Uniform pseudo-orbit tracing property
for homeomorphisms and continuous mappings

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Abstract. We prove that for every nonempty compact manifold of nonzero dimension no self-homeomorphism and no continuous self-mapping has the uniform pseudo-orbit tracing property. Several relevant counterexamples for recently studied hypotheses are indicated.

1. Introduction. A new definition emerged from previous results on the pseudo-orbit tracing property [2, 4] when K. Sakai [3] proved that every diffeomorphism on a closed smooth manifold satisfying Axiom A and strong transversality has the $C^1$-uniform pseudo-orbit tracing property (abbrev. UPOTP). An attempt has been made by R. Gu [1] to develop this result. UPOTP for continuous self-mappings of compact metric spaces was defined and two strong hypotheses were stated, namely the genericity of UPOTP in continuous self-mappings of compact manifolds and the equivalence between orbit stability and UPOTP for self-homeomorphisms of compact manifolds of dimension at least two.

However, these two hypotheses turn out to be false. In fact, given a compact nonempty manifold $M$ of positive dimension, neither any self-homeomorphism nor any continuous self-mapping of that manifold has UPOTP. The aim of this paper is to prove that some extra differential structure on the space of mappings is needed for any further study of UPOTP.

2. Notation and definitions. Let $M$ be a compact nonempty topological manifold with topology given by a metric $d$. Denote by $C(M)$ the space of all continuous self-mappings of $M$ and by $H(M)$ the space of all self-homeomorphisms of $M$. Define a metric $\tilde{d}$ on $C(M)$ and $H(M)$ by

$$\tilde{d}(f, g) = \sup_{x \in M} d(f(x), g(x)).$$

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Given \( \delta > 0 \) and \( f \in C(M) \) we call \( \{x_n\}_{n=1}^{\infty} \subset M \) a \( \delta \)-pseudo-orbit of \( f \) if
\[ d(f(x_i), x_{i+1}) < \delta \] for \( i \in \mathbb{N} \). For any \( \varepsilon > 0 \) we call \( \{x_n\}_{n=1}^{\infty} \subset M \) \( \varepsilon \)-traced by \( \{y_n\}_{n=1}^{\infty} \subset M \) if \( d(x_i, y_i) < \varepsilon \) for \( i \in \mathbb{N} \).

**Definition 2.1.** We say that \( f \in C(M) \) has UPOTP if there is \( \Delta > 0 \) such that for all \( \varepsilon > 0 \) there is \( \delta > 0 \) having the property that for all \( g \in C(M) \) such that \( d(f, g) < \Delta \) each \( \delta \)-pseudo-orbit of \( g \) can be \( \varepsilon \)-traced by some orbit of \( g \).

**Remark 2.2.** By replacing in the above definition \( C(M) \) by \( H(M) \) we get another tracing property. We shall denote it by UPOTP(H). Although UPOTP is equivalent to UPOTP(H) on \( H(M) \), there is no easy way to see that from the definition. In the next section we will discuss both cases separately.

### 3. Main result

**Proposition 3.1.** If \( \dim M = 0 \) then all \( f \in C(M) \) have UPOTP and all \( f \in H(M) \) have UPOTP(H).

**Proof.** Both properties are evident since \( M \) is a finite collection of points. \( \blacksquare \)

**Theorem 3.2** (Main result). If \( \dim M > 0 \) then no \( f \in C(M) \) has UPOTP and no \( f \in H(M) \) has UPOTP(H).

**Lemma 3.3.** If \( f \in C(M) \) has a periodic point then it does not have UPOTP. Similarly, if \( f \in H(M) \) has a periodic point then it does not have UPOTP(H).

**Proof.** Fix \( \Delta > 0 \). Denote by \( \{p_i\}_{i=1}^{k} \) any periodic orbit of \( f \in C(M) \). First we consider the case \( k > 1 \). Let \( \{U_i\}_{i=1}^{k} \) be a family of disjoint open neighbourhoods of \( \{p_i\}_{i=1}^{k} \) such that \( \operatorname{diam} U_i < \Delta \), \( U_i \) is homeomorphic to \( \mathbb{B}(0, 1) \) (an open ball in \( \mathbb{R}^n \)), \( f(U_i) \subset U_{i+1} \) for \( i=2, \ldots, k-1 \) and \( f(U_k) \subset U_1 \).

Let \( \varphi_i : \mathbb{B}(0, 1) \to U_i \) be a family of homeomorphisms such that \( \varphi_i(0) = p_i \). Fix \( \varepsilon > 0 \) such that \( f(\varphi_i(\mathbb{B}(0, \eta))) \subset U_2 \). Modify \( f \) continuously
- on \( \varphi_1(\mathbb{B}(0, \eta)) \) so that \( (\varphi_2|_{\mathbb{B}(0, 1/2)})^{-1} \circ f \circ \varphi_1|_{\mathbb{B}(0, \eta/2)} = \eta^{-1}\text{id}_{\mathbb{B}(0, \eta/2)} \) and \( f \circ \varphi_1(\mathbb{B}(0, \eta)) \subset U_2 \),
- on \( U_i \) for \( i = 2, \ldots, k-1 \) so that \( (\varphi_{i+1}|_{\mathbb{B}(0, 1/2)})^{-1} \circ f \circ \varphi_i|_{\mathbb{B}(0, 1/2)} = \text{id}_{\mathbb{B}(0, 1/2)} \) and \( f(U_i) \subset U_{i+1} \),
- on \( U_k \) so that
  \[
  (\varphi_1|_{\mathbb{B}(0, \eta/2)})^{-1} \circ f \circ \varphi_k|_{\mathbb{B}(0, 1/2)}(x)
  = \begin{cases} 
  \eta x (5/4 - (\|4x\| - 1/2)^2) & \text{for } \|x\| \leq 1/4, \\
  \eta x & \text{for } 1/4 < \|x\| \leq 1/2,
  \end{cases}
  \]
and \( f(U_k) \subset U_1 \). Denote the resulting mapping by \( f_\Delta \in C(M) \).
It is possible to derive an exact formula for $f_\Delta$, but we will not discuss that in this paper since we are only interested in its existence which is clear.

Since $\text{diam } U_1 < \Delta$ we have $\tilde{d}(f, f_\Delta) < \Delta$. Define

$$
\varepsilon = \frac{1}{2} \min\{d(p_1, M \setminus \varphi_1(\mathbb{B}(0, \eta/4))), d(\varphi_1(\mathbb{B}(0, \eta/4)), M \setminus \varphi_1(\mathbb{B}(0, \eta/2)))\}
$$

Fix $\delta > 0$. Consider the following $\delta$-pseudo-orbit of $f^k_\Delta$:

- we start at $p_1$,
- we jump from $p_1$ to some point in $\varphi_1(\mathbb{B}(0, \eta/4))$,
- we let the iterations of $f^k_\Delta$ take us closer than $\delta$ to $\varphi_1(\mathbb{B}(0, \eta/2)) \setminus \varphi_1(\mathbb{B}(0, \eta/4))$,
- we jump to $\varphi_1(\mathbb{B}(0, \eta/2)) \setminus \varphi_1(\mathbb{B}(0, \eta/4))$ at a distance of more than $\varepsilon$ from $\varphi_1(\mathbb{B}(0, \eta/4))$,
- we stay there for the rest of time.

Such an orbit descends of course from some $\delta$-pseudo-orbit of $f_\Delta$.

No orbit of $f_\Delta$ traces this $\delta$-pseudo-orbit. Indeed, due to the choice of $\varepsilon$ such a tracing orbit would have to start somewhere in $\varphi_1(\mathbb{B}(0, \eta/4))$, and so the orbit of $f^k_\Delta$ beginning at the same point would be unable to leave that set. But our $\delta$-pseudo-orbit wanders farther than $\varepsilon$ from that set and therefore there can be no $\varepsilon$-tracing.

Summarizing, for every $\Delta > 0$ we can construct $f_\Delta \in C(M)$ such that $\tilde{d}(f, f_\Delta) < \Delta$ and $\varepsilon > 0$ such that for all $\delta > 0$ there is a $\delta$-pseudo-orbit of $f_\Delta$ that cannot be $\varepsilon$-traced by any orbit of $f_\Delta$. In other words, $f$ does not have UPOTP.

We treat the case $k = 1$ (i.e. $p_1$ is a fixed point of $f$) in a similar way. Given a neighbourhood $U_1$ of $p_1$ of diameter less than $\Delta$ and a homeomorphism $\varphi_1 : \mathbb{B}(0, 1) \to U_1$ we find $\eta > 0$ such that $f(\varphi_1(\mathbb{B}(0, \eta))) \subset U_1$.

It is now enough to modify $f$ on $\varphi_1(\mathbb{B}(0, \eta))$ in the same manner that $f$ was modified on $U_1$ in the above proof. The analysis of the properties of $f_\Delta$ obtained in this way proceeds exactly as previously.

The above arguments are also valid for $f \in H(M)$ and UPOTP(H). The only difference is that $f_\Delta$ has to be a homeomorphism, so instead of continuous we need to make homeomorphic modifications of $f$.

**Lemma 3.4.** The set of all mappings having a periodic orbit is dense in $C(M)$ and $H(M)$.

**Proof.** The method we use to modify mappings in order to get a periodic point was previously applied in [1].
Let $f \in C(M)$ and $\varepsilon > 0$ be given. Fix an open cover $\{U_i\}_{i=1}^m$ of $M$ with sets of diameter less than $\varepsilon$ and homeomorphic to $\mathbb{B}(0,1)$. Fix $y \in M$. Let $k \in \mathbb{N}$ be the least such that there are $i \in \{1, \ldots, m\}$ and $j < k$ such that $f^j(y), f^k(y) \in U_i$.

Let $V$ be an open neighbourhood of $f^{k-1}(y)$ homeomorphic to $\mathbb{B}(0,1)$ such that $f(V) \subset U_i$ and $y, f(y), \ldots, f^{k-2}(y) \notin V$. Let $\varphi: \mathbb{B}(0,1) \to U_i$ and $\psi: \mathbb{B}(0,1) \to V$ be homeomorphisms such that $\psi(0) = f^{k-1}(y)$. Define $f_\varepsilon \in C(M)$ as

$$f_\varepsilon(x) = \begin{cases} \varphi(\|\psi^{-1}(x)\|\varphi^{-1}(f(x))) + (1 - \|\psi^{-1}(x)\|)\varphi^{-1}(f^2(y)) & \text{for } x \in V, \\ f(x) & \text{for } x \notin V. \end{cases}$$

The continuity of $f_\varepsilon$ is elementary.

Note that $\tilde{d}(f, f_\varepsilon) < \varepsilon$ and $f_\varepsilon$ has a periodic orbit consisting of the points $f^j(y), f_\varepsilon(f^j(y)), \ldots, f_\varepsilon^{k-1}(f^j(y)) = f^j(y)$. This proves the density.

The proof requires only a minor modification to fit the $H(M)$ case. The $f_\varepsilon$ has to be a homeomorphism, and therefore instead of the above definition we take as $f_\varepsilon$ any homeomorphic extension of $f|_{M \setminus f^{k-1}(U_i)}$ such that $f_\varepsilon(f^{k-1}(y)) = f^j(y)$.

**Proof of Theorem 3.2.** Suppose $f \in C(M)$ has UOTP with $\Delta > 0$. By Lemma 3.4 there is $g \in C(M)$ with periodic orbit such that $\tilde{d}(f,g) < \Delta/2$. By definition of UOTP, $g$ has that property with a constant $\Delta/2$, contrary to Lemma 3.3.

For the homeomorphism case, replace in the above argument $C(M)$ by $H(M)$ and UOTP by UOTP(H).

### 4. Corollaries

**Corollary 4.1.** For $\dim M > 0$, UOTP is not generic in $C(M)$ and UOTP(H) is not generic in $H(M)$ (cf. [1], p. 353).

**Definition 4.2.** We call $f \in H(M)$ orbit stable if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $g \in H(M)$ and $\tilde{d}(f,g) < \delta$ then every orbit of $f$ can be $\varepsilon$-traced by some orbit of $g$ and every orbit of $g$ can be $\varepsilon$-traced by some orbit of $f$.

**Corollary 4.3.** For $\dim M > 0$, UOTP(H) is not equivalent to orbit stability (cf. [1], p. 358).

**Proof.** For every $n > 0$ one can consider the homeomorphism $f$ on the unit sphere in $\mathbb{R}^{n+1}$ given by

$$f: S^n \ni (x_0, \ldots, x_n) \mapsto \left(\frac{x_0}{a(x_n)}, \ldots, \frac{x_{n-1}}{a(x_n)}, \frac{1}{2}(x_n + 1)^2 - 1\right) \in S^n$$
where \( a(x_n) > 0 \) is chosen in such a manner that \( \text{Im } f = S^n \). By Theorem 3.2, \( f \) does not have UPOTP(H).

Given a small \( \varepsilon > 0 \) divide \( S^n \) into three disjoint sets
\[
A = B((0, \ldots, 0, 1), \varepsilon/4), \quad C = B((0, \ldots, 0, -1), \varepsilon/4),
B = S^n \setminus (A \cup C),
\]
We can take \( \eta > 0 \) so small that if \( g \in H(S^n) \) and \( \tilde{d}(f, g) < \eta \) then
- \( g(B \cup C) \subset B \cup C \),
- \( g(C) \subset C \),
- there is \( k \in \mathbb{N} \) such that any orbit of \( g \) has no more than \( k \) points in \( B \).

Define \( \delta = \min\{\eta, \varepsilon/k\} \). Take any \( g \in H(S^n) \) such that \( \tilde{d}(f, g) < \delta \).

By Brouwer’s Theorem applied to \( g(A) \) and \( C \), \( g \) has fixed points \( a \in A \) and \( c \in C \).

If \( \{x_i\}_{i=1}^{\infty} \) is an orbit of \( g \) then it is also a \( \delta \)-pseudo-orbit of \( f \). There are three possible types of behaviour for an orbit of \( g \):
- it can stay in \( A \) for all time; then it is \( \varepsilon \)-traced by \((0, \ldots, 0, 1)\) which is a fixed point for \( f \),
- it can stay in \( C \) for all time; then it is \( \varepsilon \)-traced by \((0, \ldots, 0, -1)\) which is a fixed point for \( f \),
- there is \( m \geq 0 \) such that \( x_1, \ldots, x_{m-1} \in A \) and \( x_m \in B \). Then by the choice of \( \delta \) the orbit is \( \varepsilon \)-traced by \( f^{-m}(x_m), \ldots, f^{-1}(x_m), x_m, f(x_m), \ldots \).

Similarly, every orbit of \( f \) can be \( \varepsilon \)-traced by some orbit of \( g \). Fixed points of \( f \) are traced by \( a \) and \( c \) respectively, and any other orbit \( \{x_i\}_{i=1}^{\infty} \) such that \( x_1, \ldots, x_{m-1} \in A \) but \( x_m \in B \) is traced by \( g^{-m}(x_m), \ldots, g^{-1}(x_m), x_m, g(x_m), \ldots \). Therefore \( f \) is orbit stable. This is the desired conclusion.

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