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Local characterization of algebraic manifolds and characterization of components of the set S_f

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Abstract. We show that every n-dimensional smooth algebraic variety X can be covered by Zariski open subsets U_i which are isomorphic to closed smooth hypersurfaces in $\mathbb{C}^{n+1}$.

As an application we show that for every (pure) $n-1$-dimensional $\mathbb{C}$-uniruled variety $X \subset \mathbb{C}^m$ there is a generically-finite (even quasi-finite) polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ such that $X \subset S_f$.

This gives (together with [3]) a full characterization of irreducible components of the set S_f for generically-finite polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^m$.

1. Introduction. In Section 2 we prove the following theorem:

Let X be an n-dimensional algebraic variety and $x \in X$ be a smooth point on X. Then there is a Zariski open neighborhood $U_x \subset X$ of x which is isomorphic to a closed smooth hypersurface in $\mathbb{C}^{n+1}$.

In particular it implies that every n-dimensional smooth algebraic variety X can be covered by Zariski open subsets U_i which are isomorphic to closed smooth hypersurfaces in $\mathbb{C}^{n+1}$. Moreover, we find that every algebraic variety of dimension $n > 0$ has infinitely many pairwise non-isomorphic smooth models in $\mathbb{C}^{n+1}$.

As an application of the theorem above we give a characterization of components of the set S_f of points at which a polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is not proper. Let us recall that f is *not proper* at a point y if there is no neighborhood U of y such that $f^{-1}(\mathrm{cl}(U))$ is compact. In [3] we showed that the set S_f (if non-empty) has pure dimension $n-1$ and it is $\mathbb{C}$-*uniruled*, i.e., for every point $x \in S_f$ there is an affine parametric curve through this point. In this paper we show that, conversely, for every $\mathbb{C}$-uniruled $n-1$-dimensional variety $X \subset \mathbb{C}^m$ (where $2 \le n \le m$), there

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is a generically-finite (even quasi-finite) polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^m$ such that $X \subset S_F$. This gives (together with [3]) a full characterization of irreducible components of the set S_f.

2. Zariski open subsets which are affine hypersurfaces. We begin with the following lemma:

LEMMA 2.1. *Let X be an n-dimensional affine algebraic variety and $w \in X$ be a smooth point on X. Then there is a finite, regular birational mapping $\phi : X \to \mathbb{C}^{n+1}$ such that*

(1) $\phi^{-1}(\phi(w)) = \{w\}$,
(2) *the mapping $d_w\phi : T_wX \to \mathbb{C}^{n+1}$ is an embedding.*

Proof. We can assume that $X \subset \mathbb{C}^m$. Observe that there is a finite projection $f : X \to \mathbb{C}^n$ such that $d_w f : T_wX \to \mathbb{C}^n$ is an isomorphism. We can assume that $f : X \ni (x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) \mapsto (x_1, \ldots, x_n) \in \mathbb{C}^n$. Let $f^{-1}(f(w)) = \{w_1, \ldots, w_k\}$. By the theorem on the primitive element there is a linear form $z(x) = \sum_{i=n+1}^{m} c_i x_i$ such that the mapping $\phi := (f, z)$ is birational. Moreover, we can assume that $z(w_i) \neq z(w_j)$ for $i \neq j$ and consequently $\phi^{-1}(\phi(w)) = \{w\}$. This finishes the proof. ∎

REMARK 2.2. In particular, the point $\phi(w)$ is smooth on the variety $\phi(X)$.

For our next step it is convenient to introduce the following special birational mapping F_h.

DEFINITION 2.3. Let $h \in \mathbb{C}[x_1, \ldots, x_n]$ and define

$$F_h : \mathbb{C}^{n+1} \ni (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n, h(x_1, \ldots, x_n)x_{n+1}) \in \mathbb{C}^{n+1}.$$

The mapping F_h will be called the *h-process* given by the polynomial h. The hypersurface $V_h := \{x \in \mathbb{C}^{n+1} : h(x) = 0\}$ is the *vertical hypersurface* of the h-process F_h and the hyperplane $H_h := \{x \in \mathbb{C}^{n+1} : x_{n+1} = 0\}$ is the *horizontal hyperplane* of F_h.

LEMMA 2.4. *The h-process F_h given by a polynomial $h \in \mathbb{C}[x_1, \ldots, x_n]$ is a birational mapping and the restriction $F_h : \mathbb{C}^{n+1} \setminus V_h \to \mathbb{C}^{n+1} \setminus V_h$ is an isomorphism. Moreover, for every hypersurface*

$$X := \Big\{x \in \mathbb{C}^{n+1} : x_{n+1}^k + \sum_{j=1}^{k-1} a_j(x_1, \ldots, x_n) x_{n+1}^{k-j} = 0\Big\},$$

we have $\mathrm{cl}(F_h(X)) \cap V_h \subset V_h \cap H_h$.

Proof. Indeed, it is easy to see that the hypersurface $\mathrm{cl}(F_h(X))$ has equation $x_{n+1}^k + \sum_{j=1}^{k-1} a_j(x_1, \ldots, x_n) h^j x_{n+1}^{k-j} = 0$, and the lemma follows. ∎

Our first main result is:

THEOREM 2.5. *Let X be an n-dimensional algebraic variety and $x \in X$ be a smooth point on X. Then there is a Zariski open neighborhood $U_x \subset X$ of x which is isomorphic to a closed smooth hypersurface in $\mathbb{C}^{n+1}$.*

Proof. Let $\phi : X \to \mathbb{C}^{n+1}$ be a birational embedding as in Lemma 2.1. Set $X_1 := \phi(X)$. By a change of variable we can assume that

$$X_1 = \Big\{x \in \mathbb{C}^{n+1} : x_{n+1}^k + \sum_{j=1}^{k-1} a_j(x_1, \ldots, x_n)x_{n+1}^{k-j} = 0\Big\}.$$

Let $\pi : \mathbb{C}^{n+1} \ni (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n) \in \mathbb{C}^n$ be the projection. Let $Y = \mathrm{Sing}(X_1)$ be the singular locus of X_1 and set $Y' = \pi(Y)$. It is easy to see that Y' is contained in some hypersurface $H \subset \{x : x_{n+1} = 0\}$. The hypersurface H is described by a reduced polynomial $h \in \mathbb{C}[x_1, \ldots, x_n]$. Without restriction of generality we can assume that $\pi(x) \notin H$. Moreover, we can assume that $h(0) \neq 0$ and $0 \notin X_1$.

Now consider the h-process F_h and $X_2 := \mathrm{cl}(F_h(X_1))$. From Lemma 2.4 we see that $X_1 \setminus V_h \cong X_2 \setminus H_h$ and $0 \notin X_2$.

Consider the mapping

$$\sigma : \mathbb{C}^{n+1} \ni (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1x_{n+1}, x_2x_{n+1}, \ldots, x_nx_{n+1}, x_{n+1}) \in \mathbb{C}^{n+1}.$$

Since $h(0) \neq 0$ and $0 \notin X_1$, we have $\sigma^{-1}(X_2) = \sigma^{-1}(X_2) \setminus H_h \cong X_1 \setminus V_h \cong X \setminus \phi^{-1}(V_h)$. Hence, if we take $U := \sigma^{-1}(X_2)$ and $f = \phi^{-1} \circ {F_h}^{-1} \circ \sigma$, then $f : U \to X$ is an open embedding and $U_x := f(U)$ is a smooth affine neighborhood we are looking for. ■

COROLLARY 2.6. *Let X be a smooth n-dimensional algebraic variety. Then there is an open covering $\{U_1, \ldots, U_k\}$ of X such that every U_i is isomorphic to a closed hypersurface $S_i \subset \mathbb{C}^{n+1}$.*

COROLLARY 2.7. *Let X be an n-dimensional algebraic variety ($n > 0$). Then there are infinitely many smooth affine hypersurfaces $Y_s \subset \mathbb{C}^{n+1}$, $s \in \mathbb{N}$, such that each Y_s is birationally isomorphic to X, and Y_s is not isomorphic to $Y_{s'}$ for $s \neq s'$.*

Proof. We construct the sequence of hypersurfaces $Y_s \subset \mathbb{C}^n$, $s \in \mathbb{N}$ inductively. A hypersurface $Y_1 \subset X$ exists by Corollary 2.6. Now assume that we have a sequence $Y_k \subset Y_{k-1} \subset \ldots \subset Y_1$ (with all inclusions strict) such that all Y_i are isomorphic to hypersurfaces in $\mathbb{C}^{n+1}$. We show how to construct Y_{k+1}. Take points $a, b \in Y_k$ and let Y_{k+1} be a Zariski open neighborhood $U_a \subset Y_k \setminus \{b\}$ of a which is isomorphic to a smooth hypersurface in $\mathbb{C}^{n+1}$. Of course we have the strict inclusion $Y_{k+1} \subset Y_k$.

Now if $s \neq s'$ then $Y_s \subset Y_{s'}$ or conversely (and the inclusion is strict). We can assume that $Y_s \subset Y_{s'}$. If Y_s were isomorphic to $Y_{s'}$, then we would have an injective mapping $f : Y_{s'} \to Y_s \subset Y_{s'}$. But by [4] this means that the mapping $f : Y_{s'} \to Y_{s'}$ is an isomorphism, in particular $Y_s = Y_{s'}$, a contradiction. ∎

We also have a stronger version of Theorem 2.5:

THEOREM 2.8. *Let $X_1, \ldots, X_r$ be n-dimensional algebraic varieties. Then there are Zariski open subsets $U_i \subset X_i$ such that every U_i is isomorphic to a closed smooth hypersurface $S_i \subset \mathbb{C}^{n+1}$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.*

Proof. Let $\phi_j : X_j \to \mathbb{C}^{n+1}$ be birational embeddings as in Lemma 2.1. Define $X'_j := \phi(X_j)$. We can assume that $X'_i \neq X'_j$ for $i \neq j$. Let $X := \bigcup_{j=1}^r X_j$ and $X' := \bigcup_{j=1}^r X'_j$.

Now we follow the proof of Theorem 2.5. By changing variables we can assume that

$$X' := \Big\{x \in \mathbb{C}^{n+1} : x_{n+1}^k + \sum_{j=1}^{k-1} a_j(x_1, \ldots, x_n) x_{n+1}^{k-j} = 0\Big\}.$$

Let $\pi : \mathbb{C}^{n+1} \ni (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n) \in \mathbb{C}^n$ be the projection. Let $Y = \operatorname{Sing}(X')$ and $Y' = \pi(Y)$. Then Y' is contained in some hypersurface $H \subset \{x : x_{n+1} = 0\}$, described by a reduced polynomial $h \in \mathbb{C}[x_1, \ldots, x_n]$. Without restriction of generality we can assume that $h(0) \neq 0$ and $0 \notin X'$.

Now consider the h-process F_h and set $X'' := \operatorname{cl}(F_h(X'))$. From Lemma 2.4 we see that $X' \setminus V_h \cong X'' \setminus H_h$ and $0 \notin X''$.

Consider the mapping $\sigma : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. Since $h(0) \neq 0$ and $0 \notin X''$, we have

$$\sigma^{-1}(X'') = \sigma^{-1}(X'') \setminus H_h \cong X' \setminus V_h \cong X \setminus \phi^{-1}(V_h) \cong \bigcup_{j=1}^r (X_j \setminus \phi_j^{-1}(V_h)).$$

Hence, it is enough to take $U_j := X_j \setminus \phi_j^{-1}(V_h)$ and $S_j := \sigma^{-1} \circ F_h \circ \phi(U_j)$. ∎

3. Known results. Let us recall some facts about the set of points at which a polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is not proper (cf. [2], [3]).

DEFINITION 3.1. Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial generically-finite mapping. We say that f is *proper* at a point $y \in \mathbb{C}^n$ if there exists an open neighborhood U of y such that $\operatorname{res}_{f^{-1}(U)} f : f^{-1}(U) \to U$ is a proper map.

REMARK 3.2. A polynomial mapping f is finite if and only if it is proper at every point $y \in \mathbb{C}^m$.

In [2], [3] we have studied the set S_f of points at which a mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is not proper. To formulate the main result of this study we need the notion of a $\mathbb{C}$-uniruled variety. The following proposition was proved in [3], Proposition 5.1:

PROPOSITION 3.3. *Let X be an irreducible affine variety of dimension ≥ 1. The following conditions are equivalent*:

(1) *for every $x \in X$ there is an affine parametric line Γ_x in X through x*;

(2) *there exists a Zariski-open, non-empty subset U of X such that for every $x \in U$ there is an affine parametric line Γ_x in X through x*;

(3) *there exists a subset U of X of the second Baire category such that for every point $x \in U$ there is an affine parametric line Γ_x in X through x*;

(4) *there exists an affine variety W with $\dim W = \dim X - 1$ and a dominant polynomial mapping $\phi : W \times \mathbb{C} \to X$.*

Now we can introduce our basic definition (cf. [3]):

DEFINITION 3.4. An affine irreducible variety X is called $\mathbb{C}$-*uniruled* if it is of dimension ≥ 1 and satisfies one of the equivalent conditions (1)–(4) of Proposition 3.3. More generally, if X is an affine variety then X is called $\mathbb{C}$-*uniruled* if it has pure dimension ≥ 1 and every component of X is $\mathbb{C}$-uniruled. Additionally we assume that the empty set is $\mathbb{C}$-uniruled.

Finally we have the following description of the set S_f (cf. [3], Theorem 5.8):

PROPOSITION 3.5. *Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial generically-finite mapping. Then the set S_f of points at which the mapping f is not proper is either empty or it has pure dimension $n - 1$. Moreover, the variety S_f is $\mathbb{C}$-uniruled.*

In what follows we also need the following theorem (cf. [3], Theorem 5.4):

THEOREM 3.6. *Let X be an affine variety and $Y \subset X$ be a closed subvariety. Let $f : Y \to \mathbb{C}^n$ be a polynomial mapping. Assume that $\dim X \leq n$. Then there exists a polynomial mapping $F : X \to \mathbb{C}^n$ such that*

(1) $\mathrm{res}_Y F = f$,

(2) *the mapping $\mathrm{res}_{X \setminus Y} F : X \setminus Y \to \mathbb{C}^n$ is quasi-finite.*

4. A characterization of components of S_f. Now we can prove our second main result.

THEOREM 4.1. *Let $2 \leq n \leq m$ and let $X \subset \mathbb{C}^m$ be an $n-1$-dimensional $\mathbb{C}$-uniruled subset of $\mathbb{C}^m$. Then there is a polynomial quasi-finite mapping $F : \mathbb{C}^n \to \mathbb{C}^m$ such that $X \subset S_F$.*

Proof. First assume that $F : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ and $X = S \times \mathbb{C}$ where $S \subset \mathbb{C}^n = \{x \in \mathbb{C}^{n+1} : x_{n+1} = 0\}$. This means that the subset X is described by a polynomial $h \in \mathbb{C}[x_1, \ldots, x_n]$. Consider the mapping

$$F : \mathbb{C}^{n+1} \ni (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n, h(x_1, \ldots, x_n)x_{n+1}^2 + x_{n+1}) \in \mathbb{C}^{n+1}.$$

It is easy to see that F is a quasi-finite mapping and $S_F = X$.

We now turn to the general case. Let $X \subset \mathbb{C}^m$ be an $n-1$-dimensional $\mathbb{C}$-uniruled algebraic set. We have a decomposition $X = \bigcup_{j=1}^r X_j$, where X_j are $n-1$-dimensional $\mathbb{C}$-uniruled irreducible varieties.

From Proposition 3.3, there are affine varieties W_j with $\dim W_j = n-2$ and dominant polynomial mappings $\phi_j : W_j \times \mathbb{C} \to X_j$. By Corollary 2.7, we can assume that $W_j \subset \mathbb{C}^{n-1}$ and $W_i \cap W_j = \emptyset$ for $i \neq j$. Put $Y_j := W_j \times \mathbb{C}$. Hence $Y_j \subset \mathbb{C}^n$ and $Y_i \cap Y_j = \emptyset$ for $i \neq j$.

By the first part of our proof there is a quasi-finite polynomial mapping $G : \mathbb{C}^n \to \mathbb{C}^n$ such that $S_G = \bigcup_{j=1}^r Y_j$. Since G is quasi-finite, by the Zariski Main Theorem there is an affine variety Z which contains $\mathbb{C}^n$ as an open dense subset and a finite mapping $G_1 : Z \to \mathbb{C}^n$ such that $\mathrm{res}_{\mathbb{C}^n} G_1 = G$. Set $P := Z \setminus \mathbb{C}^n$. It is easy to see that $S_G = G_1(P)$. In particular if $P = \bigcup_{i=1}^s P_i$, then for every Y_j there is an appropriate P_i such that $Y_j = G_1(P_i)$.

Now we return to the set $X = \bigcup_{j=1}^r X_j$. Recall that we have dominant mappings $\phi_j : Y_j \to X_j$. Since $Y_i \cap Y_j = \emptyset$ for $i \neq j$, we also have the mapping $\phi = \bigcup_{i=1}^r \phi_i$. Consider the mapping $f : P \ni z \mapsto \phi \circ G_1(z) \in \mathbb{C}^m$. It is easy to see that $\mathrm{cl}(f(P)) = X$. By Theorem 3.6 we can extend f to a mapping $F_1 : Z \to \mathbb{C}^m$ such that

(1) $\mathrm{res}_P F_1 = f$,

(2) the mapping $\mathrm{res}_{Z \setminus P} F_1 : Z \setminus P \to \mathbb{C}^m$ is quasi-finite.

If we set $F = \mathrm{res}_{\mathbb{C}^n} F_1$, then the mapping F is quasi-finite and $X \subset S_F$ by the construction. ∎

Corollary 4.2. *Let $2 \leq n \leq m$ and let $X \subset \mathbb{C}^m$ be an irreducible variety. Then X is an irreducible component of the set S_F for some generically-finite polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^m$ if and only if X is $\mathbb{C}$-uniruled and* $\dim X = n-1$.

The author does not know whether the (last) inclusion in Theorem 4.1 can be replaced by equality. However in some cases it is possible:

Proposition 4.3. *Let $2 \leq n \leq m$ and let $S_1, \ldots, S_r \subset \mathbb{C}^m$ be affine $n-1$-dimensional irreducible varieties such that there are finite mappings $\phi_i : \mathbb{C}^{n-1} \to S_i$, $i = 1, \ldots, r$. Then there exists a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^m$ with finite fibers such that $S_F = \bigcup_{i=1}^r S_i$.*

P r o o f. Consider the mapping

$$G : \mathbb{C}^n \ni (x_1, \ldots, x_n) \mapsto \Big(x_1, \ldots, x_{n-1}, \Big(\prod_{i=1}^{r}(x_1 - i)\Big)x_n^2 + x_n\Big) \in \mathbb{C}^n.$$

It is easy to see that G is quasi-finite and $S_G = \{x \in \mathbb{C}^n : \prod_{i=1}^{r}(x_1 - i) = 0\}$. In particular $S_G = \bigcup_{i=1}^{r} W_i$, where $W_i \cong \mathbb{C}^{n-1}$ for $i = 1, \ldots, r$ and $W_i \cap W_j = \emptyset$ for $i \neq j$.

Now let $S_1, \ldots, S_r \subset \mathbb{C}^m$ be affine $n-1$-dimensional irreducible varieties such that there are finite mappings $\phi_i : \mathbb{C}^{n-1} \to S_i$, $i = 1, \ldots, r$. We can assume that $\phi_i : W_i \to S_i$. In particular we have a finite mapping $\phi : \bigcup_{i=1}^{r} W_i \to \bigcup_{i=1}^{r} S_i \subset \mathbb{C}^m$. By [2], Proposition 21, we can extend ϕ to a finite mapping $\Phi : \mathbb{C}^n \to \mathbb{C}^m$. Now it is enough to set $F = \Phi \circ G$. ∎

COROLLARY 4.4. *Let $2 \leq n \leq m$ and let $S_1, \ldots, S_r \subset \mathbb{C}^m$ be $n-1$-dimensional linear subspaces. Then there exists a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^m$ with finite fibers such that $S_F = \bigcup_{i=1}^{r} S_i$.*

References

[1] R. Hartshorne, *Algebraic Geometry*, Springer, New York, 1987.

[2] Z. Jelonek, *The set of points at which a polynomial map is not proper*, Ann. Polon. Math. 58 (1993), 259–266.

[3] —, *Testing sets for properness of polynomial mappings*, Math. Ann. 315 (1999), 1–35.

[4] K. Nowak, *Injective endomorphisms of algebraic varieties*, ibid. 299 (1994), 769–778.

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