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Integrable system of the heat kernel associated with logarithmic potentials

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Abstract. The heat kernel of a Sturm–Liouville operator with logarithmic potential can be described by using the Wiener integral associated with a real hyperplane arrangement. The heat kernel satisfies an infinite-dimensional analog of the Gauss–Manin connection (integrable system), generalizing a variational formula of Schläfli for the volume of a simplex in the space of constant curvature.

1. Statement of the result. The classical variational formula for a geodesic simplex due to L. Schläfli plays an important role in geometry and analysis of spaces of constant curvature. The author has extended this formula to general analytic integrals of type (2.5) (theory of Gauss–Manin connection of irregular singularity). It has an invariant expression under the group of rotations $SO(n)$. This fact enables us to go straightforward to the analysis of infinite–dimensional function spaces in the framework of P. Lévy's book [11], for example (see [1] about its history since V. Volterra). Several approaches have been investigated in this direction, for example, Gauss ensembles of random matrices, white noise analysis etc. (see [5], [7], [11], [12]).

In this note, by the use of the Feynman–Kac formula ([10]), we show that the variational formula for Gauss type integrals associated with real hyperplane arrangements gives an integrable system for a system of functionals $F_0(\tau_1, \ldots, \tau_p; I)$ including the heat kernel with logarithmic potentials, by taking suitable infinite-dimensional limits.

We consider the heat equation on the real line

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - V(x)u \tag{1.1}$$

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for the logarithmic potential

$$V(x) = -\sum_{j=1}^{m} \lambda_j \log|x - y_j| \tag{1.2}$$

of a finite number of sources $y_1, \ldots, y_m \in \mathbb{C}$ with weights $\lambda_1, \ldots, \lambda_m$.

We assume λ_j are all positive.

The heat kernel $K(t, x, 0)$ satisfying (1.1) with the condition ($\delta(x)$ means the Dirac delta function)

$$\lim_{t\downarrow 0} K(t, x, 0) = \delta(x) \tag{1.3}$$

is given by the Feynman–Kac formula:

$$K(t, x, 0) = \lim_{N\to\infty} \frac{1}{(2\pi\Delta t)^{N/2}} \times \int_{\mathbb{R}^{N-1}} \exp\Big[-\frac{1}{2\Delta}\sum_{\nu=1}^{N}(x_\nu - x_{\nu-1})^2 - \Delta t \sum_{\nu=1}^{N-1} V(x_\nu)\Big]\, dx_1 \wedge \ldots \wedge dx_{N-1} \tag{1.4}$$

for $\Delta t = t/N$, where x_0, x_N denote $0, x$ respectively.

By the change of variables $\widehat{x}_\nu = x_\nu/\sqrt{t}$, $K(t, x, 0)$ can be written as

$$K(t, x, 0) = \lim_{N\to\infty} \frac{1}{(2\pi)^{N/2}\sqrt{t}} \int_{\mathbb{R}^N} \exp\Big[-\frac{\sqrt{N}}{2}\sum_{\nu=1}^{N}(\widehat{x}_\nu - \widehat{x}_{\nu-1})^2 - \Delta t \sum_{\nu=1}^{N-1} V(\sqrt{t}\,\widehat{x}_\nu)\Big] \times \delta(\widehat{x}_N - x/\sqrt{t})\, d\widehat{x}_1 \wedge \ldots \wedge d\widehat{x}_{N-1} \wedge d\widehat{x}_N \tag{1.5}$$

where the limit on the RHS is the average defined by the Wiener integral over the set of continuous paths $\widehat{x}(\tau) = \frac{1}{\sqrt{t}}x(t\tau), 0 \le \tau \le 1$.

Let us consider a finite-dimensional approximation to the integral (1.5).

We denote by $\log_\pm u$ the logarithmic functions

$$\log_+ u = \begin{cases} \log u & \text{for } u > 0, \\ 0 & \text{for } u \le 0. \end{cases}$$

$$\log_- u = \begin{cases} \log(-u) & \text{for } u < 0, \\ 0 & \text{for } u \ge 0. \end{cases}$$

First we define the positive function $\Phi_\mu(\xi)$, $\mu > -1$, for $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ as

$$\log \Phi_\mu(\xi) = -\frac{1}{2}Q(\xi) + \mu \log_+\Big(\frac{\xi_1 + \ldots + \xi_N}{\sqrt{N}} - x^*\Big) + \frac{t}{N}\sum_{j=1}^{m}\lambda_j \sum_{\nu=1}^{N} \log\Big|\frac{\xi_1 + \ldots + \xi_\nu}{\sqrt{N}} - y_j^*\Big| + \frac{1}{2}t\log t \cdot \sum_{j=1}^{m}\lambda_j \tag{1.6}$$

for $x^* = x/\sqrt{t}$, $y_j^* = y_j/\sqrt{t}$ respectively, where $Q(\xi)$ denotes the quadratic form

$$Q(\xi) = \xi_1^2 + \ldots + \xi_N^2.$$

Similarly we define the functions $\Phi_\mu(\xi; \varepsilon_1, \ldots, \varepsilon_m)$ by

$$\begin{aligned}(1.7)\qquad \log \Phi_\mu(\xi; \varepsilon_1, \ldots, \varepsilon_m) &= -\frac{1}{2}Q(\xi) + \mu \log_+ \left(\frac{\xi_1 + \ldots + \xi_N}{\sqrt{N}} - x^* \right) \\ &+ \frac{t}{N} \sum_{j=1}^m \lambda_j \sum_{\nu=1}^N \log_{\varepsilon_j} \left(\frac{\xi_1 + \ldots + \xi_\nu}{\sqrt{N}} - y_j^* \right) + \frac{1}{2} t \log t \cdot \sum_{j=1}^m \lambda_j\end{aligned}$$

where $\varepsilon_1, \ldots, \varepsilon_m$ denote $\pm$.

By the change of variables $\widehat{x}_\nu - \widehat{x}_{\nu-1} = \xi_\nu/\sqrt{N}$, (1.5) can be rewritten by using the average E concerning Wiener integrals as

$$\begin{aligned}(1.8)\qquad K(t,x,0) &= \lim_{\mu\to 0} \mu E\Big(\exp \Big[\sum_{j=1}^m \lambda_j t \int_0^1 \log |\widehat{x}(\tau) - y_j^*| \, d\tau \\ &\quad + (\mu - 1) \log_+ (\widehat{x}(1) - x^*) + \frac{1}{2} t \log t \sum_{j=1}^m \lambda_j \Big] \Big) \\ &= \lim_{N\to\infty} \frac{1}{(2\pi)^{N/2}\sqrt{t}} \lim_{\mu\to -1} (\mu + 1) \int_{\mathbb{R}^N} \Phi_\mu(\xi) \, d\xi_1 \wedge \ldots \wedge d\xi_N\end{aligned}$$

because the function $\lim_{\mu\to 0} \mu x_+^{\mu-1}$ tends to the δ function (a procedure à la S. Watanabe (see [8], [16])).

(1.8) can also be represented as

$$(1.9)\qquad K(t,x,0) = \lim_{N\to\infty} \frac{1}{(2\pi)^{N/2}\sqrt{t}} \lim_{\mu\to -1} (\mu + 1) \int_{\mathbb{R}^N} \widehat{\Phi}_\mu(\xi) \, d\xi_1 \wedge \ldots \wedge d\xi_N$$

where

$$\begin{aligned}(1.10)\qquad \log \widehat{\Phi}_\mu(\xi) = &- \frac{1}{2}Q(\xi) + \mu \log_+ \left(\frac{\xi_1 + \ldots + \xi_N}{\sqrt{N}} - x^* \right) \\ &+ \frac{t}{N} \sum_{j=1}^m \lambda_j \sum_{\nu=1}^N \log \left| \frac{\xi_1 + \ldots + \xi_\nu - \sqrt{N} y_j^*}{\sqrt{\nu}} \right| \\ &+ \frac{1}{2}(t \log t - t) \sum_{j=1}^m \lambda_j.\end{aligned}$$

In fact,

$$\log \Phi_\mu(\xi) = \log \widehat{\Phi}_\mu(\xi) + \Big(\sum_{j=1}^m \lambda_j \Big) \Big\{ \frac{t}{2} \log t + \frac{t}{2N} \Big(-N \log N + \sum_{j=2}^N \log j \Big) \Big\}.$$

The last term on the RHS tends to $\frac{1}{2}(t\log t - t)(\sum_{j=1}^{m}\lambda_j)$ as $N\to\infty$.

$\widehat{\Phi}_\mu(\xi;\varepsilon_1,\ldots,\varepsilon_m)$ are similarly defined from $\Phi_\mu(\xi;\varepsilon_1,\ldots,\varepsilon_m)$.

For given $\tau_1,\ldots,\tau_p$ such that $0<\tau_1<\ldots<\tau_p<1$, and a continuous function $\varphi(x_1,\ldots,x_p)$ on $\mathbb{R}^p$, we denote the average of $\varphi(\widehat{x}(\tau_1),\ldots,\widehat{x}(\tau_p))$ over the set of continuous paths by

$$\begin{aligned}(1.11)\quad &\langle\varphi(\widehat{x}(\tau_1),\ldots,\widehat{x}(\tau_p))\rangle\\ &= E\bigg(\varphi(\widehat{x}(\tau_1),\ldots,\widehat{x}(\tau_p))\\ &\quad\times\exp\bigg[\sum_{j=1}^{m}\lambda_j t\int_0^1\log|\widehat{x}(\tau)-y_j^*|\,d\tau+\mu\log_+(\widehat{x}(1)-x^*)+\frac{1}{2}t\log t\sum_{j=1}^{m}\lambda_j\bigg]\bigg)\end{aligned}$$

and its average with the restriction $\widehat{x}(1)=x^*$ by

$$\langle\varphi(\widehat{x}(\tau_1),\ldots,\widehat{x}(\tau_p))\rangle_0=\lim_{\mu\downarrow 0}\mu\bigg\langle\frac{\varphi(\widehat{x}(\tau_1),\ldots,\widehat{x}(\tau_p))}{\widehat{x}(1)-x^*}\bigg\rangle.$$

We are interested in finding a complete system of differential relations with respect to the parameters $x^*,y_1^*,\ldots,y_m^*$, when t is fixed.

Definition 1. For $\tau_1,\ldots,\tau_p\in(0,1)$ such that $0<\tau_1<\ldots<\tau_p<1$, and indices $I=\{i_1,\ldots,i_p\}\subset\{1,\ldots,m\}$, which may not be distinct, we define a system of functions depending not only on $\tau_1,\ldots,\tau_p$ but also on $x^*,y_1^*,\ldots,y_m^*$ by

$$(1.12)\qquad F(\tau_1,\ldots,\tau_p;I)=\bigg\langle\frac{1}{(\widehat{x}(\tau_1)-y_{i_1}^*)\ldots(\widehat{x}(\tau_p)-y_{i_p}^*)}\bigg\rangle,$$

$$(1.13)\qquad F_0(\tau_1,\ldots,\tau_p;I)=\bigg\langle\frac{1}{(\widehat{x}(\tau_1)-y_{i_1}^*)\ldots(\widehat{x}(\tau_p)-y_{i_p}^*)}\bigg\rangle_0.$$

In particular, for $p=0$ we have

$$F_0(\phi)=\langle 1\rangle_0=\sqrt{t}\,K(t,x,0).$$

It is convenient to define these functions to be zero if one of τ_h is negative or greater than 1.

Remark 1. More precisely, the functions $F(\tau_1,\ldots,\tau_p;I)$ and $F_0(\tau_1,\ldots,\tau_p;I)$ are generalized functions with respect to $\tau_1,\ldots,\tau_p$. Integration and differentiation can be done formally by the Malliavin calculus. For details see [8], [16].

Theorem 1. *The total differentials of $F(\tau_1,\ldots,\tau_p;I)$ with respect to the parameters $x^*,y_1^*,\ldots,y_m^*$ are given by*

(1.14) $$\delta F(\phi) = \sum_{j=1}^{m} \lambda_j t \Big\{ -\delta y_j^* \int_0^1 d\tau\, F(\tau; j) + \delta(x^*) \int_0^1 d\tau\, \tau F(\tau; j) \Big\} - \mu \delta(x^*) F(\phi),$$

i.e.,

(1.15) $$\frac{\partial F(\phi)}{\partial y_j^*} = -\lambda_j t \int_0^1 d\tau\, F(\tau; j),$$

(1.16) $$\frac{\partial F(\phi)}{\partial x^*} = \sum_{j=1}^{m} \lambda_j t \int_0^1 d\tau\, \tau F(\tau; j) - \mu F(\phi),$$

and for $p \geq 1$, *we have*

(1.17) $$\delta F(\tau_1, \ldots, \tau_p; i_1, \ldots, i_p) = A_I + A_{II} + A_{III}$$

where A_I, A_{II}, A_{III} *denote the differentials with respect to* $\delta x^*, \delta y_1^*, \ldots, \delta y_m^*$:

$$A_I = \sum_{h=1}^{p} \Big\{ \delta(y_{i_{h-1}}^*) \frac{1}{\tau_h - \tau_{h-1}} - \delta(y_{i_h}^*) \frac{\tau_{h+1} - \tau_{h-1}}{(\tau_{h+1} - \tau_h)(\tau_h - \tau_{h-1})} + \delta(y_{i_{h+1}}^*) \frac{1}{\tau_{h+1} - \tau_h} \Big\} F(\tau_1, \ldots, \tau_{h-1}, \tau_{h+1}, \ldots, \tau_p; \partial_h I),$$

$$A_{II} = \sum_{k=1,\, k \notin I}^{m} \lambda_k t \sum_{h=0}^{p} \Big\{ \delta(y_{i_h}^*) \int_{\tau_h}^{\tau_{h+1}} \frac{\tau_{h+1} - \tau}{\tau_{h+1} - \tau_h}\, d\tau - \delta(y_k^*) \int_{\tau_h}^{\tau_{h+1}} d\tau + \delta(y_{i_{h+1}}^*) \int_{\tau_h}^{\tau_{h+1}} \frac{\tau - \tau_h}{\tau_{h+1} - \tau_h}\, d\tau \Big\} F(\tau, \tau_1, \ldots, \tau_p; k, I) + \mu\{-\delta(x^*) + \delta(y_{i_p}^*)\} F_0(\tau_1, \ldots, \tau_p; I),$$

$$A_{III} = \Big\{ -\sum_{h=1}^{p} y_{i_h}^* \delta(y_{i_h}^*) \frac{\tau_{h+1} - \tau_{h-1}}{(\tau_h - \tau_{h-1})(\tau_{h+1} - \tau_h)} + \sum_{h=1}^{p-1} \delta(y_{i_h}^* y_{i_{h+1}}^*) \frac{1}{\tau_{h+1} - \tau_h} \Big\} F(\tau_1, \ldots, \tau_p; I),$$

where we put $\tau_0 = 0$, $\tau_{p+1} = +\infty$, *and* $y_{i_0}^* = y_{i_{p+1}}^* = 0$.

THEOREM 2. *The total differentials of* $F_0(\tau_1, \ldots, \tau_p; I)$ *with respect to the parameters* $x^*, y_1^*, \ldots, y_m^*$ *are given by*

(1.18) $$\delta F_0(\phi) = \sum_{j=1}^{m} \lambda_j t \Big\{ -\delta y_j^* \int_0^1 d\tau\, F_0(\tau; j) + \delta(x^*) \int_0^1 d\tau\, \tau F_0(\tau; j) \Big\} - x^* \delta(x^*) F_0(\phi),$$

i.e.,

$$\frac{\partial F_0(\phi)}{\partial y_j^*} = -\lambda_j t \int_0^1 d\tau\, F_0(\tau; j), \tag{1.19}$$

$$\frac{\partial F_0(\phi)}{\partial x^*} = \sum_{j=1}^m \lambda_j t \int_0^1 d\tau\, \tau F_0(\tau; j) - x^* F_0(\phi), \tag{1.20}$$

and for $p \geq 1$, *we have*

$$\delta F_0(\tau_1, \ldots, \tau_p; i_1, \ldots, i_p) = B_I + B_{II} + B_{III} \tag{1.21}$$

where B_I, B_{II}, B_{III} *denote the differentials with respect to* $\delta x^*, \delta y_1^*, \ldots, \delta y_m^*$:

$$B_I = \sum_{h=1}^p \bigg\{ \delta(y_{i_{h-1}}^*) \frac{1}{\tau_h - \tau_{h-1}} - \delta(y_{i_h}^*) \frac{\tau_{h+1} - \tau_{h-1}}{(\tau_{h+1} - \tau_h)(\tau_h - \tau_{h-1})} + \delta(y_{i_{h+1}}^*) \frac{1}{\tau_{h+1} - \tau_h} \bigg\} F_0(\tau_1, \ldots, \tau_{h-1}, \tau_{h+1}, \ldots, \tau_p; \partial_h I),$$

$$B_{II} = \sum_{k=1}^m \lambda_k t \sum_{h=0}^p \bigg\{ \delta(y_{i_h}^*) \int_{\tau_h}^{\tau_{h+1}} \frac{\tau_{h+1} - \tau}{\tau_{h+1} - \tau_h}\, d\tau - \delta(y_k^*) \int_{\tau_h}^{\tau_{h+1}} d\tau + \delta(y_{i_{h+1}}^*) \int_{\tau_h}^{\tau_{h+1}} \frac{\tau - \tau_h}{\tau_{h+1} - \tau_h}\, d\tau \bigg\} F_0(\tau, \tau_1, \ldots, \tau_p; k, I),$$

$$B_{III} = \bigg\{ -\sum_{h=1}^p y_{i_h}^* \delta(y_{i_h}^*) \frac{\tau_{h+1} - \tau_{h-1}}{(\tau_h - \tau_{h-1})(\tau_{h+1} - \tau_h)} - x^* \delta(x^*) \frac{1}{1 - \tau_p} + \sum_{h=1}^{p-1} \delta(y_{i_h}^* y_{i_{h+1}}^*) \frac{1}{\tau_{h+1} - \tau_h} + \delta(y_{i_p}^* x^*) \frac{1}{1 - \tau_p} \bigg\} F_0(\tau_1, \ldots, \tau_p; I),$$

where we put $\tau_0 = 0$, $\tau_{p+1} = 1$ *and* $y_{i_0}^* = 0$, $y_{i_{p+1}}^* = x^*$.

Theorem 3. *The derivative of* $F_0(\tau_1, \ldots, \tau_p; I)$ *with respect to* t *is expressed as*

$$\frac{\partial}{\partial t} F_0(\tau_1, \ldots, \tau_p; I) = -V(x) F_0(\tau_1, \ldots, \tau_p; I) - \frac{1}{2} \sum_{j=1}^m \lambda_j y_j^* \int_0^1 d\tau\, F_0(\tau, \tau_1, \ldots, \tau_p; j, I). \tag{1.22}$$

Theorems 1–3 show that $\{F_0(\tau_1, \ldots, \tau_p; I)\}_{p \geq 0}$ satisfy an integrable system with respect to the variables $t, x^*, y_1^*, \ldots, y_m^*$ on the space of continuous paths $\widehat{x} : [0,1] \to \mathbb{R}$, while $\{F(\tau_1, \ldots, \tau_p; I)\}_{p \geq 0}$ satisfy an integrable system with respect to the variables $x^*, y_1^*, \ldots, y_m^*$ only.

2. Arrangement of hyperplanes and a generalized Schläfli formula. In the n-dimensional Euclidean space $\mathbb{R}^n$ we consider an arrangement $\mathcal{A}$ of m real hyperplanes H_j $(1 \le j \le m)$ defined by the inhomogeneous linear equations

$$H_j: \quad f_j(x) = 0 \tag{2.1}$$

for $f_j(x) = \sum_{j=1}^n u_{j,\nu} x_\nu + u_{j,0}$. The functions $f_j(x)$ are assumed to be normalized by $\sum_{\nu=1}^n u_{j,\nu}^2 = 1$.

The *configuration matrix* A associated with the arrangement is defined as the symmetric matrix $A = (a_{j,k})_{j,k=0}^m$ of order $m+1$, where $a_{j,k}$ denotes the inner product between the coefficients of f_j, f_k:

$$a_{j,k} = \sum_{\nu=1}^n u_{j,\nu} u_{k,\nu}, \qquad m \ge j,\ k \ge 1, \tag{2.2}$$

$$a_{j,0} = a_{0,j} = u_{j,0}, \qquad m \ge j \ge 1 \text{ and } a_{0,0} = 1. \tag{2.3}$$

Note that $a_{j,j} = 1$.

For $I = \{i_1, \ldots, i_p\},\ 0 \le i_1 < \ldots < i_p \le m$, and $J = \{j_1, \ldots, j_p\},\ 0 \le j_1 < \ldots < j_p \le m$, we denote by $A\binom{I}{J}$ the subdeterminant $\det((a_{i,j})_{i \in I, j \in J})$, in particular we write $A(I)$ in the case where $I = J$. The arrangement $\mathcal{A}$ is uniquely determined by the matrix A up to isomorphism of the n-dimensional orthogonal group $O(n)$.

Let $\lambda_1, \ldots, \lambda_m$ be real numbers and $\Phi(x)$ be the analytic function

$$\Phi(x) = \exp\bigl(-\tfrac{1}{2} Q(x)\bigr) f_1(x)^{\lambda_1} \ldots f_m(x)^{\lambda_m} \tag{2.4}$$

for $Q(x) = \sum_{\nu=1}^n x_{\nu=1}^2$. We consider the integral

$$F = \int_\Delta \Phi(x)\, dx_1 \wedge \ldots \wedge dx_n \tag{2.5}$$

over a twisted cycle Δ associated with the function $\Phi(x)$.

We also consider the system of integrals

$$F(I) = \int_\Delta \Phi(x) \frac{dx_1 \wedge \ldots \wedge dx_n}{f_{i_1} \ldots f_{i_p}}. \tag{2.6}$$

It has been proved in [3] that the functions $\{F(\phi)$ and $F(I)$, $1 \le p \le n\}$ form a complete system of integrals in the n-dimensional twisted de Rham cohomology which has dimension $\sum_{j=0}^n \binom{m}{j}$.

Moreover the following variational formula holds (see Proposition 1.3 in [3], Part I).

Proposition 1 (Generalized Schläfli formula).

$$\delta F(\phi) = \sum_{j=1}^m \lambda_j \delta a_{j,0} F(j) + \frac{1}{2} \sum_{j,k=1,\, j \ne k}^m \lambda_j \lambda_k \delta a_{j,k} F(j,k) \tag{2.7}$$

where δ denote differentials of variation.

This formula is just a generalization of the classical Schläfli formula which is a variational formula for the volume of a geodesic simplex in a space of positive constant curvature (see [2], [11], [15] etc.). In fact, to obtain Schläfli's formula from (2.7), we take as Δ the simplicial cone defined by $f_1 \geq 0, \ldots, f_n \geq 0$ and we take the limit $\lambda_j \to 0$, for all j.

We can also get the variational formulae for the function $F(I)$. However these are rather complicated and we do not reproduce them here.

Later on we only need the formulae in the cases where $a_{j,k}$ $(1 \leq j, k \leq m)$ are constants. They are described as follows (see [3], Proposition 3 in Part II, or [5], Lemma 2).

The symbols $\{k, I\}$ and $\partial_h I$ will denote the sets of indices $\{k, i_1, \ldots, i_p\}$ (addition of the index k to I) and $\{i_1, \ldots, i_{h-1}, i_{h+1}, \ldots, i_p\}$ (deletion of the hth index from I) respectively.

Proposition 2. *Assume that $a_{j,k}$ $(1 \leq j, k \leq m)$ are constants. Then*

$$\delta F(\phi) = \sum_{j=1}^{m} \lambda_j \delta a_{j,0} \cdot F(j), \tag{2.8}$$

and for $p \geq 1$, we have

$$\begin{aligned} A(I) \cdot \delta F(I) = & -\sum_{h=1}^{p} (-1)^h \delta A\begin{pmatrix} I \\ 0, \partial_h I \end{pmatrix} \cdot F(\partial_h I) \\ & + \sum_{k \notin I, k \geq 1} \lambda_k \delta A\begin{pmatrix} k, I \\ 0, I \end{pmatrix} \cdot F(k, I) + \frac{1}{2} \delta A(0, I) \cdot F(I). \end{aligned} \tag{2.9}$$

Similarly we have the recurrence relations.

Proposition 3. *Let $T_k^{\pm}$ denote the shift operators corresponding to the shift $\lambda_k \mapsto \lambda_k \pm 1$. Then*

$$T_k^- F(I) = F(k, I), \qquad k \notin I, \tag{2.10}$$

$$\begin{aligned} (\lambda_{i_1} - 1) A(I) \cdot T_{i_1}^- F(I) & \\ = & \sum_{h=1}^{p} A\begin{pmatrix} \partial_1 I \\ \partial_h I \end{pmatrix} (-1)^{h+1} \cdot F(\partial_h I) \\ & - \sum_{k \notin I} \lambda_k A\begin{pmatrix} I \\ k, \partial_1 I \end{pmatrix} \cdot F(k, I) - A\begin{pmatrix} I \\ 0, \partial_1 I \end{pmatrix} \cdot F(I). \end{aligned} \tag{2.11}$$

3. Application of the generalized Schläfli formula. We denote the normalized inhomogeneous functions appearing in (1.6) as follows:

$$f_{j,\nu}(\xi) = \frac{\xi_1 + \ldots + \xi_\nu - \sqrt{N} y_j^*}{\sqrt{\nu}}, \qquad \nu = 1, \ldots, N,$$

and

$$f_{m+1,N}(\xi) = \frac{\xi_1 + \ldots + \xi_N - \sqrt{N}x^*}{\sqrt{N}}.$$

When $j = m+1$, ν takes only the value N. In the sequel y^*_{m+1} and λ_{m+1} are identified with x^* and μ respectively.

The inner product of the coefficients of $f_{j,\nu}$ and $f_{k,\sigma}$ is given by

$$a_{j,k} = (f_{j,\nu}, f_{k,\sigma}) = \begin{cases} \nu/\sqrt{\nu\sigma} & \text{for } \nu \le \sigma, \\ \sigma/\sqrt{\nu\sigma} & \text{for } \nu \ge \sigma. \end{cases}$$

By abuse of notation, we may denote it by $a_{j,k}$ without ambiguity, since below only one function $f_{j,\nu}$ corresponds to each index j. We may also denote

$$a_{j,0} = a_{0,j} = -\sqrt{N}y^*_j/\sqrt{\nu}$$

corresponding to $f_{j,\nu}$.

The function $\widehat{\Phi}_\mu(\xi)$ can be represented as

$$\begin{aligned}(3.1)\quad & \widehat{\Phi}_\mu(\xi) \\ &= \exp\Big[-\frac{1}{2}Q(\xi) + \frac{1}{2}(t\log t - t)\sum_{j=1}^{m}\lambda_j\Big] \cdot \Big\{\prod_{j=1}^{m}\prod_{\nu=1}^{N} f_{j,\nu}(\xi)^{\lambda_j t/N}\Big\} f_{m+1,N}(\xi)^\mu_+.\end{aligned}$$

Then the following lemmas can be proved by a direct computation. It is a remarkable fact that every subdeterminant is non-negative.

LEMMA 1. *Let p pairs of indices $I = \{(i_1,\nu_1),\ldots,(i_p,\nu_p)\}$, $\{i_1,\ldots,i_p\} \subset \{1,\ldots,m+1\}$ and $0 < \nu_1 < \ldots < \nu_p \le N+1$, be given (we assume $\nu_p = N+1$ if $i_p = m+1$). Then*

$$A(I) = \frac{(\nu_2 - \nu_1)\ldots(\nu_p - \nu_{p-1})}{\nu_2\ldots\nu_p},$$

and for $1 \le k,\ h \le p$,

$$A\binom{\partial_k I}{\partial_h I}/A(I) = \begin{cases} 0, & |h-k| > 1, \\ \dfrac{\sqrt{\nu_h\nu_{h+1}}}{\nu_{h+1}-\nu_h}, & k = h+1, \\ \dfrac{\sqrt{\nu_h\nu_{h-1}}}{\nu_h - \nu_{h-1}}, & k = h-1, \\ \dfrac{\nu_h(\nu_{h+1}-\nu_{h-1})}{(\nu_{h+1}-\nu_h)(\nu_h-\nu_{h-1})}, & k = h. \end{cases}$$

LEMMA 2.

$$(3.3)\quad A(0,I)/A(I) = 1 - \sum_{h=1}^{p}\frac{N(y^*_{i_h})^2(\nu_{h+1}-\nu_{h-1})}{(\nu_{h+1}-\nu_h)(\nu_h-\nu_{h-1})} + 2\sum_{h=1}^{p-1}\frac{Ny^*_{i_h}y^*_{i_{h+1}}}{\nu_{h+1}-\nu_h}.$$

Lemma 3. *For* $1 \le h \le p$,

$$A\begin{pmatrix} I \\ 0, \partial_h I\end{pmatrix}/A(I) = (-1)^{h-1} y^*_{i_{h-1}} \frac{\sqrt{N\nu_h}}{\nu_h - \nu_{h-1}} + (-1)^h y^*_{i_h} \frac{\sqrt{N\nu_h}(\nu_{h+1}-\nu_{h-1})}{(\nu_{h+1}-\nu_h)(\nu_h-\nu_{h-1})} + (-1)^{h+1} y^*_{i_{h+1}} \frac{\sqrt{N\nu_h}}{\nu_{h+1}-\nu_h} \tag{3.4}$$

where ν_0 *and* ν_{p+1} *are set to be* 0 *and* $+\infty$ *respectively.*

Lemma 4. *Assume that the index* k *corresponds to the function* $f_{k,\sigma}$. *Then*

$$A\begin{pmatrix} k, I \\ 0, I\end{pmatrix}/A(I) = y^*_{i_h} \frac{\sqrt{N}(\nu_{h+1}-\sigma)}{\sqrt{\sigma}(\nu_{h+1}-\nu_h)} - y^*_k \frac{\sqrt{N}}{\sqrt{\sigma}} + y^*_{i_{h+1}} \frac{\sqrt{N}(\sigma-\nu_h)}{\sqrt{\sigma}(\nu_{h+1}-\nu_h)} \tag{3.5}$$

if $\nu_h \le \sigma \le \nu_{h+1}$.

We can now apply the formulae (2.8), (2.9) to (1.8). We denote by $\varphi(I)$ ($I \subset \{1,2,\ldots,m+1\}$) the following integrals:

$$\varphi(I) = \int_{\mathbb{R}^N} \widehat{\Phi}_\mu(\xi) \frac{1}{f_{i_1,\nu_1}(\xi) f_{i_2,\nu_2}(\xi)\ldots f_{i_p,\nu_p}(\xi)}\, d\xi_1 \wedge \ldots \wedge d\xi_N. \tag{3.6}$$

From Lemmas 1–4 and (2.8), (2.9), we deduce the following formulae.

Proposition 4.

$$\delta\varphi(I) = X_I + X_{II} + X_{III} \tag{3.7}$$

where

$$X_I = \sum_{h=1}^{p} \Big\{ \delta(y^*_{i_{h-1}}) \frac{\sqrt{N\nu_h}}{\nu_h-\nu_{h-1}} - \delta(y^*_{i_h}) \frac{\sqrt{N\nu_h}(\nu_{h+1}-\nu_{h-1})}{(\nu_{h+1}-\nu_h)(\nu_h-\nu_{h-1})} + \delta(y^*_{i_{h+1}}) \frac{\sqrt{N\nu_h}}{\nu_{h+1}-\nu_h} \Big\} \varphi(\partial_h I), \tag{3.8}$$

$$X_{II} = \frac{1}{N} \sum_{k \notin I,\, 1 \le k \le m+1} \lambda_k \sum_{h=0}^{p} \sum_{\nu_h < \sigma < \nu_{h+1}} \Big\{ \delta(y^*_{i_h}) \frac{\sqrt{N}(\nu_{h+1}-\sigma)}{\sqrt{\sigma}(\nu_{h+1}-\nu_h)} - \delta(y^*_k)\sqrt{\frac{N}{\sigma}} + \delta(y^*_{i_{h+1}}) \frac{\sqrt{N}(\sigma-\nu_h)}{\sqrt{\sigma}(\nu_{h+1}-\nu_h)} \Big\} \varphi(k, I) + \frac{1}{N} \sum_{k=1}^{m} \lambda_k \sum_{h=1,\, i_h \ne k}^{p} \delta \log(y^*_{i_h} - y^*_k)(\varphi(I) - \varphi(k, \partial_h I)), \tag{3.9}$$

$$X_{III} = -\sum_{h=1}^{p} y^*_{i_h}\delta(y^*_{i_h})\frac{N(\nu_{h+1}-\nu_{h-1})}{(\nu_{h+1}-\nu_h)(\nu_h-\nu_{h-1})} + \sum_{h=1}^{p-1}\delta(y^*_{i_h}y^*_{i_{h+1}})\frac{N}{\nu_{h+1}-\nu_h}\varphi(I) \tag{3.10}$$

where we put $\nu_0 = 0$, $\nu_{p+1} = +\infty$.

Proof of Theorems 1 and 2. Let us take the limit $N\to\infty$ of (3.7). When ν/N tends to the value τ, the function $(\xi_1+\ldots+\xi_\nu)/\sqrt{N}$ tends almost surely to the value $\widehat{x}(\tau)$ at τ of a continuous path $\widehat{x}$.

This implies that if $\nu_1/N\to\tau_1,\ldots,\nu_p/N\to\tau_p$, then $\varphi(I)\frac{N^{p/2}}{(2\pi)^{N/2}\sqrt{\nu_1\ldots\nu_p}}$ in Proposition 4 ($I\subset\{1,\ldots,m\}$) tends to $F(\tau_1,\ldots,\tau_p;I)$ defined in Definition 1, i.e.,

$$\lim_{N\to\infty}\varphi(I)\frac{N^{p/2}}{(2\pi)^{N/2}\sqrt{\nu_1\ldots\nu_p}} = F(\tau_1,\ldots,\tau_p;I). \tag{3.11}$$

In the same way, we have

$$\lim_{N\to\infty}\lim_{\mu\downarrow 0}\varphi(I,m+1)\frac{N^{p/2}}{(2\pi)^{N/2}\sqrt{\nu_1\ldots\nu_p}} = F_0(\tau_1,\ldots,\tau_p;I). \tag{3.12}$$

We multiply both sides of (3.7) by $\frac{N^{p/2}}{(2\pi)^{N/2}\sqrt{\nu_1\ldots\nu_p}}$ and take the limit $N\to\infty$. Then the sum $\frac{1}{N}\sum_{\nu_h<\sigma<\nu_{h+1}}$ tends to the integral $\int_{\tau_h}^{\tau_{h+1}}d\tau$, whence (3.7) for $\varphi(I)$ and $\varphi(I,m+1)$ tend to the equations (1.13), (1.16) and (1.17), (1.20) respectively.

On the other hand the last term on the RHS of (3.9) tends to zero. Theorems 1 and 2 have thus been proved. ∎

Proof of Theorem 3. First note the following equality. Since $x(t\tau) = \sqrt{t}\,\widehat{x}(\tau)$, $y_j = \sqrt{t}\,y^*_j$ we have

$$\log|x(t)-y_j| = \int_0^1 \log|\widehat{x}(\tau)-y^*_j|\,d\tau + \frac{1}{2}(1+\log t) + \frac{1}{2}y^*_j\int_0^1\frac{d\tau}{\widehat{x}(\tau)-y^*_j}.$$

In fact as generalized Wiener functionals (Malliavin calculus), we have

$$\begin{aligned}\log|x(t)-y_j| &= \frac{d}{dt}\int_0^t \log|x(s)-y_j|\,ds\\ &= \frac{d}{dt}\Big\{t\int_0^1\log|\widehat{x}(\tau)-\frac{y_j}{\sqrt{t}}|\,d\tau + \frac{1}{2}t\log t\Big\},\end{aligned}$$

which gives the above equality by the Leibniz rule. See Remark 1.

This implies

$$\sum_{j=1}^{m} \lambda_j \int_0^1 \log|\widehat{x}(\tau) - y_j^*|\, d\tau + \frac{1}{2}(\log t + 1) \sum_{j=1}^{m} \lambda_j$$
$$= \sum_{j=1}^{m} \lambda_j \log|x(t) - y_j| - \frac{1}{2}\lambda_j y_j^* \int_0^1 \frac{d\tau}{\widehat{x}(\tau) - y_j^*}.$$

We use the identity $V(x) = -\sum_{j=1}^{m} \lambda_j \log|x(t) - y_j|$ and get the equality (1.22). ■

Assume now that *the indices in I are all distinct*. One can prove that the formulae in Theorems 1, 2 are still valid when we consider the functions $\widehat{\Phi}_\mu(\xi; \varepsilon_1, \ldots, \varepsilon_m)$ in place of $\widehat{\Phi}_\mu(\xi)$ (the latter is equal to the sum over all $\varepsilon_1, \ldots, \varepsilon_m$.)

Let us use the same notations $F(\tau_1, \ldots, \tau_p; I)$, $F_0(\tau_1, \ldots, \tau_p; I)$ as in Theorems 1 and 2 in the case of $\widehat{\Phi}_\mu(\xi; \varepsilon_1, \ldots, \varepsilon_m)$.

Let $T^-_{\lambda_k t}$ denote the shift operators corresponding to the shifts $\lambda_k t \mapsto \lambda_k t - 1$. Relations between the partial differentiation $\partial/\partial y_k^*$ and the shift operators $T^-_{\lambda_k t}$ (contiguous relations) are given as follows.

When $k \notin I$,

$$\text{(3.13)} \qquad \frac{\partial}{\partial y_k^*} F(\tau_1, \ldots, \tau_p; I) = -\lambda_k t \int_0^1 d\tau F(\tau, \tau_1, \ldots, \tau_p; k, I)$$
$$= -\varepsilon_k \lambda_k t^{3/2} T^-_{\lambda_k t} F(\tau_1, \ldots, \tau_p; I),$$

while if $k \in I$, then

$$\text{(3.14)} \qquad \frac{\partial}{\partial y_{i_h}^*} F(\tau_1, \ldots, \tau_p; I)$$
$$= -(\lambda_{i_h} t - 1) \left\langle \frac{1}{\widehat{x}(\tau_h - y_{i_h}^*) \widehat{x}(\tau_1 - y_{i_1}^*) \ldots \widehat{x}(\tau_p - y_{i_p}^*)} \right\rangle$$
$$- \lambda_{i_h} t \int_0^1 F(\tau, \tau_1, \ldots, \tau_p; i_h, I)$$
$$= -(\lambda_{i_h} t - 1) \left\langle \frac{1}{\widehat{x}(\tau_h - y_{i_h}^*) \widehat{x}(\tau_1 - y_{i_1}^*) \ldots \widehat{x}(\tau_p - y_{i_p}^*)} \right\rangle$$
$$- \varepsilon_{i_h} \lambda_{i_h} t^{3/2} T^-_{\lambda_{i_h} t} F(\tau_1, \ldots, \tau_p; I).$$

The same relations are also valid for $F_0(\tau_1, \ldots, \tau_p; I)$.

Remark 2. The formulae of Gauss–Manin connections for (2.5) including (2.7) which have been obtained in [3] seem to have another application.

It may be possible to extend the arguments discussed in this note to the Schrödinger operators of the one-dimensional many body system

$$-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}+V(x_1,\dots,x_n)$$

where $V(x_1,\dots,x_n)$ denotes the potential

$$V(x_1,\dots,x_n)=-\sum_{1\le j<k\le n}\lambda_{j,k}\log|x_j-x_k|-\sum_{1\le j\le n}\sum_{1\le k\le m}\mu_{j,k}\log|x_j-y_k|.$$

More generally in the complex domain $\mathbb{C}$, one can consider the operators (each $z_j=x_j+iy_j, w_j\in\mathbb{C}$)

$$-\frac{1}{2}\sum_{j=1}^{n}\left(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2}\right)+V(z_1,\dots,z_n)$$

where

$$V(z_1,\dots,z_n)=-\sum_{1\le j<k\le n}\lambda_{j,k}\log|z_j-z_k|-\sum_{1\le j\le n}\sum_{1\le k\le m}\mu_{j,k}\log|z_j-w_k|.$$

One may possibly obtain similar results to Theorems 1–3, although the formulae would be more complicated (see [4] for a similar argument).

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