_p}$ are integrable for any natural numbers $1\le i_1\le\ldots\le i_p\le k$ and for any $p\in\{1,\ldots,k\}$. We start with

Theorem 1. *Let S be a Killing tensor with constant eigenfunctions $\lambda_1,\ldots,\lambda_k$ and integrable almost product structure given by its eigendistributions $D_i=\ker(S-\lambda_i\,\mathrm{Id})$. Then $\nabla S=0$.*

P r o o f. Note that (see [J-1]) if S is a Killing tensor then for $X \in \Gamma(D_i)$ we have

$$\nabla S(X,X) = -\frac{1}{2}\nabla\lambda_i \|X\|^2 \tag{2.1}$$

and for $Y \in \Gamma(D_j)$,

$$\langle \nabla_X X, Y\rangle = \frac{1}{2}\frac{Y\lambda_i}{\lambda_j - \lambda_i}\|X\|^2. \tag{2.2}$$

If D_i are integrable then for all $X, Y \in \Gamma(D_i)$ we obtain (see [J-1]) $\nabla S(X,Y) = \nabla S(Y,X)$ and consequently

$$\nabla S(X,Y) = -\frac{1}{2}\nabla\lambda_i\langle X, Y\rangle. \tag{2.3}$$

From (2.2) it follows that the distributions D_i are autoparallel (i.e. $\nabla_X Y \in \Gamma(D_i)$ if $X, Y \in \Gamma(D_i)$). Note also that $M_S = M$. Next we prove the following lemma:

LEMMA A. *Let $(D_1, \ldots, D_k)$ be an integrable almost product structure on M such that $D_i \perp D_j$ if $i \neq j$ and*

$$TM = \bigoplus_{i=1}^{k} D_i.$$

Then $\nabla_X Y \in \Gamma(D_i \oplus D_j)$ if $i \neq j$ and $X \in \Gamma(D_i), Y \in \Gamma(D_j)$. Additionally if each D_i is autoparallel then each D_i is parallel.

P r o o f. Since the almost product structure $(D_1, \ldots, D_k)$ is integrable, for $i = 1, \ldots, k$ and every point $x_0 \in M$ we can find local coordinates $(x_1, \ldots, x_n)$ in a neighborhood U of x_0 (see [K-N]) such that

$$D_i|_U = \operatorname{span}\left\{\frac{\partial}{\partial x_{k_i}}, \ldots, \frac{\partial}{\partial x_{k_i+n_i}}\right\}$$

where $\dim D_i = n_i + 1$ and $k_1 = 1 < k_2 < \ldots < k_k < n$ are natural numbers. In what follows we write $\partial_i = \partial/\partial x_i$. Assume that p, q, r are pairwise different numbers and $\partial_i \in \Gamma(D_p)$, $\partial_j \in \Gamma(D_q)$, $\partial_l \in \Gamma(D_r)$. Then from the Koszul formula it easily follows that $\langle \nabla_{\partial_i}\partial_j, \partial_l\rangle = 0$. Hence $\nabla_X Y \in \Gamma(D_i \oplus D_j)$ if $X \in \Gamma(D_j)$ and $Y \in \Gamma(D_i)$. We show that in fact $\nabla_X Y \in \Gamma(D_i)$ if each D_i is autoparallel. Assume that $X' \in \Gamma(D_j)$ and $i \neq j$. Then $\langle X', Y\rangle = 0$. Thus

$$\langle \nabla_X X', Y\rangle + \langle X', \nabla_X Y\rangle = 0. \tag{2.4}$$

The distribution D_j is autoparallel, hence from (2.4) it follows that $\langle X', \nabla_X Y\rangle = 0$ and consequently for any section $Y \in \Gamma(D_i)$ and any $X \in \mathfrak{X}(M)$ we have $\nabla_X Y \in \Gamma(D_i)$. ■

Let now $X \in \mathfrak{X}(M)$ and let $Y \in \Gamma(D_j)$ be an arbitrary local section of D_j. Then $SY = \lambda_j Y$ and consequently

$$\nabla S(X,Y) = -(S - \lambda_j \operatorname{Id})(\nabla_X Y) = 0. \tag{2.5}$$

From (2.5) it is clear that $\nabla S = 0$, which finishes the proof of the theorem. ∎

Now we prove Theorem 2 which shows that there is a close relation between Riemannian warped products and certain Killing tensors. Recall (see [H], [N]) that if (M_i, g_i) for $i = 0, 1, \ldots, k$ are Riemannian manifolds and $f_1, \ldots, f_k$ are smooth positive functions on M_0 then the *warped product* $M = M_0 \times_{f_1} M_1 \times \ldots \times_{f_k} M_k$ is the Riemannian manifold (M, g) where $M = M_0 \times M_1 \times \ldots \times M_k$ and

$$g(X,Y) = g_0(p_0(X), p_0(Y)) + \sum_{i=1}^{k} f_i^2 g_i(p_i(X), p_i(Y))$$

where $p_i : TM \to TM_i$ is the natural projection.

THEOREM 2. *Assume that (M, g) is a complete simply connected Riemannian manifold and S is a Killing tensor on M with $k+1$ distinct eigenfunctions $\lambda_0 = \mu, \lambda_1, \ldots, \lambda_k$ and eigendistributions $D_i = \ker(S - \lambda_i \operatorname{Id})$. If*

(a) *the almost product structure $(D_0, D_1, \ldots, D_k)$ is integrable,*

(b) *$\lambda_0 = \mu$ is constant,*

(c) *the λ_i satisfy the condition $\bigoplus_{i \neq 0, j} D_i \subset \ker d\lambda_j$,*

then

$$M = M_0 \times_{f_1} M_1 \times_{f_2} M_2 \times .. \times_{f_k} M_k$$

where $TM_i = D_i$ and $f_i^2 = |\lambda_i - \mu|$.

Proof. Note that D_0 is an autoparallel foliation, which follows from general properties of $\mathcal{A}$-tensors (see [J-1]). In fact in [J-1] it is proved that an eigendistribution $D = \ker(S - \lambda \operatorname{Id})$ of S is autoparallel if and only if D is integrable and λ is constant. From Proposition 1 it follows that the D_i are totally umbilical. Note that $\xi_0 = 0$ and since $D_j \subset \ker d\lambda_j$ for any $\mathcal{A}$-tensor S, taking account of (1.2) we get $\xi_i = -\frac{1}{2} p_0 \nabla \ln |\lambda_i - \mu| = -\frac{1}{2} \nabla \ln |\lambda_i - \mu|$ where $h_i(X,Y) = \langle X, Y \rangle \xi_i$ is the second fundamental form of the foliation D_i. It is clear that $\xi_i \in \Gamma(D_0)$. Define (we follow [H]) a new metric $\overline{g}$ on M by

$$\overline{g}(X,Y) = g(p_0(X), p_0(Y)) + \sum_{i=1}^{k} f_i^{-2} g(p_i(X), p_i(Y)) \tag{2.6}$$

where $f_i = \sqrt{|\lambda_i - \mu|}$ and let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{g}$. We now show that the distributions D_i are autoparallel with respect to $\overline{\nabla}$. Let $X, Y \in \Gamma(D_i)$ and $Z \in \Gamma(D_j)$ where $i \neq j$. Set $f_0 = 1$. We consider two cases:

(a) $j \neq 0$. From the Koszul formula, taking account of (2.6) we get

$$\begin{aligned}2\overline{g}(\overline{\nabla}_X X, Z) &= -Z\overline{g}(X,X) - 2\overline{g}([X,Z],X)\\ &= f_i^{-2}(-Zg(X,Y) - 2g([X,Z],X)) = 2f_i^2(g(\nabla_X X, Z)) = 0.\end{aligned}$$

(b) $j = 0$. Then

$$\begin{aligned}2\overline{g}(\overline{\nabla}_X X, Z) &= -Z\overline{g}(X,X) - 2\overline{g}([X,Z],X)\\ &= 2Zf_i f_i^{-3} g(X,X) - f_i^{-2}Zg(X,X) - f_i^{-2}g([X,Z],X)\\ &= f_i^{-2}(2Z\ln f_i g(X,X) - Zg(X,X) - 2g([X,Z],X)\\ &= f_i^{-2}(2Z\ln f_i g(X,X) + 2g(\nabla_X X, Z)) = 0.\end{aligned}$$

Hence $\overline{\nabla}_X X \in \Gamma(D_i)$ if $X \in \Gamma(D_i)$ and each D_i is autoparallel. From Lemma A it follows that each D_i is parallel. The final result now follows from the de Rham theorem (see [K-N]). Since M is complete and simply connected we have

$$(M, \overline{g}) = (M_0, \overline{g}_0) \times (M_1, \overline{g}_1) \times \ldots \times (M_k, \overline{g}_k)$$

where $\overline{g}_i = f_i^{-2} p_i^* g$. Hence $g = \overline{g}_0 + \sum_{i=1}^k f_i^2 \overline{g}_i$, which completes the proof. ∎

Remark. Note that $\lambda_i = \mu + \varepsilon_i f_i^2$ where $\varepsilon_i \in \{-1, 1\}$. Consequently, $\operatorname{tr} S = \sum_{i=1}^k \varepsilon_i n_i f_i^2 + (\sum_{i=0}^k n_i)\mu$ where $n_i = \dim M_i$. Thus the trace of S is constant if and only if $\sum_{i=1}^k \varepsilon_i n_i f_i^2$ is constant. In particular if $k = 1$ then $\operatorname{tr} S$ is constant only if f_1 is constant. Note also that if $\mu = 0$ then we simply have

$$M = M_0 \times_{\sqrt{|\lambda_1|}} M_1 \times \ldots \times_{\sqrt{|\lambda_k|}} M_k.$$

Corollary. *Let $S \in \operatorname{End}(M)$ be a Killing tensor on a complete, simply connected Riemannian manifold M which has exactly three eigenfunctions $\lambda_0, \lambda_1, \lambda_2$ (i.e. $E_S = 3$). Assume that the almost product structure (D_0, D_1, D_2) given by eigendistributions D_i of S is integrable, the trace $\operatorname{tr} S$ is constant and $\lambda_0 = \mu \in \mathbb{R}$ is constant. Then*

$$M = M_0 \times_{f_1} M_1 \times_{f_2} M_2$$

where $TM_i = D_i = \ker(S - \lambda_i \operatorname{Id})$ and $f_i^2 = |\lambda_i - \mu|$.

Proof. It suffices to prove that condition (c) in the statement of Theorem 2 holds in our case. Note that if $n_i = \dim D_i$ then the function $g = n_0\mu + n_1\lambda_1 + n_2\lambda_2$ is constant. Hence

$$(*) \qquad n_1 \nabla\lambda_1 + n_2 \nabla\lambda_2 = 0.$$

Since $S \in \mathcal{A}$ it follows that $D_i \subset \ker d\lambda_i$. Thus $\nabla\lambda_i \perp D_i$. From $(*)$ it follows that $\nabla\lambda_1 \perp D_2$ and $\nabla\lambda_2 \perp D_1$, which means that condition (c) is satisfied. ∎

The following fact will be useful.

LEMMA B. *Assume that the distributions D_p are as above and $i \neq j$ and $j \neq 0$. If X, Y are local sections of D_i, D_j respectively then $\nabla_X Y \in \Gamma(D_j)$.*

Proof. Lemma A implies that $\nabla_X Y \in \Gamma(D_i \oplus D_j)$. Assume that $X \in \Gamma(D_i)$, $Y \in \Gamma(D_j)$, $Z \in \Gamma(D_i)$. Hence $\langle Z, Y\rangle = 0$ and consequently $\langle \nabla_X Z, Y\rangle + \langle Z, \nabla_X Y\rangle = 0$. Since $\nabla_X Z \in \Gamma(D_0 \oplus D_i)$ if $i \neq 0$ and $\nabla_X Z \in \Gamma(D_0)$ if $i = 0$ and consequently $\langle \nabla_X Z, Y\rangle = 0$, it follows that $\langle Z, \nabla_X Y\rangle = 0$, which completes the proof. ∎

REMARK. To prove Lemma B we have only used the facts that $TM = \bigoplus D_i$, $(D_0, D_1, \ldots, D_k)$ is an integrable almost product structure, $D_i \perp D_j$ if $i \neq j$, the D_i are totally umbilical, D_0 is autoparallel and the normal mean curvature ξ_i of D_i is a section of D_0, i.e. $\xi_i \in \Gamma(D_0)$.

Conversely, the following theorem holds:

THEOREM 3. *Assume that (M_i, g_i) for $i = 0, 1, \ldots, k$ are Riemannian manifolds and $f_i \in C^\infty(M_0, \mathbb{R}_+)$, $i \in \{1, \ldots, k\}$ are positive, smooth functions on M_0. Let (M, g) be the warped product manifold*

$$M = M_0 \times_{f_1} M_1 \times_{f_2} M_2 \times \ldots \times_{f_k} M_k$$

and define a $(1,1)$ tensor on M by

$$SX = \lambda_i X \quad \text{if } X \in D_i = TM_i \subset TM$$

where $\lambda_0 = \mu \in \mathbb{R}$, $\lambda_i = \mu + C_i f_i^2$ for a certain real number μ and real numbers $C_i \neq 0$, $i = 1, \ldots, k$. Then S is a Killing tensor on (M, g).

Proof. Define $D_i = TM_i \subset \bigoplus TM_j$. Note that the almost product structure $(D_0, D_1, \ldots, D_k)$ is integrable. Denote by ∇ the Levi-Civita connection of g. Since M is a warped product it follows that the distribution D_0 is autoparallel and each distribution D_i, $i > 0$, is totally umbilical and spherical with mean curvature normal $\xi_i = -\nabla \ln f_i = -\frac{1}{2}\nabla \ln|\lambda_i - \mu| \in \Gamma(D_0)$. Note that S is a well defined, smooth $(1,1)$ tensor on M. Consequently, if $X, Y \in \Gamma(D_i)$ and $i > 0$ then from the equality $SX = \lambda_i X$ we obtain

$$\begin{aligned}
0 &= \nabla S(Y, X) + (S - \lambda_i \,\mathrm{Id})(\nabla_Y X) \\
&= \nabla S(Y, X) + (S - \lambda_i \,\mathrm{Id})\left(-\frac{1}{2}\langle X, Y\rangle \frac{\nabla \lambda_i}{\lambda_i - \mu}\right) \\
&= \nabla S(Y, X) + \frac{1}{2}\langle Y, X\rangle \nabla \lambda_i.
\end{aligned}$$

Thus for every $X, Y \in \Gamma(D_i)$, $i > 0$, we obtain the formula

$$\nabla S(X, Y) = -\frac{1}{2}\langle X, Y\rangle \nabla \lambda_i. \tag{2.7}$$

Assume that $X \in \Gamma(D_i)$, $Y \in \Gamma(D_j)$ and $i \neq j$, $j \neq 0$. From Lemma B we get $\nabla_X Y \in \Gamma(D_j)$. Since $\nabla S(X,Y) + (S - \lambda_j \operatorname{Id})(\nabla_X Y) = 0$ we get

$$\nabla S(X,Y) = 0 \quad \text{if } X \in D_i,\ Y \in D_j,\ i \neq j,\ i,j \neq 0. \tag{2.8}$$

Assume now that $X \in \Gamma(D_0)$ and $Y \in \Gamma(D_i)$, $i > 0$. Then

$$\nabla S(X,Y) + (S - \lambda_i \operatorname{Id})(\nabla_X Y) = (X\lambda_i)Y.$$

From Lemma B it follows that $\nabla_X Y \in \Gamma(D_i)$ and hence

$$\nabla S(X,Y) = (X\lambda_i)Y \quad \text{if } X \in \Gamma(D_0),\ Y \in \Gamma(D_i),\ i > 0. \tag{2.9}$$

Note also that $\nabla S(X,Y) = 0$ if $X, Y \in \Gamma(D_0)$.

Our present aim is to show that the tensor S is an $\mathcal{A}$-tensor or equivalently that the tensor $\Phi(X,Y) = \langle SX, Y\rangle$ is a Killing tensor, which means that

$$\mathfrak{C}_{X,Y,Z} \nabla_X \Phi(Y,Z) = 0 \tag{A}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. We shall consider several cases.

(i) If $X \in D_i, Y \in D_j, Z \in D_k$ and i, j, k are pairwise different and different from 0, then from (2.8) it follows that $\nabla_X \Phi(Y,Z) = 0$ and consequently condition (A) holds.

(ii) If $X \in D_0$, $Y \in D_i$, $Z \in D_p$ and $i \neq p$, then from (2.9) we obtain $\langle \nabla S(X,Y), Z\rangle = 0$. Similarly using (2.7) we get $\langle \nabla S(Z,X), Y\rangle = \langle X, \nabla S(Z,Y)\rangle = 0$. Finally $\langle \nabla S(Y,Z), X\rangle = 0$ since $\nabla S(Y,Z) = 0$.

(iii) Now assume that $X = Y \in \Gamma(D_i)$ and $Z \in \Gamma(D_j)$. We shall consider three subcases.

(a) Assume that $i > 0$, $j > 0$. Then from (2.7), $\langle \nabla S(X,X), Z\rangle = 0$ and $\langle \nabla S(Z,X), X\rangle = -\langle (S - \lambda_i)(\nabla_Z X), X\rangle = 0$, thus $\mathfrak{C}_{X,Y,Z} \nabla_X \Phi(Y,Z) = 2\langle \nabla S(X,X), Z\rangle + \langle \nabla S(Z,X), X\rangle = 0$. Note that we did not assume here that $i \neq j$.

(b) If $i > 0$ and $j = 0$ then $\langle \nabla S(X,X), Z\rangle = -\frac{1}{2} Z\lambda_i \|X\|^2$ and from (2.9) we have $\nabla S(Z,X) = (Z\lambda_i)X$. Thus

$$\mathfrak{C}_{X,Y,Z} \nabla_X \Phi(Y,Z) = 2\langle \nabla S(X,X), Z\rangle + \langle \nabla S(Z,X), X\rangle = 0.$$

(c) If $i = 0$ then $\nabla S(X,X) = 0$ and

$$\langle \nabla S(Z,X), X\rangle = -\langle (S - \mu \operatorname{Id})(\nabla_Z X), X\rangle = 0.$$

From (i)–(iii) it follows that $\mathfrak{C}_{X,Y,Z} \nabla_X \Phi(Y,Z) = 0$ for any $X, Y, Z \in \mathfrak{X}(M)$, which completes the proof of Theorem 3. ∎

Remark. Note that we do not assume here that $\ker(S - \lambda_i \operatorname{Id}) = D_i$, i.e. it may happen that some of the eigenfunctions λ_i coincide at some points $x_0 \in M$ ($E_S(x_0) < k+1$ for some $x_0 \in M$). However if M_0 is compact we can always choose C_i in such a way that all λ_i are different at every point $x \in M$

(i.e. $E_S(x) = k+1$ for every $x \in M$). Note that if $\lambda_i(x_0) = \lambda_j(x_0)$ then $f_i^2(x_0) = \alpha f_j^2(x_0)$ for some $\alpha > 0$. If M_0 is compact and $\alpha > \sup f_i^2/\inf f_j^2$ then $\alpha f_j^2(x) - f_i^2(x) > 0$ for every $x \in M$. Thus if M_0 is compact then we can choose by induction C_i in such a way that $E_S(x) = k+1$ for every $x \in M$. If M is not compact we can still choose C_i in such a way that $E_S = k+1$ on an open and dense subset $U = M_S$ of M.

3. The structure of $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold on a warped product. In this section we shall find conditions under which the warped product

$$M = M_0 \times_{f_1} M_1 \times_{f_2} M_2 \times \ldots \times_{f_k} M_k$$

is an $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold. We shall prove that in this case every manifold M_i, $i > 0$, has to be an Einstein space and obtain a system of nonlinear partial differential equations on the warping functions $f_1, \ldots, f_k$ such that every solution $f_1, \ldots, f_k$ of this system gives the warped product $M = M_0 \times_{f_1} M_1 \times_{f_2} M_2 \times \ldots \times_{f_k} M_k$ which is an $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold. In [G] it is proved that every $\mathcal{A}$-manifold has constant scalar curvature. Hence an $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold is an $\mathcal{A}$-manifold if and only if it has constant scalar curvature.

Let us recall that a submersion $p : (M, g) \to (N, g_*)$ is called a *Riemannian submersion* if it preserves the lengths of horizontal vectors (see [O'N]). We denote by V the distribution of vertical vectors (i.e. those tangent to the fibers $F_x = p^{-1}(x)$, $x \in N$) and by H the horizontal distribution which is an orthogonal complement of V in TM. Define the O'Neill tensors T, A as follows:

$$T(E,F) = \mathcal{H}(\nabla_{\mathcal{V}E}\mathcal{V}F) + \mathcal{V}(\nabla_{\mathcal{V}E}\mathcal{H}F),$$
$$A(E,F) = \mathcal{V}(\nabla_{\mathcal{H}E}\mathcal{H}F) + \mathcal{H}(\nabla_{\mathcal{H}E}\mathcal{V}F),$$

where $\mathcal{H}, \mathcal{V}$ denote the orthogonal projections on H, V respectively. Our present aim is to describe the Ricci tensor of the warped product M. We start with

LEMMA C. *Let $p : (M, g) \to (N, g_*)$ be a Riemannian submersion and $f \in C^\infty(N)$. Set $F = f \circ p \in C^\infty(M)$. Then the Hessian H^F of F has the following properties*:

(a) $H^F(X^*, Y^*) = H^f(X, Y) \circ p$,
(b) $H^F(X^*, V) = -g(A(\nabla f, X^*), V)$,
(c) $H^F(U, V) = -g(\nabla f, T(U, V))$,

where $X, Y \in \mathfrak{X}(M)$, $U, V \in \Gamma(V)$ and X^* denotes the horizontal lift of X.

Proof. This follows from the following calculations:

$$\text{(a)}\quad \begin{aligned} H^F(X^*, Y^*) &= g(D_{X^*}\nabla F, Y^*) = X^* g(\nabla F, Y^*) - g(\nabla F, D_{X^*}Y^*) \\ &= X g_*(\nabla f, Y) - g_*(\nabla f, \nabla_X Y) = H^f(X, Y) \circ p, \end{aligned}$$

$$\begin{aligned}
\text{(b)}\quad H^F(X^*,V) &= g(D_{X^*}\nabla F,V) = -g(\nabla F, D_{X^*}V)\\
&= -g(\nabla F,\mathcal{H}(D_{X^*}V)) = -g_*(\nabla F, A(X^*,V))\\
&= g(A(X^*,\nabla F),V) = -g(A(\nabla F,X^*),V),\\
\text{(c)}\quad H^F(U,V) &= g(D_U\nabla F,V) = -g(\nabla F,D_UV)\\
&= -g(\nabla F,\mathcal{H}(D_UV)) - g_*(\nabla F,T(U,V)),
\end{aligned}$$

where D denotes the Levi-Civita connection of g and ∇ the Levi-Civita connection of g_*. ■

Corollary. *The Laplacian $\Delta F = \operatorname{tr}_g H^F$ of F with respect to g equals*

$$\Delta F = \Delta f - \langle \nabla f, \operatorname{tr}_g T\rangle.$$

If we take $M = M_0 \times_{f_1} M_1$ and $N = M_0$ then $A = 0$ and $T(U,V) = -\nabla \ln f_1 \langle U,V\rangle$. Consequently, for any $f \in C^\infty(M_0)$ Lemma C shows that if $F = f\circ p$ then

$$\begin{aligned}
&H^F(X_0^*,Y_0^*) = H^f(X_0,Y_0),\\
(3.1)\qquad &H^F(X_0^*,X_1^*) = 0,\\
&H^F(X_1^*,Y_1^*) = \langle \nabla \ln f_1, \nabla f\rangle \langle X_1^*, Y_1^*\rangle,
\end{aligned}$$

where $X_0, Y_0 \in \mathfrak{X}(M_0)$, $X_0, Y_1 \in \mathfrak{X}(M_1)$, $*$ denotes the lift of X to $X^* \in \mathfrak{X}(M_0 \times M_1)$ and $\langle\, ,\rangle$ denotes the warped product metric on M. In what follows we shall not distinguish X from X^*. Note also that in view of (3.1), $\Delta F = \Delta f + n_1\langle \nabla \ln f_1, \nabla f\rangle$. Let $M = M_0 \times_{f_1} M_1 \times_{f_2} M_2 \times \ldots \times_{f_k} M_k$ and write

$$\overline{M}_i = M_0 \times_{f_1} M_1 \times_{f_2} M_2 \times \ldots \times_{f_i} M_i$$

for $i > 0$. Then $\overline{M}_{i+1} = \overline{M}_i \times_{f_{i+1}} M_{i+1}$. Hence by an easy induction taking account of (3.1) we obtain for any $f \in C^\infty(M_0)$ the following formulas for H^F where $F = f\circ p \in C^\infty(M)$ and $X_i, Y_i \in \mathfrak{X}(M_i)$:

$$\begin{aligned}
&H^F(X_i,Y_i) = \langle \nabla f, \nabla \ln f_i\rangle\langle X_i,Y_i\rangle \quad \text{if } i\neq 0,\\
(3.2)\qquad &H^F(X_0,Y_0) = H^f(X_0,Y_0),\\
&H^F(X_i,X_j) = 0 \quad \text{if } i \neq j.
\end{aligned}$$

Recall that if $M = M_0 \times_{f_1} M_1$ then (see for example [B]) we have the following formulas for the Ricci tensor Ric of M:

$$\begin{aligned}
&\operatorname{Ric}(X_0,Y_0) = \operatorname{Ric}^0(X_0,Y_0) - \frac{n}{f_1}H^{f_1}(X_0,Y_0),\\
(3.3)\qquad &\operatorname{Ric}(X_0,X_1) = 0,\\
&\operatorname{Ric}(X_1,Y_1) = \operatorname{Ric}^1(X_1,Y_1) - \left(\frac{\Delta f_1}{f_1} + (n-1)\frac{\|\nabla f_1\|^2}{f_1^2}\right)\langle X_1,Y_1\rangle,
\end{aligned}$$

where $\Delta f = \operatorname{tr}_g H^f$ and $n = \dim M_1$. Hence, taking account of (3.2) and (3.3), by an easy induction we obtain

LEMMA D. *Let* $M = M_0 \times_{f_1} M_1 \times_{f_2} M_2 \times \ldots \times_{f_k} M_k$ *and assume that* $X_i, Y_i \in \mathfrak{X}(M_i)$ *for* $i = 0, 1, \ldots, k$ *are arbitrary vector fields. Then the Ricci tensor of* (M, g) *is given by*

$$\mathrm{Ric}(X_0, Y_0) = \mathrm{Ric}^0(X_0, Y_0) - \sum_{i=1}^{k} \frac{n_i}{f_i} H^{f_i}(X_0, Y_0),$$

$$\mathrm{Ric}(X_i, Y_i) = \mathrm{Ric}^i(X_i, Y_i) - \left(\frac{\Delta f_i}{f_i} + \sum_{j=1}^{k} n_j \langle \nabla \ln f_i, \nabla \ln f_j \rangle - \|\nabla \ln f_i\|^2 \right) \langle X_i, Y_i \rangle,$$

where $i > 0$ *and* $\mathrm{Ric}(X_i, X_j) = 0$ *if* $i \neq j$ (*here* Ric, Ric^i *denote the Ricci tensors of* (M, g) *and* (M_i, g_i) *respectively and* $n_i = \dim M_i$). *In particular if* $k = 2$ *then*

$$\mathrm{Ric}(X_0, Y_0) = \mathrm{Ric}^0(X_0, Y_0) - \frac{n_1}{f_1} H^{f_1}(X_0, Y_0) - \frac{n_2}{f_2} H^{f_2}(X_0, Y_0),$$

$$\mathrm{Ric}(X_1, Y_1) = \mathrm{Ric}^1(X_1, Y_1) - \left(\frac{\Delta f_1}{f_1} + (n_1 - 1)\|\nabla \ln f_1\|^2 + n_2 \langle \nabla \ln f_1, \nabla \ln f_2 \rangle \right) \langle X_1, Y_1 \rangle,$$

$$\mathrm{Ric}(X_2, Y_2) = \mathrm{Ric}^2(X_2, Y_2) - \left(\frac{\Delta f_2}{f_2} + (n_2 - 1)\|\nabla \ln f_2\|^2 + n_1 \langle \nabla \ln f_1, \nabla \ln f_2 \rangle \right) \langle X_2, Y_2 \rangle.$$

Assume that the Ricci endomorphism S of (M, g) is such that for a certain function $s \in C^\infty(M)$ the tensor $S - s\,\mathrm{Id}$ satisfies the conditions of Theorem 3 (i.e. $\mathrm{Ric}(X_i, Y) = (\lambda_i + s)\langle X_i, Y \rangle$ where $\lambda_0 = \mu$ and $\lambda_i = \mu + C_i f_i^2$, $i > 0$). Then

$$\mathrm{Ric}^0(X_0, Y_0) = \sum_{i=1}^{k} \frac{n_i}{f_i} H^{f_i}(X_0, Y_0) + (\mu + s)\langle X_0, Y_0 \rangle, \tag{3.4i}$$

$$\mathrm{Ric}^i(X_i, Y_i) = \Big(\mu + s + C_i f_i^2 + \Delta \ln f_i + \sum_{j=1}^{k} n_j \langle \nabla \ln f_i, \nabla \ln f_j \rangle \Big) \langle X_i, Y_i \rangle \tag{3.4ii}$$

for some $C_i \in \mathbb{R} - \{0\}$.

LEMMA E. *Let* (M, g) *be a Riemannian manifold of dimension* n. *Then* M *is an* $\mathcal{A} \oplus \mathcal{C}^\perp$*-manifold if and only if there exists a function* $s \in C^\infty(M)$ *such that*

$$S - s\,\mathrm{Id} \in \mathcal{A}. \tag{$*$}$$

If $(*)$ *holds then* $ds = \frac{2}{n+2} d\tau$ *where* τ *is the scalar curvature of* (M, g).

P r o o f. From $(*)$ we get

$$\mathfrak{C}_{X,Y,Z}\nabla_X\varrho(Y,Z)=\mathfrak{C}_{X,Y,Z}Xsg(Y,Z).$$

Hence

$$2\nabla_X\varrho(X,Y)+\nabla_Y\varrho(X,X)=2Xsg(X,Y)+Ysg(X,X). \tag{3.5}$$

Set $\delta\varrho(Y)=\operatorname{tr}_g\nabla_{.}\varrho(\cdot,Y)$. Then $\delta\varrho=\frac{1}{2}d\tau$.

On the other hand taking account of (3.5) we have

$$2\delta\varrho+\operatorname{tr}\nabla_Y\varrho(\cdot,\cdot)=2g(\nabla s,Y)+nYs. \tag{3.6}$$

Since $\operatorname{tr}\nabla_Y\varrho(\cdot,\cdot)=Y\tau$ we finally obtain $2d\tau=(n+2)ds$. ∎

Taking account of Theorem 3 and Lemma E we get

COROLLARY. *If* (M_i,g_i) *and* f_i *satisfy equations* (3.4i), (3.4ii) *then* (M,g) *is an* $\mathcal{A}\oplus\mathcal{C}^\perp$*-manifold and all the manifolds* (M_i,g_i) *are Einstein for* $i>0$, *i.e.* $\operatorname{Ric}^i=\tau_ig_i$ *where* $\tau_i\in\mathbb{R}$.

Summarizing we get

THEOREM 4. *Assume that a simply connected complete* $\mathcal{A}\oplus\mathcal{C}^\perp$*-manifold* (M,g) *has Ricci tensor* S, *with* $k+1$ *eigenfunctions* $\lambda_0,\lambda_1,\ldots,\lambda_k$, *such that for* $s=\frac{2}{n+2}\tau$ *the tensor* $S-s\operatorname{Id}$ *satisfies the assumptions of Theorem* 2. *Then*

$$M=M_0\times_{f_1}M_1\times_{f_2}M_2\times\ldots\times_{f_k}M_k$$

where (M_i,g_i) *are Einstein manifolds* ($\operatorname{Ric}^i=\tau_ig_i$ *for* $i>0$) *of dimensions* $\dim M_i=n_i$ *and the Ricci tensor* Ric^0 *of* (M_0,g_0) *satisfies*

$$\operatorname{Ric}^0(X,Y)=(\mu+s)g_0(X,Y)+\sum_{i=1}^k\frac{n_i}{f_i}H^{f_i}(X,Y) \tag{3.7}$$

and $\lambda_0=\mu+s$ *where* $\mu\in\mathbb{R}$, $\lambda_i=\mu+s+C_if_i^2$ *for* $i>0$ *where* $C_i\in\mathbb{R}-\{0\}$. *The functions* f_i *additionally satisfy the following* k *equations*:

$$\Delta\ln f_i+\sum_{j=1}^k n_j\langle\nabla\ln f_i,\nabla\ln f_j\rangle+\mu+s+C_if_i^2-\frac{\tau_i}{f_i^2}=0. \tag{3.8}$$

Conversely, *assume that* (M_i,g_i) *are Einstein with* $\operatorname{Ric}^i=\tau_ig_i$ *for* $i>0$, $\dim M_i=n_i$ *and functions* $f_1,\ldots,f_k$ *satisfy equations* (3.4i), (3.4ii) *for some* $C_i\in\mathbb{R}-\{0\}$ *and* $s\in C^\infty(M)$. *Then* $M=M_0\times_{f_1}M_1\times\ldots\times_{f_k}M_k\in\mathcal{A}\oplus\mathcal{C}^\perp$ *and* $ds=\frac{2}{n+2}d\tau$ *where* τ *is the scalar curvature of* (M,g).

P r o o f. Note that

$$\begin{aligned} H^{\ln f}(X,Y) &= \langle D_X \nabla \ln f, Y\rangle \\ &= \left\langle D_X \frac{\nabla f}{f}, Y\right\rangle = \left\langle \frac{1}{f} D_X \nabla f, Y\right\rangle + \left\langle -\frac{Xf}{f^2}\nabla f, Y\right\rangle \\ &= -X\ln f Y \ln f + \frac{1}{f} H^f(X,Y). \end{aligned}$$

Hence $\Delta \ln f = -\|\nabla \ln f\|^2 + \frac{1}{f}\Delta f$ and the assertion follows from Lemmas D and E. ∎

Corollary. *Let (M,g) be a complete, simply connected $\mathcal{A}$-manifold whose Ricci tensor S has exactly three eigenfunctions $\lambda_0, \lambda_1, \lambda_2$ (i.e. $E_S = 3$). Assume that all eigendistributions D_i of S form an integrable almost product structure and $\lambda_0 = \mu \in \mathbb{R}$ is constant. Then*

$$M = M_0 \times_{f_1} M_1 \times_{f_2} M_2$$

where $TM_i = D_i = \ker(S - \lambda_i \operatorname{Id})$ and $f_i^2 = |\lambda_i - \mu|$ for $i > 0$. The manifolds M_1, M_2 are Einstein spaces ($\mathrm{Ric}^i = \tau_i g_i$, $i = 1,2$) *and the warping functions f_1, f_2 satisfy equations* (3.7) *and* (3.8) *with $k = 2$.*

In the book [B] many examples of $\mathcal{A} \oplus \mathcal{C}^\perp$-manifolds are given, including compact twisted warped products. However the description of these examples is rather complicated. All of them have harmonic Weyl tensor so they are $\mathcal{C}^\perp$-manifolds in Gray's notation (see [G]). Now we give examples of complete $\mathcal{A} \oplus \mathcal{C}^\perp$-manifolds (M,g) with $M = \mathbb{R}^n$ for every $n > 2$ which are of a very simple explicit form. These manifolds are conformally flat so in fact they are also $\mathcal{C}^\perp$-manifolds. Take $M_0 = \mathbb{R}_+$ and $k = 1$. We shall write $f_1 = f$ and $C_1 = C$, $\tau_1 = \tau$. Equations (3.7) and (3.8) are

$$\frac{nf''}{f} = -(\mu + s), \tag{3.9a}$$

$$\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2} - \frac{\tau}{f^2} + \mu + s + Cf^2 = 0. \tag{3.9b}$$

Consequently, we get

$$-(n-1)\left(\frac{f''}{f} - \frac{(f')^2}{f^2}\right) = \frac{\tau}{f^2} - Cf^2. \tag{3.10}$$

From (3.10) we obtain $(n-1)(\ln f)'' = Cf^2 - \tau f^{-2}$. Write $g = \ln f$ and let $C = \tau$. Hence we get

$$g'' = \frac{2\tau}{n-1}\sinh 2g. \tag{3.11}$$

Integrating (3.11) we have

$$\frac{1}{2}(g')^2 = \frac{\tau}{n-1}\cosh 2g + D_1 = \frac{\tau}{n-1}(\cosh 2g + D_0) \tag{3.12}$$

where $D_0 \in \mathbb{R}$. Take $D_0 = -1$. Then equation (3.12) reads

$$(g')^2 = \frac{4\tau}{n-1}(\sinh g)^2.$$

Thus $\tau > 0$ and consequently

$$g' = 2\varepsilon\sqrt{\frac{\tau}{n-1}}\sinh g, \tag{3.13}$$

where $\varepsilon \in \{-1, 1\}$. Take $\varepsilon = -1$. Integrating equation (3.13) we get

$$\tanh\left(\frac{g}{2}\right) = E_0 \exp\left(-2\sqrt{\frac{\tau}{n-1}}t\right), \tag{3.14}$$

where $E_0 \in \mathbb{R} - \{0\}$. From (3.14) we obtain, taking $E_0 = -1$, and $M_1 = S^n$ with the standard metric of constant sectional curvature,

$$f(t) = \tanh\left(\sqrt{\frac{\tau}{n-1}}t\right). \tag{3.15}$$

Our metric on $M = \mathbb{R}_+ \times S^n$ is

$$g_\lambda = dt^2 + \phi(t)^2\lambda\,\mathrm{can} \tag{3.16}$$

where can is the standard metric on S^n with sectional curvature 1,

$$\phi(t) = \tanh\left(\sqrt{\frac{\tau}{n-1}}t\right) \quad \text{and} \quad \lambda = \frac{n-1}{\tau}.$$

Hence $\phi'(0) = 1/\sqrt{\lambda}$. Thus $\phi'(0) = \lambda$ if $\lambda = 1$. In view of Lemma 9.114, p. 269 of [B], the metric g_λ for $\lambda = \lambda_0 = 1$ extends to a C^∞ metric $\overline{g}_{\lambda_0}$ on $\overline{M} = \mathbb{R}^{n+1}$ for $n > 1$. Since $M = \mathbb{R}^{n+1} - \{0\}$ and (M, g_{λ_0}) is an $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold it follows that $(\overline{M}, \overline{g}_{\lambda_0})$ is also an $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold. We shall show that $(\overline{M}, \overline{g}_{\lambda_0})$ is complete. To this end recall the following

Lemma F. *Assume that d_1, d_2 are metrics on the space M. If (M, d_1) is a complete metric space and there exist positive constants C_1, C_2 such that for every $x, y \in M$,*

$$C_1 d_1(x, y) \le d_2(x, y) \le C_2 d_1(x, y)$$

then the metric space (M, d_2) is also complete. Every Cauchy sequence in (M, d_1) is a Cauchy sequence in (M, d_2) and vice versa.

Observe that on M,

$$g_{\lambda_0} = dt^2 + (\tanh t)^2\mathrm{can}. \tag{3.17}$$

Note that the metric $g_n = dt^2 + (\sinh t)^2 \operatorname{can}$ on $M = \mathbb{R}_+ \times S^n$ can be extended to a complete metric on the hyperbolic space $\overline{M} = H^{n+1}$ (see [B], 9.111, p. 268). Let $p : \overline{M} = \mathbb{R}^{n+1} \to \mathbb{R}_+ \cup \{0\}$ be defined by

$$p(x_1, \ldots, x_{n+1}) = \sqrt{x_1^2 + \ldots + x_{n+1}^2}.$$

Then p is an extension to $\overline{M}$ of the natural projection $p : \mathbb{R}_+ \times S^n \to \mathbb{R}_+$. We shall denote by d_0 the metric induced on $\overline{M}$ by the Riemannian metric tensor g_{λ_0}, and by $|a-b|$ the natural metric on $\mathbb{R}$. Note that in view of (3.17), $|p(x)-p(y)| \le d_0(p(x), p(y))$. Hence if (x_n) is a Cauchy sequence in $(\overline{M}, d_0)$ then $(p(x_n))$ is a Cauchy sequence in $(\mathbb{R}, |\,|)$. In particular $p(x_n)$ is bounded, i.e. there exists $K > 0$ such that $p(x_n) < K$ for every $n \in \mathbb{N}$. It follows that $\cosh(p(x_n)) < L = \cosh K$. Note that on $D = \{x \in M : p(x) < K\}$ we have

$$\frac{1}{L^2}(du^2 + (\sinh u)^2 \operatorname{can}) < g_{\lambda_0} < (du^2 + (\sinh u)^2 \operatorname{can}).$$

Hence $L^{-1} d_n(x, y) < d_0(x, y) < d_n(x, y)$ for every $x, y \in D$ where d_n denotes the complete hyperbolic metric on H^{n+1} induced by $du^2 + (\sinh u)^2 \operatorname{can}$. Taking account of Lemma F we see that the Cauchy sequence x_n in $(\overline{M}, d_0)$ is convergent, which means that $(\overline{M}, d_0)$ is complete. Hence we have proved

THEOREM 5. *For every $n \ge 2$ the metric $g_{\lambda_0} = dt^2 + (\tanh t)^2 \operatorname{can}$ on the space $M = \mathbb{R}_+ \times S^n$ extends to a smooth complete metric $\overline{g}_{\lambda_0}$ on $\overline{M} = \mathbb{R}^{n+1}$ such that $(\mathbb{R}^{n+1}, \overline{g}_{\lambda_0})$ is an $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold.*

Finally we shall construct compact examples of $\mathcal{A} \oplus \mathcal{C}^\perp$-manifolds with more than two eigenvalues of the Ricci tensor. As in [B], we shall also consider twisted warped products (note that in [B] all the examples have only two eigenvalues and the construction is different from ours). Consider the equations (3.7) and (3.8) and take $k = 2$, $M_0 = \mathbb{R}$. The manifolds M_1, M_2 are assumed to be Einstein with $\dim M_1 = n_1 = n$, $\dim M_2 = n_2 = m$ where $m, n > 1$. Equations (3.7) and (3.8) are

$$\frac{nf''}{f} + \frac{mg''}{g} = -(\mu + s), \tag{3.18a}$$

$$\frac{f''}{f} + (n-1)\frac{(f')^2}{f^2} + m\frac{g'f'}{fg} - \frac{\tau_1}{f^2} + \mu + s + C_1 f^2 = 0, \tag{3.18b}$$

$$\frac{g''}{g} + (m-1)\frac{(g')^2}{g^2} + n\frac{f'g'}{fg} - \frac{\tau_2}{g^2} + \mu + s + C_2 g^2 = 0. \tag{3.18c}$$

Assume that $f = 1/g$. Then $\ln f = -\ln g$ and

$$\frac{f'g'}{fg} = -\left(\frac{f'}{f}\right)^2 = -\left(\frac{g'}{g}\right)^2.$$

Hence we obtain as before (see the solution of (3.9))

$$-(n-1)(\ln f)'' - m(\ln g)'' - 2m\left(\frac{f'}{f}\right)^2 = \frac{\tau_1}{f^2} - C_1 f^2, \tag{3.19a}$$

$$-n(\ln f)'' - (m-1)(\ln g)'' - 2n\left(\frac{f'}{f}\right)^2 = \frac{\tau_2}{g^2} - C_2 g^2. \tag{3.19b}$$

Consequently,

$$(m-n+1)\frac{f''}{f} - (3m-n+1)\left(\frac{f'}{f}\right)^2 = \frac{\tau_1}{f^2} - C_1 f^2, \tag{3.20a}$$

$$(m-n-1)\frac{f''}{f} - (m+n+1)\left(\frac{f'}{f}\right)^2 = \tau_2 f^2 - \frac{C_2}{f^2}. \tag{3.20b}$$

Thus equations (3.19) are

$$\frac{f''}{f} - \left(\frac{3m-n+1}{m-n+1}\right)\left(\frac{f'}{f}\right)^2 = \frac{1}{m-n+1}\left(\frac{\tau_1}{f^2} - C_1 f^2\right), \tag{3.21a}$$

$$\frac{f''}{f} - \left(\frac{m+n+1}{m-n-1}\right)\left(\frac{f'}{f}\right)^2 = \frac{1}{m-n-1}\left(\tau_2 f^2 - \frac{C_2}{f^2}\right). \tag{3.21b}$$

Set $k = m - n$. Hence $k \in \mathbb{Z}$. Note that

$$\frac{3m-n+1}{m-n+1} = \frac{m+n-1}{m-n-1} \tag{3.22}$$

if and only if

$$k^2 + k - 2m = 0. \tag{3.23}$$

Consequently, (3.22) holds if and only if $m = \frac{1}{2}k(k+1)$, $n = \frac{1}{2}k(k-1)$ where (we have assumed that $m, n > 1$) $k \in \mathbb{Z}$ and $|k| > 2$. We can assume that $k > 2$ since the case $k < -2$ is obtained from the first case on replacing f by g. We shall assume that the above conditions on m and n are satisfied. Then

$$\frac{3m-n+1}{m-n+1} = \frac{m+n-1}{m-n-1} = k+1. \tag{3.24}$$

Set $a = k + 1$. Assume also that C_1, C_2 satisfy

$$\frac{\tau_1}{k+1} = -\frac{C_2}{k-1} \quad \text{and} \quad \frac{\tau_2}{k-1} = -\frac{C_1}{k+1}.$$

Hence the equations (3.21) reduce to

$$\frac{f''}{f} - a\left(\frac{f'}{f}\right)^2 = \frac{c}{f^2} + df^2 \tag{A}$$

where $a = k+1, c = \frac{\tau_1}{k+1}, d = \frac{\tau_2}{k-1}$. Write $(f')^2 = P(f^2)$ where $P \in C^\infty(\mathbb{R})$.

Then P satisfies the equation

$$P'(x) - \frac{a}{x}P(x) = \frac{c}{x} + dx. \tag{3.25}$$

Hence

$$P(x) = C_0 x^a + \frac{d}{2-a}x^2 - \frac{c}{a}$$

where $C_0 \in \mathbb{R}$. Consequently,

$$(f')^2 = C_0 f^{2a} + \frac{d}{2-a}f^4 - \frac{c}{a}. \tag{3.26}$$

From (3.26) and (A) it follows that f satisfies

$$f'' = \frac{2d}{2-a}f^3 + aC_0 f^{2a-1} \tag{3.27}$$

and (3.26) is a first integral of (3.27). Every solution of equations (3.26) and (3.27) satisfies (A). Note that (3.26) can be written as $(f')^2 = Q(f)$ where $Q(x) = C_0 x^{2a} + Dx^4 + E$ and

$$D = \frac{d}{2-a} = -\frac{\tau_2}{(k-1)^2}, \qquad E = -\frac{c}{a} = -\frac{\tau_1}{(k+1)^2}.$$

Lemma G. *Consider the equations*

$$f'' = \frac{1}{2}Q'(f), \tag{F1}$$

$$(f')^2 = Q(f), \tag{F2}$$

where $Q(x) = C_0 x^{2a} + Dx^4 + E$ *and* $a = k+1 \geq 4$. *Then the equations* (F1), (F2) *admit a periodic nonconstant positive solution* f *if and only if* $C_0 < 0, E < 0, D > 0$ *and*

$$Q(x_0) = x_0^4 D\left(\frac{k-1}{k+1}\right) + E > 0$$

where x_0 *is a positive root of the equation* $Q'(x) = 0$, *thus*

$$x_0 = \left(-\frac{2D}{C_0 a}\right)^{1/(2k-2)}.$$

Proof. We shall use Lemma 16.37, p. 445 of [B]. The equation $Q'(x) = 0$ has a positive solution if and only if $C_0 D < 0$. From Lemma 16.37 of [B] it follows that $C_0 < 0$. Hence $D > 0$. Note that in our case the range of a solution f is $\mathrm{im} f = [x_1, x_2]$ where $x_1 < x_2$ are positive roots of Q. The polynomial Q has two positive roots if $Q(0) = E < 0$ and $Q(x_0) > 0$. ∎

From Lemma G it follows that equation (A) has a periodic, nonconstant and positive solution if $\tau_1 > 0, \tau_2 < 0$ and $\tau_1 \in (0, \alpha)$ where

$$\alpha = -x_0^4 \tau_2 \frac{k+1}{k-1} = |\tau_2|^{(k+1)/(k-1)} \left(\frac{2}{(k-1)^2(k+1)|C_0|} \right)^{2/(k-1)} \frac{k+1}{k-1}.$$

Take compact Einstein manifolds $(M_1, g_1), (M_2, g_2)$ whose Ricci tensors ϱ_1, ϱ_2 satisfy $\varrho_i = \tau_i g_i$ and τ_i are as above. We also assume that $\dim M_1 = \frac{1}{2}k(k-1)$ and $\dim M_2 = \frac{1}{2}k(k+1)$. Let f be a positive periodic solution of (A). Then the $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold $M = \mathbb{R} \times_f M_1 \times_{1/f} M_2$ has compact quotients $\overline{M} = M/\mathbb{Z}$ with a twisted warped product metric (for the details see [B], 16.26, p. 441). Such quotients are compact $\mathcal{A} \oplus \mathcal{C}^\perp$-manifolds whose Ricci tensor has three different eigenfunctions (at least on an open and dense subset U of $\overline{M}$).

THEOREM 6. *Let (M_1, g_1), (M_2, g_2) be compact Einstein manifolds*, $\dim M_1 = \frac{1}{2}k(k-1)$, $\dim M_2 = \frac{1}{2}k(k+1)$ *where $k \in \mathbb{N}$, $k > 2$. Assume that the Ricci tensors ϱ_i of (M_i, g_i) satisfy $\varrho_i = \tau_i g_i$ where $\tau_1 > 0$ and $\tau_2 < 0$. Then there exists a positive, periodic nonconstant function $f \in C^\infty(\mathbb{R})$ such that $M = \mathbb{R} \times_f M_1 \times_{1/f} M_2$ is an $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold. If $A_1 \in \mathrm{Iso}(M_1)$ and $A_2 \in \mathrm{Iso}(M_2)$ are isometries of (M_i, g_i), $i = 1, 2$, respectively, then the mapping $A : M \to M$ defined by $A(t, x_1, x_2) = (t + T, A_1(x_1), A_2(x_2))$, where T is the period of f, is an isometry of (M, g). The quotient $\overline{M} = M/\mathbb{Z}A$ with the twisted warped product metric $\overline{g}$ is a compact $\mathcal{A} \oplus \mathcal{C}^\perp$-manifold whose Rici tensor has three eigenfunctions.*

Proof. Choose $C_0 < 0$ such that

$$\tau_1 < |\tau_2|^{(k+1)/(k-1)} \left(\frac{2}{(k-1)^2(k+1)|C_0|} \right)^{2/(k-1)} \frac{k+1}{k-1}.$$

Then our theorem follows from Lemma G and from [B], 16.26 (p. 441). ∎

REMARK. It is easy to show that the above examples do *not* have harmonic Weyl tensor, i.e. that the tensor $\varrho(X, Y) - \frac{\tau}{2n-2} g(X, Y)$ where $\tau = \mathrm{tr}\, \varrho$ is not a Codazzi tensor. Thus $(\overline{M}, \overline{g}) \in \mathcal{A} \oplus \mathcal{C}^\perp$ and $(\overline{M}, \overline{g}) \notin \mathcal{C}^\perp$. Since the scalar curvature of $(\overline{M}, \overline{g})$ is not constant we also have $(\overline{M}, \overline{g}) \notin \mathcal{A}$. As far as the author knows these are the first known examples of this kind (see [B], p. 433). Note that from the author's paper [J-2] it follows that the examples of compact Einstein–Weyl 4-manifolds given in [M-P-P-S] are also of this kind.

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