ANNALES
POLONICI MATHEMATICI
LXXV.1 (2000)

Uniform pseudo-orbit tracing property for homeomorphisms and continuous mappings

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Abstract. We prove that for every nonempty compact manifold of nonzero dimension no self-homeomorphism and no continuous self-mapping has the uniform pseudo-orbit tracing property. Several relevant counterexamples for recently studied hypotheses are indicated.

1. Introduction. A new definition emerged from previous results on the pseudo-orbit tracing property [2, 4] when K. Sakai [3] proved that every diffeomorphism on a closed smooth manifold satisfying Axiom A and strong transversality has the C^1-uniform pseudo-orbit tracing property (abbrev. UPOTP). An attempt has been made by R. Gu [1] to develop this result. UPOTP for continuous self-mappings of compact metric spaces was defined and two strong hypotheses were stated, namely the genericity of UPOTP in continuous self-mappings of compact manifolds and the equivalence between orbit stability and UPOTP for self-homeomorphisms of compact manifolds of dimension at least two.

However, these two hypotheses turn out to be false. In fact, given a compact nonempty manifold M of positive dimension, neither any self-homeomorphism nor any continuous self-mapping of that manifold has UPOTP. The aim of this paper is to prove that some extra differential structure on the space of mappings is needed for any further study of UPOTP.

2. Notation and definitions. Let M be a compact nonempty topological manifold with topology given by a metric d. Denote by $C(M)$ the space of all continuous self-mappings of M and by $H(M)$ the space of all self-homeomorphisms of M. Define a metric $\widetilde{d}$ on $C(M)$ and $H(M)$ by

$$\widetilde{d}(f,g) = \sup_{x\in M} d(f(x),g(x)).$$

2000 *Mathematics Subject Classification*: Primary 34C40.
Key words and phrases: discrete dynamical system on manifold, pseudo-orbit tracing property, orbit stability.

Given $\delta > 0$ and $f \in C(M)$ we call $\{x_n\}_{n=1}^{\infty} \subset M$ a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in \mathbb{N}$. For any $\varepsilon > 0$ we call $\{x_n\}_{n=1}^{\infty} \subset M$ *ε-traced* by $\{y_n\}_{n=1}^{\infty} \subset M$ if $d(x_i, y_i) < \varepsilon$ for $i \in \mathbb{N}$.

Definition 2.1. We say that $f \in C(M)$ has UPOTP if there is $\Delta > 0$ such that for all $\varepsilon > 0$ there is $\delta > 0$ having the property that for all $g \in C(M)$ such that $\widetilde{d}(f, g) < \Delta$ each δ-pseudo-orbit of g can be ε-traced by some orbit of g.

Remark 2.2. By replacing in the above definition $C(M)$ by $H(M)$ we get another tracing property. We shall denote it by UPOTP(H). Although UPOTP is equivalent to UPOTP(H) on $H(M)$, there is no easy way to see that from the definition. In the next section we will discuss both cases separately.

3. Main result

Proposition 3.1. *If* $\dim M = 0$ *then all* $f \in C(M)$ *have UPOTP and all* $f \in H(M)$ *have UPOTP(H).*

Proof. Both properties are evident since M is a finite collection of points. ∎

Theorem 3.2 (Main result). *If* $\dim M > 0$ *then no* $f \in C(M)$ *has UPOTP and no* $f \in H(M)$ *has UPOTP(H).*

Lemma 3.3. *If* $f \in C(M)$ *has a periodic point then it does not have UPOTP. Similarly, if* $f \in H(M)$ *has a periodic point then it does not have UPOTP(H).*

Proof. Fix $\Delta > 0$. Denote by $\{p_i\}_{i=1}^{k}$ any periodic orbit of $f \in C(M)$. First we consider the case $k > 1$. Let $\{U_i\}_{i=1}^{k}$ be a family of disjoint open neighbourhoods of $\{p_i\}_{i=1}^{k}$ such that $\operatorname{diam} U_i < \Delta$, U_i is homeomorphic to $\mathbb{B}(0,1)$ (an open ball in $\mathbb{R}^n$), $f(U_i) \subset U_{i+1}$ for $i = 2, \ldots, k-1$ and $f(U_k) \subset U_1$. Let $\varphi_i : \mathbb{B}(0,1) \to U_i$ be a family of homeomorphisms such that $\varphi_i(0) = p_i$. Fix $\eta > 0$ such that $f(\varphi_1(\mathbb{B}(0,\eta))) \subset U_2$. Modify f continuously

- on $\varphi_1(\mathbb{B}(0,\eta))$ so that $(\varphi_2|_{\mathbb{B}(0,1/2)})^{-1} \circ f \circ \varphi_1|_{\mathbb{B}(0,\eta/2)} = \eta^{-1} \mathrm{id}_{\mathbb{B}(0,\eta/2)}$ and $f \circ \varphi_1(\mathbb{B}(0,\eta)) \subset U_2$,
- on U_i for $i = 2, \ldots, k-1$ so that $(\varphi_{i+1}|_{\mathbb{B}(0,1/2)})^{-1} \circ f \circ \varphi_i|_{\mathbb{B}(0,1/2)} = \mathrm{id}_{\mathbb{B}(0,1/2)}$ and $f(U_i) \subset U_{i+1}$,
- on U_k so that

$$(\varphi_1|_{\mathbb{B}(0,\eta/2)})^{-1} \circ f \circ \varphi_k|_{\mathbb{B}(0,1/2)}(x) = \begin{cases} \eta x(5/4 - (\|4x\| - 1/2)^2) & \text{for } \|x\| \le 1/4, \\ \eta x & \text{for } 1/4 < \|x\| \le 1/2, \end{cases}$$

and $f(U_k) \subset U_1$. Denote the resulting mapping by $f_\Delta \in C(M)$.

It is possible to derive an exact formula for f_Δ, but we will not discuss that in this paper since we are only interested in its existence which is clear.

Since $\operatorname{diam} U_i < \Delta$ we have $\widetilde{d}(f, f_\Delta) < \Delta$. Define

$$\varepsilon = \frac{1}{2}\min\{d(p_1, M\backslash\varphi_1(\mathbb{B}(0,\eta/4))), d(\varphi_1(\mathbb{B}(0,\eta/4)), M\backslash\varphi_1(\mathbb{B}(0,\eta/2)))\}$$

Fix $\delta > 0$. Consider the following δ-pseudo-orbit of f_Δ^k:

- we start at p_1,
- we jump from p_1 to some point in $\varphi_1(\mathbb{B}(0,\eta/4))$,
- we let the iterations of f_Δ^k take us closer than δ to $\varphi_1(\mathbb{B}(0,\eta/2))\backslash\varphi_1(\mathbb{B}(0,\eta/4))$,
- we jump to $\varphi_1(\mathbb{B}(0,\eta/2))\backslash\varphi_1(\mathbb{B}(0,\eta/4))$,
- by a finite number of jumps we get to a point in $\varphi_1(\mathbb{B}(0,\eta/2))\backslash\varphi_1(\mathbb{B}(0,\eta/4))$ at a distance of more than ε from $\varphi_1(\mathbb{B}(0,\eta/4))$,
- we stay there for the rest of time.

Such an orbit descends of course from some δ-pseudo-orbit of f_Δ.

No orbit of f_Δ traces this δ-pseudo-orbit. Indeed, due to the choice of ε such a tracing orbit would have to start somewhere in $\varphi_1(\mathbb{B}(0,\eta/4))$, and so the orbit of f_Δ^k beginning at the same point would be unable to leave that set. But our δ-pseudo-orbit wanders farther than ε from that set and therefore there can be no ε-tracing.

Summarizing, for every $\Delta > 0$ we can construct $f_\Delta \in C(M)$ such that $\widetilde{d}(f, f_\Delta) < \Delta$ and $\varepsilon > 0$ such that for all $\delta > 0$ there is a δ-pseudo-orbit of f_Δ that cannot be ε-traced by any orbit of f_Δ. In other words, f does not have UPOTP.

We treat the case $k = 1$ (i.e. p_1 is a fixed point of f) in a similar way. Given a neighbourhood U_1 of p_1 of diameter less than Δ and a homeomorphism $\varphi_1 : \mathbb{B}(0,1) \to U_1$ we find $\eta > 0$ such that $f(\varphi_1(\mathbb{B}(0,\eta))) \subset U_1$. It is now enough to modify f on $\varphi_1(\mathbb{B}(0,\eta))$ in the same manner that f was modified on U_k in the above proof. The analysis of the properties of f_Δ obtained in this way proceeds exactly as previously.

The above arguments are also valid for $f \in H(M)$ and UPOTP(H). The only difference is that f_Δ has to be a homeomorphism, so instead of continuous we need to make homeomorphic modifications of f. ∎

LEMMA 3.4. *The set of all mappings having a periodic orbit is dense in $C(M)$ and $H(M)$.*

Proof. The method we use to modify mappings in order to get a periodic point was previously applied in [1].

Let $f \in C(M)$ and $\varepsilon > 0$ be given. Fix an open cover $\{U_i\}_{i=1}^m$ of M with sets of diameter less than ε and homeomorphic to $\mathbb{B}(0,1)$. Fix $y \in M$. Let $k \in \mathbb{N}$ be the least such that there are $i \in \{1,\ldots,m\}$ and $j < k$ such that $f^j(y), f^k(y) \in U_i$.

Let V be an open neighbourhood of $f^{k-1}(y)$ homeomorphic to $\mathbb{B}(0,1)$ such that $f(V) \subset U_i$ and $y, f(y), \ldots, f^{k-2}(y) \notin V$. Let $\varphi : \mathbb{B}(0,1) \to U_i$ and $\psi : \mathbb{B}(0,1) \to V$ be homeomorphisms such that $\psi(0) = f^{k-1}(y)$. Define $f_\varepsilon \in C(M)$ as

$$f_\varepsilon(x) = \begin{cases} \varphi(\|\psi^{-1}(x)\|\varphi^{-1}(f(x)) + (1 - \|\psi^{-1}(x)\|)\varphi^{-1}(f^j(y))) & \text{for } x \in V, \\ f(x) & \text{for } x \notin V. \end{cases}$$

The continuity of f_ε is elementary.

Note that $\widetilde{d}(f, f_\varepsilon) < \varepsilon$ and f_ε has a periodic orbit consisting of the points $f^j(y), f_\varepsilon(f^j(y)), \ldots, f_\varepsilon^{k-j}(f^j(y)) = f^j(y)$. This proves the density.

The proof requires only a minor modification to fit the $H(M)$ case. The f_ε has to be a homeomorphism, and therefore instead of the above definition we take as f_ε any homeomorphic extension of $f|_{M \setminus f^{-1}(U_i)}$ such that $f_\varepsilon(f^{k-1}(y)) = f^j(y)$. ∎

Proof of Theorem 3.2. Suppose $f \in C(M)$ has UPOTP with $\Delta > 0$. By Lemma 3.4 there is $g \in C(M)$ with periodic orbit such that $\widetilde{d}(f, g) < \Delta/2$. By definition of UPOTP, g has that property with a constant $\Delta/2$, contrary to Lemma 3.3.

For the homeomorphism case, replace in the above argument $C(M)$ by $H(M)$ and UPOTP by UPOTP(H). ∎

4. Corollaries

COROLLARY 4.1. *For* $\dim M > 0$, *UPOTP is not generic in* $C(M)$ *and UPOTP(H) is not generic in* $H(M)$ (cf. [1], p. 353).

DEFINITION 4.2. We call $f \in H(M)$ *orbit stable* if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $g \in H(M)$ and $\widetilde{d}(f, g) < \delta$ then every orbit of f can be ε-traced by some orbit of g and every orbit of g can be ε-traced by some orbit of f.

COROLLARY 4.3. *For* $\dim M > 0$, *UPOTP(H) is not equivalent to orbit stability* (cf. [1], p. 358).

P r o o f. For every $n > 0$ one can consider the homeomorphism f on the unit sphere in $\mathbb{R}^{n+1}$ given by

$$f : S^n \ni (x_0, \ldots, x_n) \mapsto \left(\frac{x_0}{a(x_n)}, \ldots, \frac{x_{n-1}}{a(x_n)}, \frac{1}{2}(x_n + 1)^2 - 1 \right) \in S^n$$

where $a(x_n) > 0$ is chosen in such a manner that $\operatorname{Im} f = S^n$. By Theorem 3.2, f does not have UPOTP(H).

Given a small $\varepsilon > 0$ divide S^n into three disjoint sets

$$A = \mathbb{B}((0,\dots,0,1),\varepsilon/4), \qquad C = \mathbb{B}((0,\dots,0,-1),\varepsilon/4),$$
$$B = S^n \setminus (A \cup C),$$

We can take $\eta > 0$ so small that if $g \in H(S^n)$ and $\widetilde{d}(f,g) < \eta$ then

- $g(B \cup C) \subset B \cup C$,
- $g(C) \subset C$,
- there is $k \in \mathbb{N}$ such that any orbit of g has no more than k points in B.

Define $\delta = \min\{\eta, \varepsilon/k\}$. Take any $g \in H(S^n)$ such that $\widetilde{d}(f,g) < \delta$.

By Brouwer's Theorem applied to $g(A)$ and C, g has fixed points $a \in A$ and $c \in C$.

If $\{x_i\}_{i=1}^{\infty}$ is an orbit of g then it is also a δ-pseudo-orbit of f. There are three possible types of behaviour for an orbit of g:

• it can stay in A for all time; then it is ε-traced by $(0,\dots,0,1)$ which is a fixed point for f,

• it can stay in C for all time; then it is ε-traced by $(0,\dots,0,-1)$ which is a fixed point for f,

• there is $m \geq 0$ such that $x_1,\dots,x_{m-1} \in A$ but $x_m \in B$. Then by the choice of δ the orbit is ε-traced by $f^{-m}(x_m),\dots,f^{-1}(x_m), x_m, f(x_m),\dots$

Similarly, every orbit of f can be ε-traced by some orbit of g. Fixed points of f are traced by a and c respectively, and any other orbit $\{x_i\}_{i=1}^{\infty}$ such that $x_1,\dots,x_{m-1} \in A$ but $x_m \in B$ is traced by $g^{-m}(x_m),\dots,g^{-1}(x_m), x_m$, $g(x_m),\dots$ Therefore f is orbit stable. This is the desired conclusion. ■

Acknowledgements. I wish to express my gratitude to Prof. Jerzy Ombach for bringing the subject to my attention.

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Reçu par la Rédaction le 25.6.1999
Révisé le 10.3.2000