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Singular holomorphic functions for which all fibre-integrals are smooth

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Abstract. For a germ $(X,0)$ of normal complex space of dimension $n+1$ with an isolated singularity at 0 and a germ $f:(X,0)\to(\mathbb{C},0)$ of holomorphic function with $df(x)\neq 0$ for $x\neq 0$, the fibre-integrals

$$s\mapsto \int\limits_{f=s} \varrho\omega'\wedge\overline{\omega''},\qquad \varrho\in C_c^\infty(X),\ \omega',\omega''\in\Omega_X^n,$$

are C^∞ on $\mathbb{C}^*$ and have an asymptotic expansion at 0. Even when f is singular, it may happen that all these fibre-integrals are C^∞. We study such maps and build a family of examples where also fibre-integrals for $\omega',\omega''\in\underline{\omega}_X$, the Grothendieck sheaf, are C^∞.

0. Introduction. Let $(X,0)$ be a germ of normal complex space of dimension $n+1$ with an isolated singularity at 0 and let $f:(X,0)\to(\mathbb{C},0)$ be a germ of holomorphic function such that $df(x)\neq 0$ for $x\neq 0$.

In a previous paper [B-M 99], we have explained how eigenvalues of the monodromy M of f acting on $H^n(F)$, where F is the Milnor fibre of f, contribute to create poles of the meromorphic extension of the current $\lambda\mapsto \Gamma(\lambda)^{-1}\int_X|f|^{2\lambda}\square$. For eigenvalues different from 1, our results generalize those of the first author [B 84] for smooth X. But for the eigenvalue 1 of M, poles of the above current appear at negative integers if, and only if, 1 is also an eigenvalue of the monodromy of f acting in the quotient $H^n(F)/J$, where J is the image of the map $H^n(X\setminus\{0\})\to H^n(F)$ induced by restriction. (See Example 3 for explicit computation of the image.) When this restriction is surjective (which implies $M=1$ and is therefore a very strong hypothesis), it follows that $\lambda\mapsto\int_X|f|^{2\lambda}\square$ has only simple poles on the negative integers.

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Using inverse Mellin transform, we deduce that, for all $\varphi \in C_{\mathrm{c}}^{\infty}(X)^{n,n}$, the fibre-integrals

$$s \mapsto \int_{f=s} \varphi \tag{1}$$

are of class C^{∞} (because

$$\int_X |f|^{2\lambda} \varphi \wedge \frac{df}{f} \wedge \frac{d\overline{f}}{\overline{f}} = \int_{\mathbb{C}} |s|^{2\lambda} \int_{f=s} \varphi \frac{ds}{s} \wedge \frac{d\overline{s}}{\overline{s}}$$

by Fubini's theorem). In this situation, fibre-integrals of C^{∞} forms of type (n,n) do not detect the singularity of the map $f : X \to \mathbb{C}$, that is, X not smooth or $df(0) = 0$.

In a more general context, asymptotic expansions at 0 of functions (1) give rise to a finitely generated $\mathbb{C}[[s,\overline{s}]]$-module $\mathcal{M}$ (see Theorem 1 below). Because X and f have an isolated singularity at 0, this module is generated by 1 and the asymptotic expansions of the following functions:

$$s \mapsto \int_{f=s} \varrho\omega' \wedge \overline{\omega''}, \qquad \omega', \omega'' \in \Omega_X^n, \tag{2}$$

where $\varrho \in C_{\mathrm{c}}^{\infty}(X)$ is equal to 1 near 0. Indeed, for any integer $N > 0$, there exist $\omega_l', \omega_l'' \in \Omega_X^n$ and $L(N) \in \mathbb{N}$ such that

$$\varphi - \sum_{l=1}^{L(N)} \varrho\omega_l' \wedge \overline{\omega_l''}$$

is flat of order N at 0; because the coefficients of the non-C^{∞} terms of the asymptotic expansion are currents carried by 0 and because $\mathcal{M}$ is of finite type (cf. Theorem 1 below), the assertion follows.

When $f : X \to \mathbb{C}$ satisfies $J = H^n(F)$ the considerations above may be written briefly as

$$\mathcal{M} = \mathbb{C}[[s,\overline{s}]].$$

In order to detect the singularity of the map $f : X \to \mathbb{C}$ with fibre-integrals, we then consider

$$s \mapsto \int_{f=s} \varrho\omega' \wedge \overline{\omega''}, \qquad \omega', \omega'' \in \underline{\omega}_X^n, \tag{3}$$

where $\underline{\omega}_X^n$ is the direct image sheaf on X of holomorphic n-forms on the nonsingular part $X^* = X \setminus \{0\}$. The germs at 0 of these new fibre-integrals, to which we add the function 1, have the structure of a $\mathbb{C}\{s,\overline{s}\}$-module; this module tensored by $\mathbb{C}[[s,\overline{s}]]$ gives a $\mathbb{C}[[s,\overline{s}]]$-module $\mathcal{N}$ containing $\mathcal{M}$. When X is smooth, we have $\mathcal{N} = \mathcal{M}$ because, by Hartogs, $\underline{\omega}_X^n = \Omega_X^n$. In general we know that there exists an integer ν such that $|s|^{2\nu}\mathcal{N} \subseteq \mathcal{M}$, because

$f^\nu \underline{\omega}_X^n \subseteq \Omega_X^n$ (see Remark 1.2 of [B-M 99]). The trivial inclusion $\mathcal{M} \subseteq \mathcal{N}$ is strict in general as shown in Example 2.

Proposition 6 below shows that we may have $\mathcal{N} = \mathcal{M} = \mathbb{C}[[s,\bar{s}]]$ for a small but nonempty class of singular maps $f : X \to \mathbb{C}$. The invariant $\mathcal{N}$ is therefore not fine enough to detect the singularity of $f : X \to \mathbb{C}$. It is then natural to widen the class of fibre-integrals under consideration. To this end, if ω' and ω'' belong to $\underline{\omega}_X^{n+1}$ we look at

$$s \mapsto \Big(\int_{f=s} \varrho \, \frac{\omega'}{df} \wedge \overline{\frac{\omega''}{df}} \Big) \, ds \wedge d\bar{s}. \tag{4}$$

The $(1,1)$-form above is nothing but the direct image $f_*(\varrho\omega' \wedge \overline{\omega''})$. The asymptotic expansions of forms (4) generate a module $\mathcal{N}^{1,1}$ on $\mathbb{C}[[s,\bar{s}]]$. In case X is smooth, this new module can be deduced from $\mathcal{N} = \mathcal{M}$ by means of the following relation:

$$f_*(\varrho\underline{\omega}_X^{n+1} \wedge \overline{\Omega_X^n}) \subseteq d' f_*(\varrho \Omega_X^n \wedge \overline{\Omega_X^n}). \tag{5}$$

Indeed, for $\omega' \in \Omega_X^{n+1}$ and $\omega'' \in \Omega_X^n$, we may write, using the holomorphic de Rham lemma,

$$\omega' = d'\omega_1 \quad \text{with } \omega_1 \in \Omega_X^n.$$

Hence

$$f_*(\varrho\omega' \wedge \overline{\omega''}) = d' f_*(\varrho\omega_1 \wedge \overline{\omega''}) - f_*(d'\varrho \wedge \omega_1 \wedge \overline{\omega''}).$$

But because ϱ is identically 1 near 0, the direct image $f_*(d'\varrho \wedge \omega_1 \wedge \overline{\omega''})$ belongs to $C_c^\infty(\mathbb{C}^*)^{1,0} \subseteq d' C^\infty(\mathbb{C})^{0,0}$ and hence $f_*(\varrho\omega_1 \wedge \overline{\omega''})$ belongs to $d'(f_*(\varrho\Omega_X^n \wedge \overline{\Omega_X^n}))$. Relation (5) implies that for X smooth we have $\mathcal{N}^{1,1} = d'd''\mathcal{N} (= d'd''\mathcal{M})$. This equality does not hold in the example of $X = \{x^2 + y^3 = z^6\}$ and $f = z$ (see (18) and Proposition 6) where $\mathcal{M} = \mathcal{N} = \mathbb{C}[[s,\bar{s}]]$ and $\mathcal{N}^{1,1}$ contains $(ds \wedge d\bar{s})/s\bar{s}$! When X is singular, the holomorphic de Rham lemma is not valid and in fact relation (5) is no longer true.

It turns out that there exist very few maps for which $\mathcal{M} = \mathcal{N} = \mathbb{C}[[s,\bar{s}]]$ and $\mathcal{N}^{1,1} = \mathbb{C}[[s,\bar{s}]] ds \wedge d\bar{s}$. We describe the construction of a class of examples presenting that feature in Section 3 and conclude with very explicit singularities.

1. Asymptotic expansion of fibre-integrals. Let us start with a version of the asymptotic expansion theorem for fibre-integrals of forms of type (2), (3) or (4). We do not assume that X and f have an isolated singularity here.

Theorem 1. *Let X be a reduced analytic space of pure dimension $n+1 \geq 2$ and let $f : X \to \mathbb{C}$ be a holomorphic function satisfying*

(i) $\mathrm{Sing}(X) \subseteq f^{-1}(0)$;
(ii) $df(x) = 0 \Rightarrow f(x) = 0$ *for* $x \in X \setminus \mathrm{Sing}(X)$.

Then, for any $\varrho \in C^\infty_c(X)$ *and* $(\omega', \omega'') \in \underline{\omega}^{n+p}_X \times \underline{\omega}^{n+q}_X$, *where* $p, q \in \{0,1\}$, *the direct image* $f_*(\varrho\,\omega' \wedge \overline{\omega''})$ *is a* (p,q)*-current of class* C^∞ *on* $\mathbb{C}^*$, *admitting, as* $s \to 0$, *an asymptotic expansion that belongs to*

$$\bigoplus_{r\in R,\, k\in[0,n]} \mathbb{C}[[s,\overline{s}]]\,|s|^{2(r-\nu)}\log^k|s| \left(\frac{ds}{s}\right)^p \wedge \left(\frac{d\overline{s}}{\overline{s}}\right)^q \qquad \textit{if } p+q>0,$$

$$\bigoplus_{r\in R,\, k\in[0,n]} \mathbb{C}[[s,\overline{s}]]\,|s|^{2(r-\nu)}\log^k|s| + \mathbb{C}[[s,\overline{s}]]\,|s|^{-2\nu} \qquad \textit{if } p+q=0,$$

where ν *is an integer*, R *is a finite subset of* $]0,1] \cap \mathbb{Q}$ *that only depends on* X, f *and* $\mathrm{supp}\,\varrho$. *This asymptotic expansion may be differentiated termwise.*

REMARK. In the second expression, terms of type $|s|^{-2\nu}\log^k|s|$ are not permitted if $k > 0$.

Proof. By [B 78] the sheaf $\underline{\omega}^{n+p}_X$ is coherent and $\mathrm{supp}\,\underline{\omega}^{n+p}_X/\Omega^{n+p}_X$ is contained in $\mathrm{Sing}(X)$ for any p; the Nullstellensatz gives locally an integer ν such that $f^\nu\underline{\omega}^{n+p}_X \subseteq \Omega^{n+p}_X/\text{torsion}$, using (i). There exist therefore two forms $\zeta' \in \Omega^{n+p}_X$ and $\zeta'' \in \Omega^{n+q}_X$ such that

$$\omega' = \zeta'/f^\nu \quad \text{and} \quad \omega'' = \zeta''/f^\nu.$$

Because

$$f_*(\varrho\omega' \wedge \overline{\omega''})(s) = f_*(\varrho\zeta'/f^\nu \wedge \overline{\zeta''}/\overline{f}^\nu)(s) = \frac{1}{|s|^{2\nu}} f_*(\varrho\zeta' \wedge \overline{\zeta''})(s),$$

it is enough to prove that the asymptotic expansion of $f_*(\varrho\zeta' \wedge \overline{\zeta''})$ belongs to

$$\bigoplus_{r\in R,\, k\in[0,n]} \mathbb{C}[[s,\overline{s}]]\,|s|^{2r}\log^k|s| \left(\frac{ds}{s}\right)^p \wedge \left(\frac{d\overline{s}}{\overline{s}}\right)^q \qquad \text{if } p+q>0,$$

$$\bigoplus_{r\in R,\, k\in[0,n]} \mathbb{C}[[s,\overline{s}]]\,|s|^{2r}\log^k|s| + \mathbb{C}[[s,\overline{s}]] \qquad \text{if } p+q=0.$$

Let us desingularize X. We are reduced to proving the result when X is nonsingular and ω', ω'' are holomorphic; therefore $\phi := \varrho\omega' \wedge \overline{\omega''}$ belongs to $C^\infty_c(X)^{n+p,n+q}$. Using a partition of unity, we may even assume X to be an open subset of $\mathbb{C}^{n+1}$.

Consider the case $p = q = 1$. From the definition of direct images, when $\phi \in C^\infty_c(X')^{n+1,n+1}$, where $X' = X \setminus f^{-1}(0)$, we have

$$\int_{X'} |f|^{2\lambda}\phi = \int_{\mathbb{C}^*} |s|^{2\lambda} f_*\phi(s). \tag{6}$$

Indeed, set $\psi(s)=|s|^{2\lambda}$ in the relation $\langle f_*\phi,\psi\rangle = \langle \phi, f^*\psi\rangle$. It follows that for $\phi\in C_c^\infty(X)^{n+1,n+1}$, the form $f_*\phi|_{\mathbb{C}^*}$ is equal to $\mathfrak{M}^{-1}(\lambda\mapsto \int_X |f|^{2\lambda}\phi)$ where $\mathfrak{M}$ is the complex Mellin transform defined by $\mathfrak{M}\alpha(\lambda) = \int_{\mathbb{C}} |s|^{2\lambda}\alpha(s)$ for $\alpha\in C_c^\infty(\mathbb{C}^*)^{1,1}$. It is well known that $\lambda\mapsto \int_X |f|^{2\lambda}\phi$ admits a meromorphic extension to $\mathbb{C}$ with poles at strictly negative rationals contained in $-R-\mathbb{N}$ with some finite $R\subset\,]0,1]$. Moreover $\lambda\mapsto\int_X |f|^{2\lambda}\phi$ is rapidly decreasing on $\{\operatorname{Re}\lambda = \text{const}\}$. Considering also the meromorphic extension of $\lambda\mapsto \int_X |f|^{2\lambda}\overline{f}^m\phi$ for $m\in\mathbb{Z}$ and taking the inverse Mellin transform we get the desired asymptotic expansion (see [B-M 89]).

In case $p=q=0$ instead of (6) we write

$$(7)\qquad \int_{X'} |f|^{2\lambda}\varphi\wedge\frac{df}{f}\wedge\frac{d\overline{f}}{\overline{f}} = \int_{\mathbb{C}^*} |s|^{2\lambda} f_*\varphi(s)\frac{ds}{s}\wedge\frac{d\overline{s}}{\overline{s}},\qquad \varphi\in C_c^\infty(X)^{n,n}.$$

When f has only normal crossings we may write in an appropriate coordinate system

$$f(z) = z_0^{\alpha_0}\ldots z_n^{\alpha_n}$$

and so

$$\begin{aligned}\frac{df}{f}\wedge\frac{d\overline{f}}{\overline{f}} &= \left(\alpha_0\frac{dz_0}{z_0}+\ldots+\alpha_n\frac{dz_n}{z_n}\right)\wedge\left(\alpha_0\frac{d\overline{z}_0}{\overline{z}_0}+\ldots+\alpha_n\frac{d\overline{z}_n}{\overline{z}_n}\right)\\ &= \sum\alpha_j\alpha_k\frac{dz}{z}_j\wedge\frac{d\overline{z}_k}{\overline{z}_k}.\end{aligned}$$

When $j\neq k$, the form

$$z\mapsto |z_0^{\alpha_0}\ldots z_n^{\alpha_n}|^{2\lambda}\varphi(z)\wedge\frac{dz_j}{z_j}\wedge\frac{d\overline{z}_k}{\overline{z}_k}$$

is integrable for $\operatorname{Re}\lambda\geq 0$; the pole at $\lambda=0$ of

$$\lambda\mapsto\int |f|^{2\lambda}\varphi\wedge\frac{df}{f}\wedge\frac{d\overline{f}}{\overline{f}}$$

is therefore created by terms of the type

$$\int |z_0^{\alpha_0}\ldots z_n^{\alpha_n}|^{2\lambda}\varphi(z)\wedge\frac{dz_j}{z_j}\wedge\frac{d\overline{z}_j}{\overline{z}_j}$$

and it is simple. The result follows by taking the Mellin transform and (7).

The case $p=1$, $q=0$ is similar. ■

Example 2. *Computation of* $\mathcal{M}$ *and* $\mathcal{N}$ *for*

$$X=\{(x,y,z)\in\mathbb{C}^3 \mid xy=z^2\}\quad\textit{and}\quad f(x,y,z)=z.$$

Using a Taylor expansion, we see that for $\varphi\in C_c^\infty(\mathbb{C}^3)^{1,1}$,

$$\int_{X,z=s}\varphi = \int_{xy=s^2}\varphi(x,y,s) = \sum_{p+q\leq N-1} s^p\overline{s}^q\int_{xy=s^2}\psi_{pq} + O(|s|^N)$$

where $\psi_{pq} \in C^\infty_c(\mathbb{C}^2)^{1,1}$. The asymptotic expansion of $\int_{xy=\sigma} \psi_{pq}$ belongs to the module $\mathbb{C}[[\sigma,\overline{\sigma}]] \oplus \mathbb{C}[[\sigma,\overline{\sigma}]]\,|\sigma|^2 \log|\sigma|$ because the monodromy of the map $(x,y) \mapsto xy$ is the identity (see [B 85]). Therefore

$$\mathcal{M} = \mathbb{C}[[s,\overline{s}]] \oplus \mathbb{C}[[s,\overline{s}]]\,|s|^4 \log|s|.$$

Take $\omega = (xdy - ydx)/z$. Then ω belongs to $\underline{\omega}^1_X$ because $x\omega$ and $y\omega$ are holomorphic. Standard computations give

$$\int_{X,z=s} \varrho\omega \wedge \overline{\omega} \sim |s|^2 \log|s|$$

and so

$$\mathcal{N} = \mathbb{C}[[s,\overline{s}]] \oplus \mathbb{C}[[s,\overline{s}]]\,|s|^2 \log|s|.$$

2. Occurrence of logarithmic terms. Let us recall the following consequence of Theorem 6.4 of [B-M 99] that guarantees the occurrence of a term $s^m\overline{s}^{m+j}\log|s|$ in the asymptotic expansion of fibre-integrals for (n,n)-forms. We assume that $(X,0)$ is a germ of normal complex space of dimension $n+1$ with an isolated singularity at 0 and denote by $f:(X,0)\to(\mathbb{C},0)$ a germ of holomorphic function such that $df(x)\neq 0$ for $x\neq 0$.

Let J be the image of the restriction map $H^n(X\setminus\{0\}) \to H^n(F)$, where F is the Milnor fibre of f.

Theorem. *Suppose ω is a holomorphic n-form on X that satisfies*

$$d\omega = m\frac{df}{f}\wedge\omega \quad \text{with some } m\in\mathbb{N}.$$

Then the following two properties are equivalent:

(i) *there exist $j\in\mathbb{Z}$ and $\omega''\in H^0(X,\Omega^n_X)$ such that the asymptotic expansion of the function $s\mapsto \int_{f=s}\varrho\omega\wedge\overline{\omega''}$ contains the term $s^m\overline{s}^{m+j}\log|s|$*;

(ii) *the class of ω/f^m in $H^n(F)^M$ does not belong to J.*

Remark. Using the decomposition of ω'' in a Jordan basis of the Gauss–Manin system of f, it is possible to choose ω'' so as to have

$$\int_{f=s} \varrho\omega\wedge\overline{\omega''} \equiv s^m\overline{s}^{m+j}\log|s| \pmod{\mathbb{C}[[s,\overline{s}]]},$$

after increasing j if necessary.

In the next example, we compute J and $H^n(F)^M$.

Example 3. *For the singularity $X=\{x^2+y^3+z^6=0\}\subset\mathbb{C}^3$ and $f:X\to\mathbb{C}$ given by $f(x,y,z)=x$ we have*

$$0 \subsetneq J \subsetneq H^1(F)^M = H^1(F)_1.$$

Proof. Here, $n=1$ and the Milnor fibre of f is $F=\{(1,y,z)\in\mathbb{C}^3 \mid y^3+z^6=-1\}$; it is therefore also the Milnor fibre of $g:\mathbb{C}^2\to\mathbb{C}$ given by

$g(y,z) = y^3 + z^6$. By Milnor, $\dim H^1(F) = 10$. The corresponding monodromy M_g is diagonal with eigenvalues $e^{2i\pi 1/2}$ (2), $e^{2i\pi 2/3}$, $e^{2i\pi 5/6}$ (2), $e^{2i\pi}$ (2), $e^{2i\pi 7/6}$ (2), $e^{2i\pi 4/3}$ (the number in parentheses indicates multiplicity).

The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{C} \\ \pi\downarrow & & \downarrow\tau \\ \mathbb{C}^2 & \xrightarrow{g} & \mathbb{C} \end{array}$$

where $\pi(x,y,z) = (y,z)$ and $\tau(x) = x^2$ shows that $M_f = M_g^2$ and its eigenvalues are $e^{2i\pi}$ (4), $e^{2i\pi 2/3}$ (3), $e^{2i\pi 4/3}$ (3). Here $H^1(F)^{M_f} = H^1(F)_1$ has dimension 4 and $\dim H^1(X^* \setminus X_0^*) = 5$, where $X_0 = \{x = 0, y^3 + z^6 = 0\}$. To check this last equality, remember that $H^1(X^* \setminus X_0^*) \cong H^1(F)_1 \oplus \mathbb{C}\frac{df}{f}$.

Let

$$\omega_1 = \frac{zdy - 2ydz}{x} \quad \text{and} \quad \omega_2 = \frac{yz^5dy - 2y^2z^4dz}{x^3} = \frac{yz^4}{x^2}\omega_1.$$

Then ω_1 and ω_2 give classes in $H^1(X \setminus X_0)$ which extend to X^*. The other three generators of $H^1(X \setminus X_0)$,

$$\omega_3 = \frac{yz}{x}\omega_1, \quad \omega_4 = \frac{y^3}{x}\omega_1 \quad \text{and} \quad \omega_5 = \frac{dx}{x},$$

do not "extend" to X^*. ∎

Remark. For the same singularity but with $f(x,y,z) = y$, it is easy to see that fibre-integrals of C^∞ forms are not always C^∞. As a consequence the wave front set of the integration current on $X \subset \mathbb{C}^3$ contains $\{0\} \times \mathbb{C}^3$ because it contains a cotangent vector $(0,0,0;0,1,0)$ that does not belong to the closure of the conormal space to X^*.

3. A class of singularities with smooth fibre-integrals. In this section, we consider the following situation. Let $g \in \mathcal{O}_{\mathbb{C}^{n+1}}$ have an isolated singularity at 0, $g(0) = 0$. Denote by M_g the monodromy of g at 0 and suppose M_g does not have the eigenvalue 1, that is,

(8) $$M_g - 1 \text{ is invertible,}$$

or, equivalently, the intersection form on $H^n(G)$, where G is the Milnor fibre of g, is nondegenerate (see [A-G-Z-V], p. 410).

Assume also the existence of an integer $N > 0$ such that

(9) $$M_g^N = 1.$$

This last hypothesis implies that M_g diagonalizes.

Let $\sigma(g)$ denotes the Arnold exponent of g and $R(g) \subset]0,1[$ its spectrum modulo 1. Hypothesis (9) yields

(10) $$N \cdot R(g) \subset \mathbb{N}^*.$$

By classical results (cf. [B 85]), fibre-integrals with respect to g have asymptotic expansions at 0 of the following type, for $\varrho \in C_c^\infty(\mathbb{C}^{n+1})$ equal to 1 in a neighbourhood of $0 \in \mathbb{C}^{n+1}$:

$$\begin{aligned} &\eta', \eta'' \in \Omega^n_{\mathbb{C}^{n+1}} \Rightarrow \int_{g=t} \varrho\eta' \wedge \overline{\eta''} \in \sum_{r\in R(g)} \mathbb{C}[[t,\overline{t}]]\,|t|^{2r}, \\ &\zeta', \zeta'' \in \Omega^{n+1}_{\mathbb{C}^{n+1}} \Rightarrow \int_{g=t} \varrho\frac{\zeta'}{dg} \wedge \overline{\frac{\zeta''}{dg}} \in \sum_{r\in R(g)} \mathbb{C}[[t,\overline{t}]]\,|t|^{2r-2}. \end{aligned} \tag{11}$$

There are no logarithmic terms because $1 \notin \operatorname{spec} M_g$ and all Jordan blocks of M_g have size 1.

Let us now define the analytic space X and the holomorphic function f we will study in this section:

$$\text{(H)} \quad \begin{aligned} &X = \{(x,s) \in \mathbb{C}^{n+2} \mid g(x) = s^N\}, \quad \text{where } g \text{ satisfies (8) and (9)}, \\ &f(x,s) = s. \end{aligned}$$

Observe that the hypersurface X has an isolated singularity at 0 because $df \wedge (dg - Ns^{N-1}ds) = 0$ implies $ds \wedge dg = 0$.

The commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{C} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tau} \\ \mathbb{C}^{n+1} & \xrightarrow{g} & \mathbb{C} \end{array}$$

where $\pi(x,s) = x$ and $\tau(s) = s^N$, shows that the fibres of f and g are isomorphic because $f^{-1}(s) = g^{-1}(s^N) \times \{s\}$; it also explains why $M_f = M_g^N = 1$. On $X' := X \setminus \{s = 0\}$, we have

$$N\frac{ds}{s} = \frac{dg}{g}. \tag{12}$$

As a consequence, for any $\eta \in \Omega^n_{\mathbb{C}^{n+1}}$ such that $d\eta = r\frac{dg}{g} \wedge \eta$, the following formula holds:

$$d\left(\frac{\pi^*\eta}{s^m}\right) = (rN - m)\frac{ds}{s} \wedge \frac{\pi^*\eta}{s^m}, \qquad m \in \mathbb{N}. \tag{13}$$

For holomorphic forms on X we use the obvious decomposition

$$\begin{aligned} \Omega^n_X &= \mathcal{O}_X\pi^*\Omega^n_{\mathbb{C}^{n+1}} + \mathcal{O}_X\pi^*\Omega^{n-1}_{\mathbb{C}^{n+1}} \wedge ds, \\ \Omega^{n+1}_X &= \mathcal{O}_X\pi^*\Omega^{n+1}_{\mathbb{C}^{n+1}} + \mathcal{O}_X\pi^*\Omega^n_{\mathbb{C}^{n+1}} \wedge ds. \end{aligned} \tag{14}$$

For the sheaves $\underline{\omega}^n_X$ and $\underline{\omega}^{n+1}_X$ we need a lemma.

Lemma 4. *Under hypothesis* (H), *we have*

$$\underline{\omega}^n_X \subseteq \frac{1}{s^{N-1}}\mathcal{O}_X\pi^*\Omega^n_{\mathbb{C}^{n+1}}, \qquad \underline{\omega}^{n+1}_X \subseteq \frac{1}{s^{N-1}}\mathcal{O}_X\pi^* dx_0 \wedge \ldots \wedge dx_n.$$

Proof. Let $j = n$ or $n+1$. Any section of $\underline{\omega}_X^j$ near the origin belongs to $\Omega_X^j[s^{-1}]$ because there exists $\nu \in \mathbb{N}$ such that $s^\nu \underline{\omega}_X^j$ is contained in Ω_X^j modulo torsion. On the other hand, on X' we have

$$ds = \frac{1}{Ns^{N-1}} dg,$$

by (12). So any section of $\underline{\omega}_X^j$ is a finite sum of elements of the type $\pi^*\alpha/s^k$ where $\alpha \in \Omega_{\mathbb{C}^{n+1}}^j$ and $k \in \mathbb{Z}$. Suppose $k \geq N$; because the sections of $\underline{\omega}_X^j$ have the trace property (see [B 78]), we get

$$\operatorname{trace}_\pi \left(\frac{\pi^*\alpha}{s^N} \right) = N \frac{\alpha}{g} \in \Omega_{\mathbb{C}^{n+1}}^j.$$

Hence $\alpha = g\beta$ with $\beta \in \Omega_{\mathbb{C}^{n+1}}^j$ and so $\pi^*\alpha/s^k = \pi^*\beta/s^{k-N}$. Iterating this process we are reduced to $k \leq N-1$, proving the inclusions. ∎

Remark. The second inclusion is in fact an equality.

Proposition 5. *Assume* (H). *Then for* $(p,q) \in \{0,1\}$ *we have*

$$\mathcal{M}^{p,q} = \mathbb{C}[[s,\overline{s}]]\, ds^p \wedge d\overline{s}^q.$$

Proof. Case 1: $p = q = 0$. Thanks to (14) we only need to show that fibre-integrals for $\pi^*\eta' \wedge \pi^*\overline{\eta''}$ are C^∞ for $\eta', \eta'' \in \Omega_{\mathbb{C}^{n+1}}^n$. Indeed, the second term in (14) does not contribute and $\mathcal{O}_X \subseteq \mathbb{C}[[s]]\pi^*\mathcal{O}_{\mathbb{C}^{n+1}}$ explains how $\mathcal{O}_X$ coefficients are treated. But

$$\int\limits_{f^{-1}(s)\cap X} \varrho\pi^*\eta' \wedge \pi^*\overline{\eta''} = \int\limits_{g=s^N} \varrho\eta' \wedge \overline{\eta''} \in \sum_{r\in R(g)} \mathbb{C}[[s^N,\overline{s}^N]]\,|s|^{2rN} \subseteq \mathbb{C}[[s,\overline{s}]]$$

by (11) and (10).

Case 2: $p = q = 1$. The term containing ds in formula (14) produces a C^∞ term after fibre-integration, from the first part of the proof. Now for $\zeta \in \Omega_{\mathbb{C}^{n+1}}^{n+1}$, we have

$$\frac{\pi^*\zeta}{ds} = \frac{1}{s}\pi^*\left(\frac{Ng\zeta}{dg}\right) = Ns^{N-1}\pi^*\left(\frac{\zeta}{dg}\right)$$

from (13) and hence

$$\int\limits_{f^{-1}(s)} \varrho\, \frac{\pi^*\zeta'}{ds} \wedge \frac{\pi^*\overline{\zeta''}}{d\overline{s}} = N^2|s|^{2N-2} \int\limits_{g=s^N} \varrho\, \frac{\zeta'}{dg} \wedge \frac{\overline{\zeta''}}{d\overline{g}}$$
$$\in \sum_{r\in R(g)} \mathbb{C}[[s^N,\overline{s}^N]]|s|^{2rN-2};$$

this fibre-integral is C^∞ because $N\sigma(g) - 1 \geq 0$ from $N\sigma(g) \in \mathbb{N}^*$.

Other cases are left to the reader. ∎

Remarks. 1) The cutoff function ϱ need not be compactly supported in s, that is why it only depends on x in the above calculations. In fact the f-proper forms and the compactly supported ones give the same asymptotic expansions modulo $\mathbb{C}[[s,\bar{s}]]$.

2) Proposition 5 and Corollary 6.5 of [B-M 99] show that $\dim H^n(X^*) = \dim H^n(F)$. In our situation (H), this dimension is easily computable because F is isomorphic to the Milnor fibre of g.

Proposition 6. *Under hypothesis* (H), *the following implications hold*:

(a) $N\sigma(g) \geq N-1 \Rightarrow \mathcal{N}_f = \mathbb{C}[[s,\bar{s}]]$;

(b) $\sigma(g) > 1 \Leftrightarrow \mathcal{N}_f^{1,1} = \mathbb{C}[[s,\bar{s}]]\, ds \wedge d\bar{s}$.

The converse of (a) *is true for quasi-homogeneous* g.

Remark. Because $\sigma(g)$ is not an integer, $\sigma(g) > 1$ is equivalent to $\sigma(g) \geq 1$. On the other hand, because $N\sigma(g)$ is an integer, $\sigma(g) \geq 1$ is equivalent to $\sigma(g) > (N-1)/N$.

Proof of Proposition 6. (a)($\Rightarrow$) Let $\eta', \eta'' \in \underline{\omega}_X^n$. By Lemma 4, there exist $\eta_j', \eta_k'' \in \Omega^n_{\mathbb{C}^{n+1}}$ such that

$$\eta' = \frac{1}{s^{N-1}} \sum_{j=0}^{\infty} s^j \pi^* \eta_j', \qquad \eta'' = \frac{1}{s^{N-1}} \sum_{k=0}^{\infty} s^k \pi^* \eta_k''.$$

Therefore

$$\int_{f^{-1}(s)\cap X} \varrho \pi^* \eta' \wedge \pi^* \overline{\eta''} = \sum_{j,k\geq 0} \int_{g=s^N} \varrho s^{j-N+1} \bar{s}^{k-N+1} \eta_j' \wedge \overline{\eta_k''}$$
$$\in \mathbb{C}[[s,\bar{s}]]\, |s|^{2N\sigma(g)-2N+2} \subseteq \mathbb{C}[[s,\bar{s}]].$$

(a)($\Leftarrow$) When g is quasi-homogeneous, from [L] we get the existence of $\omega \in \Omega^n_{\mathbb{C}^{n+1}}$ such that

$$\int_{g=t} \varrho \omega \wedge \overline{\omega} = |t|^{2\sigma(g)} + o(|t|^{2\sigma(g)}). \tag{15}$$

It is possible to choose ω such that

$$d\omega = \sigma(g) \frac{dg}{g} \wedge \omega. \tag{16}$$

Consider $\eta = (1/s^{N-1})\pi^*\omega$; we check η belongs to $\underline{\omega}_X^n$. By [B 78], it is enough to see that for all $j \in [0, N-1]$,

$$\operatorname{trace}_\pi \left(\frac{s^j}{s^{N-1}} \pi^* \omega \right) \in \Omega^n_{\mathbb{C}^{n+1}} \quad \text{and} \quad \operatorname{trace}_\pi \left(\frac{s^j}{s^{N-1}} ds \wedge \pi^* \omega \right) \in \Omega^{n+1}_{\mathbb{C}^{n+1}}.$$

The first trace vanishes for $j < N-1$, and it is equal to $N\omega$ when $j = N-1$. The second trace is nonzero only for $j = N-2$ and then it is equal to $\frac{dg}{g} \wedge \omega$.

Relation (16) implies $\eta \in \underline{\omega}_X^n$.

Integrating along fibres, we get, from (15),

$$\int_{f^{-1}(s)\cap X} \varrho\eta \wedge \overline{\eta} = |s|^{2N\sigma(g)-2N+2}(1+o(1)).$$

In order that this integral be C^∞, we must have $N\sigma(g) \geq N-1$.

(b)($\Rightarrow$) For $\zeta \in \Omega^{n+1}_{\mathbb{C}^{n+1}}$, we have by (12),

$$\frac{1}{s^{N-1}}\frac{\pi^*\zeta}{ds} = \frac{1}{s^N}\pi^*\left(\frac{Ng\zeta}{dg}\right) = N\pi^*\left(\frac{\zeta}{dg}\right).$$

Taking fibre-integrals gives

$$\int_{f^{-1}(s)} \varrho\,\frac{\pi^*\zeta'}{ds} \wedge \frac{\pi^*\overline{\zeta''}}{d\overline{s}} = N^2 \int_{g=s^N} \varrho\,\frac{\zeta'}{dg} \wedge \frac{\overline{\zeta''}}{d\overline{g}} \in \sum_{r\in R(g)} \mathbb{C}[[s^N, \overline{s}^N]]\,|s|^{2N(r-1)};$$

this fibre-integral is C^∞. It remains to use $\mathbb{C}[[s,\overline{s}]]$-linearity and Lemma 4.

(b)($\Leftarrow$) Following [L], take a holomorphic $(n+1)$-form Ω on $\mathbb{C}^{n+1}$ such that

$$\int_{g=t} \varrho\,\frac{\Omega}{dg} \wedge \frac{\overline{\Omega}}{d\overline{g}} = |t|^{2\sigma(g)-2} + o(|t|^{2\sigma(g)-2}).$$

With $\zeta := \frac{1}{Ns^{N-1}}\pi^*\Omega \in \underline{\omega}_X^{n+1}$ we have

$$\int_{f^{-1}(s)\cap X} \varrho\,\frac{\zeta}{ds} \wedge \frac{\overline{\zeta}}{d\overline{s}} = |s|^{2N(\sigma(g)-1)}(1+o(1)).$$

If this integral is C^∞ then $\sigma(g) \geq 1$. ∎

4. Explicit examples. We present here explicit examples of singularities X and functions f for which all fibre-integrals are C^∞; integration of forms in $\underline{\omega}_X^{n+1}$ is allowed.

To fulfill conditions (8) and (9), we look for Fermat's singularities

$$g(x) = x_0^{p_0} + \ldots + x_n^{p_n}$$

where $p_0, \ldots, p_n$ are integers ≥ 2 that satisfy

$$\frac{a_0}{p_0} + \ldots + \frac{a_n}{p_n} \notin \mathbb{N} \tag{17}$$

for all $a_j \in \mathbb{N}$ with $0 < a_j < p_j$. We take $N = \mathrm{lcm}(p_0, \ldots, p_n)$ in (9). A sufficient condition for (17) is

$$\exists j \in [0,n] \quad \text{such that} \quad (p_j, p_k) = 1,\ \forall k \neq j.$$

For n even, the following condition is also sufficient:

$$\forall j,\ \forall k: \quad (p_j, p_k) = 2 \text{ if } j \neq k.$$

Case $n = 1$. Condition (17) is equivalent to $(p_0, p_1) = 1$ and so $N = p_0 p_1$. The smallest values of $p_0 \le p_1$ are $2, 3$, so that

$$\frac{1}{p_0} \le \frac{1}{2}, \qquad \frac{1}{p_1} \le \frac{1}{3}, \qquad \frac{1}{N} \le \frac{1}{6}.$$

Hence

$$\frac{1}{p_0} + \frac{1}{p_1} + \frac{1}{N} \le 1 \quad \text{or} \quad \frac{1}{p_0} + \frac{1}{p_1} \le \frac{N-1}{N}$$

and the inequalities are strict if $p_0 > 2$ or $p_1 > 3$. Hence the only X for which Proposition 6 applies is

$$X = \{x_0^2 + x_1^3 = s^6\}. \tag{18}$$

Case $n = 2$. We take $p_0 \le p_1 \le p_2$ and notice that

$$\frac{1}{p_0} + \frac{1}{p_1} + \frac{1}{p_2} \ge \frac{N-1}{N}$$

with $N = \operatorname{lcm}(p_0, p_1, p_2)$ may be satisfied, because $N \ge p_2$, only if

$$p_2 \le \frac{2p_0 p_1}{p_0 p_1 - p_0 - p_1}.$$

This remark enables us to easily eliminate many values of p_0, p_1, p_2 satisfying (17).

p_0	p_1	p_2	N	$\sigma(g)$		$\frac{N-1}{N}$
2	2	$2k$	$2k$	$\frac{2k+1}{2k}$	$>$	$\frac{2k-1}{2k}$
2	2	$2k+1$	$2k+1$	$\frac{2k+2}{2k+1}$	$>$	$\frac{2k}{2k+1}$
2	3	3	6	$\frac{5}{6}$	$=$	$\frac{5}{6}$
2	3	4	12	$\frac{13}{12}$	$>$	$\frac{11}{12}$
2	3	5	30	$\frac{31}{30}$	$>$	$\frac{29}{30}$
2	3	7	42	$\frac{41}{42}$	$=$	$\frac{41}{42}$
2	3	8	24	$\frac{23}{24}$	$=$	$\frac{23}{24}$
2	3	9	18	$\frac{17}{18}$	$=$	$\frac{17}{18}$
2	4	5	20	$\frac{19}{20}$	$=$	$\frac{19}{20}$
2	5	5	10	$\frac{9}{10}$	$=$	$\frac{9}{10}$
3	3	4	12	$\frac{11}{12}$	$=$	$\frac{11}{12}$

In the table above, we give all triples p_0, p_1, p_2 for which Proposition 6 applies, that is, (17) and $\sigma(g) \ge (N-1)/N$ hold. So there are only few examples where $\sigma(g) > 1$, i.e., examples for which all fibre-integrals are smooth.

REMARK. Using the Thom–Sebastiani result, it is easy to see that if g and N satisfy conditions (8) and (9), then the function G defined by $G(x, y, z_0, \ldots, z_n) = x^2 + y^2 + g(z_0, \ldots, z_n)$ gives also an example with the same N. So the first two series of examples in the table come from trivial examples in dimension 1 ($n = 0$).

Application. The wave front set of the integration current on the quadratic cone $X = \{x_0^2 + \ldots + x_n^2 = s^2\}$, for n even, is equal to the closure of the conormal space to X^*. For n odd this wave front set contains $\{0\} \times \mathbb{C}^{n+2}$. This follows, for n even, from the fact that fibre-integrals with respect to $\xi_0 x_0 + \ldots + \xi_n x_n + \eta s$ are C^∞ if $\xi_0^2 + \ldots + \xi_n^2 \neq \eta^2$ because a linear change of coordinates leaving the cone fixed reduces to our situation. Same argument for n odd.

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