

Mellin analysis of partial differential equations in papers of B. Ziemian

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Abstract. Existence and regularity theorems for Fuchsian type differential operators and the theory of second microlocalization are presented.

1. Introduction. The study of solutions to singular elliptic partial differential equations is quite different from that of solutions to elliptic equations. In the latter case solutions are completely controlled by the behaviour of their Fourier transforms at infinity (first wave front set). In the case of corner equations the Fourier transform is replaced by the multidimensional Mellin transform. Using special spaces of distributions (subspaces of Mellin transformable distributions) one can measure local regularity of solutions to such equations.

Let $a, b \in \mathbb{R}^n$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$. Then $a < b$ denotes $a_j < b_j$ for $j = 1, \dots, n$. Let $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x > 0\}$, $I = (0, \mathbf{r}] = \{x \in \mathbb{R}_+^n : x \leq \mathbf{r}\}$, $\mathbf{r} = (r, \dots, r) \in \mathbb{R}_+^n$, $\mathbf{r} < \mathbf{1}$, $\mathbf{1} = (1, \dots, 1)$. We write

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) = (D_1, \dots, D_n), \quad x \frac{\partial}{\partial x} = \left(x_1 \frac{\partial}{\partial x_1}, \dots, x_n \frac{\partial}{\partial x_n} \right).$$

For $y \in \mathbb{R}^n$ we define $\mu(y) = (e^{-y_1}, \dots, e^{-y_n})$.

Let $a \in \mathbb{R}^n$. We denote by $M_a = M_a(I)$ the complex vector space of functions $\varphi \in C^\infty(I)$ such that for $\nu \in \mathbb{N}_0^n$,

$$\varrho_{a,\nu}(\varphi) = \sup_{x \in I} |x^{a+\nu+1} D^\nu \varphi(x)| < \infty$$

with the topology given by the seminorms $\varrho_{a,\nu}$. The space $M_{(\omega)} = M_{(\omega)}(I)$ for $\omega \in (\mathbb{R} \cup \{\infty\})^n$ is the inductive limit

$$M_{(\omega)} = \varinjlim_{a < \omega} M_a$$

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and $M'_{(\omega)}$ is the dual of $M_{(\omega)}$. The space $M' = \bigcup_{\omega} M'_{(\omega)} \subset \mathcal{D}'(\mathbb{R}_+^n)$ is called the space of *Mellin (transformable) distributions on I* . If $u \in M'_{(\omega)}$ then the *Mellin transform* of u is defined by

$$\mathcal{M}u(z) = u[x^{-z-1}] \quad \text{for } z \in \mathbb{C}^n, \operatorname{Re} z < \omega.$$

Since $\mathcal{M}(x \frac{\partial}{\partial x} u)(z) = z \mathcal{M}u(z)$ the Mellin transformation is well suited for studying Fuchsian type differential operators of the form $R = R(x, x\partial/\partial x)$ where $R(x, z)$ is a polynomial of degree m in $z \in \mathbb{C}^n$. The space of *Mellin distributions on \mathbb{R}_+^n* for every $\alpha \in \mathbb{R}^n$ is defined as the dual \mathfrak{M}'_{α} of the space

$$\mathfrak{M}_{\alpha} = \mathfrak{M}_{\alpha}(\mathbb{R}_+^n) = \{\sigma \in C^{\infty}(\mathbb{R}_+^n) : (x^{\alpha+1}\sigma) \circ \mu \in S(\mathbb{R}^n)\}$$

with the topology induced from the space $S(\mathbb{R}^n)$. The *Fourier–Mellin transform* of $u \in \mathfrak{M}'_{\alpha}$ is the distribution

$$\mathcal{M}_{\alpha}u = F^{-1}(e^{\alpha s}(u \circ \mu)) \in S'(\mathbb{R}^n)$$

where the inverse Fourier transform F^{-1} is defined as

$$F^{-1}\psi(\xi) = \int_{\mathbb{R}^n} e^{is\xi} \psi(s) ds \quad \text{for } \psi \in S(\mathbb{R}^n).$$

2. Local existence and regularity theorems. Let R be a second order partial differential operator on an open set $0 \in U \subset \mathbb{R}^2$,

$$R = \sum_{|\nu| \leq 2} a_{\nu}(x) \left(x_1 \frac{\partial}{\partial x_1} \right)^{\nu_1} \left(x_2 \frac{\partial}{\partial x_2} \right)^{\nu_2}$$

where a_{ν} are smooth functions on U and

- 1) R has a *regular singularity* at zero: $a_{20}(0) \cdot a_{02}(0) \neq 0$, $a_{\nu}(0) = 0$ otherwise,
- 2) R is *degenerate elliptic* in U : $a_{20}(x) \cdot a_{02}(x) > 0$ for $x \in U$.

THEOREM 1 (Z. Szmydt and B. Ziemian [2]). *Let $w \in \mathcal{D}'(U)$ be of order (s_1, s_2) about zero (i.e. $w = D_1^{s_1} D_2^{s_2} f$, f continuous on U). Then for every proper subcone $\Gamma \subset \mathbb{R}_+^2$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that $\alpha_1 + \alpha_2 < -s_1 - s_2$ there exists a neighbourhood V of zero in \mathbb{R}^2 and a Mellin distribution $u \in \mathfrak{M}'_{\alpha}$ such that*

$$Ru = w \quad \text{on } \Gamma \cap V.$$

In [3], the cone Γ is replaced by a set $\Gamma_{\varrho} \subset \mathbb{R}_+^2$, $\varrho = (\varrho_1, \varrho_2) \in \mathbb{R}_+^2$, $\varrho_1 < 1 < \varrho_2$, which is tangent to $\{x_1 = 0\}$ of order $1/\varrho_1 - 1$ and to $\{x_2 = 0\}$ of order $\varrho_2 - 1$ (explicitly $\Gamma_{\varrho} = \{(x_1, x_2) : x_1 > 0, x_1^{\varrho_2} < x_2 < x_1^{\varrho_1}\}$), and R is a differential operator as in Theorem 1.

THEOREM 2 (Z. Szmydt and B. Ziemian [3]). *Let $w \in \mathcal{D}'(U)$ be of order (s_1, s_2) about zero. Then for every $\Gamma_{\varrho} \subset \mathbb{R}_+^2$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ such*

that

$$\alpha_1 + \varrho_2 \alpha_2 < -s_1 - \varrho_1 s_2, \quad \alpha_1 + \varrho_2 \alpha_2 < -s_1 - \varrho_2 s_2$$

there exists a neighbourhood V of zero in \mathbb{R}^2 and a Mellin distribution $u \in \mathfrak{M}'_\alpha$ such that

$$Ru = w \quad \text{on } \Gamma_\varrho \cap V.$$

A continuation of [2] and [3] is [5].

Denote by $M'_{(\omega)}{}^s$ the subspace of $u \in M'_{(\omega)}$ such that $\mathcal{M}u(z)$ is $O(\prod_{j=1}^n (1 + |\operatorname{Im} z_j|)^s)$ as $|\operatorname{Im} z| \rightarrow +\infty$ locally uniformly in $\{\operatorname{Re} z = x < \omega\}$. Let $\Omega \subset \mathbb{R}_+^{n-1}$ be an arbitrary open bounded set and Γ_K ($K = \overline{\Omega}$) be the “curved cone”: $\Gamma_K = \psi(\mathbb{R}_+^1 \times K)$ where

$$\psi : \mathbb{R}_+^1 \times K \ni (t, \varrho) \mapsto x = (t, t^{\varrho_1}, \dots, t^{\varrho_{n-1}}) \in \mathbb{R}_+^n.$$

Let $u \in M'$. We say that $u \in M'_{(\omega)}{}^s$ Γ_K -locally at zero if there exists a cut-off function κ subordinate to Γ_K such that $\kappa u \in M'_{(\omega)}{}^s$. Let $R = R(x, x\partial/\partial x)$ where $R(x, z)$ is a polynomial of degree m in z with smooth coefficients on U , $P(z) = R(0, z)$ (the principal Mellin symbol of R at zero). By assumption

$$R = P\left(x \frac{\partial}{\partial x}\right) - Q\left(x, x \frac{\partial}{\partial x}\right),$$

$$Q\left(x, x \frac{\partial}{\partial x}\right) = x_1 Q_1\left(x, x \frac{\partial}{\partial x}\right) + \dots + x_n Q_n\left(x, x \frac{\partial}{\partial x}\right),$$

where Q_1, \dots, Q_n are differential operators of order m and for every $\alpha \in \mathbb{R}^n$ there exist A and c_0 such that (ellipticity condition)

$$|P(\alpha + i\beta)| > c_0 \prod_{j=1}^n (1 + |\beta_j|)^m \quad \text{for } |\beta| \geq A.$$

For $\alpha \in \mathbb{R}^n$ define $l(\alpha, \varrho) = \alpha_1 + \varrho_1 \alpha_2 + \dots + \varrho_{n-1} \alpha_n$ for $\varrho \in K$.

THEOREM 3 (Z. Szmydt and B. Ziemian [4], [5]). *Suppose $w \in M'_{(\omega)}{}^s$, where $\omega \in \mathbb{R}^n$, $s \in \mathbb{R}$ and $\max_{\varrho \in K} l(\alpha - \omega, \varrho) < 0$. Then there exist $u \in M'_{(\alpha)}{}^{s-m}$ Γ_K -locally and an open neighbourhood W of zero in \mathbb{R}^n such that*

$$Ru = w \quad \text{on } \Gamma_\Omega \cap W.$$

Regularity of solutions to degenerate elliptic operators is also studied in [6] and [7]. Such operators were studied e.g. by M. Kashiwara and R. Melrose. These operators are 2-microelliptic in direction $\delta x \in \mathbb{R}_+^n$ (cf. [8]).

Let Ω^1 be an open subset of \mathbb{C} such that together with every $\zeta \in \Omega^1$ it also contains the half-line $\zeta + \mathbb{R}_-$ (Ω^1 is an \mathbb{R}_- -connected open subset of \mathbb{C}) such that for every $\zeta \in \Omega^1$ the set of points not in Ω^1 whose real part is less

than $\operatorname{Re} \zeta$ is compact. In applications $\Omega^1 = \mathbb{C} \setminus \bigcup_{j=1}^k L_j$ where $L_j = \xi_j + \mathbb{R}_+$ for some ξ_1, \dots, ξ_k . Let $\zeta = Az$, where

$$\zeta_1 = z_1 + \dots + z_n, \quad \zeta_j = z_j \quad \text{for } j = 2, \dots, n.$$

Set $\Omega = A^{-1}(\Omega^1 \times \mathbb{C}^{n-1})$ and let $s \in \mathbb{R}$. We say that a Mellin distribution u with support in a proper cone $\Gamma \subset \mathbb{R}_+^n$ belongs to $M(\Omega; s)$ if $\mathcal{M}u$ is holomorphic in Ω and for any open neighbourhood W of $\mathbb{C} \setminus \Omega^1$,

$$|(\mathcal{M} \circ A^{-1})(\alpha + i\beta)| \leq C(1 + |\beta|)^s \quad \text{for } \alpha + i\beta \in \Lambda = (\mathbb{C} \setminus W) \times \mathbb{C}^{n-1}$$

where $C = C(W, \alpha)$ is locally bounded in $\alpha \in \operatorname{Re} \Lambda$. Let P be a polynomial in \mathbb{C}^n , and $\tilde{\alpha} \in \mathbb{R}^n$. Set $\tilde{a} = A\tilde{\alpha}$ and $\mathcal{P}(\zeta) = P \circ A^{-1}(\zeta)$, and let $(1/\mathcal{P})_{\tilde{a}}$ be a regularization to a distribution in $S'(\mathbb{R}^n)$ of the function $b \mapsto (1/\mathcal{P})(\tilde{a} + ib)$, $\zeta = (\zeta_1, \zeta')$, $\zeta' \in \mathbb{C}^{n-1}$. Let $\Omega^1 \subset \mathbb{C}$ be the largest subset such that the function

$$\zeta \mapsto \tilde{\mathcal{C}}(\zeta) = (2\pi)^{-n} (1/\mathcal{P})_{\tilde{a}} [F(a + i\gamma) K(\zeta - \tilde{a} - i\gamma)] \quad \text{for } \operatorname{Re} \zeta_1 < \tilde{a}_1, \zeta' \in \mathbb{C}^{n-1}$$

extends to a holomorphic function on $(\Omega^1 \cap \tilde{\Omega}^1) \times \mathbb{C}^{n-1}$; here F is a holomorphic function on $\tilde{\Omega}^1$ such that for any open neighbourhood W of $\mathbb{C} \setminus \tilde{\Omega}^1$ there exist constants C and M such that

$$|F(a + ib)| \leq C(1 + |b|)^M \quad \text{on } (\mathbb{C} \setminus W) \times \mathbb{C}^{n-1}$$

locally uniformly in $a \in \operatorname{Re} \tilde{\Omega}^1 \times \mathbb{R}^{n-1}$, and $\tilde{\Omega}^1$ is an open subset of \mathbb{C} with the same properties as Ω^1 , $K(\zeta) = (\mathcal{M}\kappa) \circ A^{-1}(\zeta)$, $\kappa = \varphi \cdot \tilde{\kappa}$ is a conical cut-off function, i.e. φ is a test function, $\varphi \equiv 1$ near the origin and $\tilde{\kappa} \in C^\infty(\mathbb{R}_+^n)$ is homogeneous of order 0, supported by a proper subcone of \mathbb{R}_+^n . The *radial characteristic set* $\operatorname{Char}_{\tilde{\alpha}} P$ is defined by

$$\operatorname{Char}_{\tilde{\alpha}} P = \mathbb{C}^n \setminus A^{-1}(\Omega^1 \times \mathbb{C}^{n-1}).$$

This set depends on the geometry of the zero set $\{P(z) = 0\}$. If P is a polynomial in one variable $z \in \mathbb{C}$, then for $\alpha \in \mathbb{R}$, $\operatorname{Char}_\alpha P = \{z \in \mathbb{C} : P(z) = 0, \operatorname{Re} z \geq \alpha\}$.

Let $\delta x_0 \in \mathbb{R}_+^n$ and $u \in \mathfrak{M}'_\omega$ for some ω . We say that $u \in M(\Omega; \varrho)$ 2-locally at $(0, \delta x_0)$ if there exists a conical cut-off function κ with $\tilde{\kappa}(\delta x_0) \neq 0$ and $\kappa u \in M(\Omega; \varrho)$. As in [4], $R = P - Q$, $P(z) = R(0, z)$, where R is a singular elliptic operator of order m .

THEOREM 4 (B. Ziemian). *Let $w \in M(\Omega; s)$ 2-locally at $(0; \delta x_0)$. If $u \in \mathfrak{M}'_\alpha$ satisfies*

$$Pu = Q(\kappa u) + \kappa w \quad \text{in } \mathbb{R}_+^n$$

then $\kappa u \in M(\Omega \setminus \bigcup_{j=0}^\infty \{\operatorname{Char}_{\tilde{\alpha}} P + j\}; s - m)$, where $j = (j, 0, \dots, 0) \in \mathbb{N}_0^n$.

It is worth mentioning that usually solutions to the equation $Ru = w$ have continuous radial asymptotics even though the right-hand side has a discrete asymptotic expansion.

As we see the notion of the radial characteristic set plays an important role in regularity theory of solutions to corner elliptic equations. Below we describe this set for a class of polynomials in \mathbb{C}^2 .

Let $P(z) = \sum_{|\nu|=m} b_\nu z^\nu$ be a polynomial of order m in two complex variables with constant coefficients and let

$$\mathcal{P}(\zeta_1, \zeta_2) = P(\zeta_1 - \zeta_2, \zeta_2) = a_m \prod_{j=1}^m (\zeta_2 - c_j(\zeta_1)).$$

By assumption P is non-characteristic in $(-1, 1)$ and \mathcal{P} has no multiple polynomial factors. Define the *discriminant* of \mathcal{P} by

$$\Delta = \prod_{j < k} (c_j(\zeta_1) - c_k(\zeta_1)).$$

Fix $\tilde{\alpha} \in \mathbb{R}^2$ and let $\tilde{a} = A\tilde{\alpha}$. Let θ_ν ($\nu = 1, \dots, N$) be all points in \mathbb{C} such that $\Delta(\theta_\nu) = 0$. Denote by B_μ ($\mu = 1, \dots, M$) all points in \mathbb{R} such that for some j , $\operatorname{Re} c_j(\tilde{a}_1 + iB_\mu) = \tilde{a}_2$. Define

$$\begin{aligned} L_\mu &= \mathbb{R} + iB_\mu, & \tilde{L}_\nu &= \overline{\mathbb{R}}_+ + \theta_\nu, & Z &= \{\zeta_1 : \Delta(\zeta_1) = 0\}, \\ L &= \bigcup_{\mu=1}^M L_\mu \cup \bigcup_{\nu=1}^N \tilde{L}_\nu \cup Z, & L_{\tilde{a}} &= L \cap \{\operatorname{Re} \zeta_1 \geq \tilde{a}_1\}. \end{aligned}$$

In this situation we can write $\operatorname{Char}_{\tilde{\alpha}} P = A^{-1}(L_{\tilde{a}} \times \mathbb{C})$, and a refinement of the regularity theorem can be proved.

3. Second microlocalization. It is well known that non-linear singularities do not have to propagate along bicharacteristic curves. In other words different non-linear waves may interact producing new singular waves. The most fundamental example here is the case of a semilinear wave operator in two space variables $\square u = f(x, u)$, $x \in \mathbb{R}^3$. To study such waves we need more information on the way the solution approaches the singular set. This is achieved by a further localization of solution called the second microlocalization. It was studied by M. Kashiwara, J. M. Bony and others. In the case when the singular set reduces to the origin, second microlocalization consists (roughly speaking) in localizing u (in addition to microlocalizing it) to a conical neighbourhood in the x space of a vector $\delta\tilde{x}$. The effect is then measured in terms of the scale of suitable weighted Sobolev spaces. For instance one may use the space $SP(s, s')$.

Let $u \in \mathcal{D}'(\mathbb{R}^n \setminus 0)$, $u \equiv 0$ in the unit ball. Let $s + s'$ be a non-negative integer. We say that $u \in SP(s, s')$ if

$$\|x\|^{-s+|\lambda|} D^\lambda u \in L^2(\mathbb{R}^n) \quad \text{for } 0 \leq |\lambda| \leq s + s', \lambda \in \mathbb{R}^n, |\lambda| = \lambda_1 + \dots + \lambda_n.$$

For remaining (s, s') the spaces $SP(s, s')$ are defined by duality and interpolation. Fix $\delta\tilde{x} \in \mathbb{R}_+^n$, $\tilde{\xi} \in \mathbb{R}^n$ and let:

1° $\varrho \in C^\infty(i(\mathbb{R}^n \setminus \{0\}))$ be a homogeneous function of order zero defined in a conical neighbourhood of $i\tilde{\beta} = i\delta\tilde{x}\tilde{\xi}$, $\varrho(i\tilde{\beta}) \neq 0$. The function ϱ is extended to $\mathbb{R}^n + i(\mathbb{R}^n \setminus \{0\})$ by putting $\varrho(z) = \varrho(\text{Im } z)$.

2° $\kappa = \varphi \cdot \kappa'$ where φ is a C_0^∞ bump function at zero and $\kappa' \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is a cut-off function in a conical neighbourhood of $\delta\tilde{x}$, i.e., $\text{supp } \kappa' \subset \mathbb{R}_+^n$, κ' is homogeneous of order zero and $\kappa'(\delta\tilde{x}) \neq 0$.

3° $\chi \in C^\infty(i(\mathbb{R}^n \setminus \{0\}))$ be a function with $\text{supp } \chi \subset \{i\tau : 1/4 \leq \|\tau\|\}$ and $\chi(i\tau) \equiv 1$ for $\|\tau\| \geq 1/2$. The function χ is extended to $\mathbb{R}^n + i(\mathbb{R}^n \setminus \{0\})$ by putting $\chi(z) = \chi(\text{Im } z)$.

DEFINITION 1 (J. M. Bony [1]). Let $(0, \xi) \in T_0^*(\mathbb{R}^n)$, $\delta\tilde{x} \in \mathbb{R}_+^n$, $\delta\tilde{\xi} = 0$. Let $u \in SP(s, -\infty) = \bigcup_\sigma SP(s, \sigma)$. We say that u belongs to $SP(s, s')$ 2-microlocally at $(\tilde{x}, \tilde{\xi}, \delta\tilde{x}, \delta\tilde{\xi}) = (0, \tilde{\xi}, \delta\tilde{x}, 0)$ if there exist functions ϱ, κ, χ satisfying conditions 1°, 2°, 3° respectively such that

$$\tilde{P}(x, D)u = \chi(xD)\varrho(xD)\kappa(x)u \in SP(s, s') \quad (xD = (x_1D_1, \dots, x_nD_n)).$$

If $u \in SP(s, -\infty)$ we define its $SP(s, s')$ -second wave front set (denoted by $2WF_{SP(s, s')}$) as a closed subset of the space

$$(T^*\mathbb{R}^n \setminus T_0^*(\mathbb{R}^n)) \cup N_{T_0^*(\mathbb{R}^n)}T(T^*\mathbb{R}^n)$$

consisting of the points $(x, \xi) \notin T_0^*(\mathbb{R}^n)$ such that $u \notin SP(s + s', 0)$ microlocally at (x, ξ) (observe that outside of zero, $SP(s, s') = SP(s + s', 0)$ coincides with the usual Sobolev space) and of the points $(0, \xi, \delta x, 0)$ such that $u \notin SP(s, s')$ 2-microlocally.

This definition however is not practical and one would like to have a definition in terms of growth of a suitable transformation. Clearly the Fourier transformation is not convenient here since the spaces $SP(s, s')$ are not isotropic. Instead one can use the approach based on the Mellin transformation (Bony uses in this situation complicated techniques of Paley–Littlewood decompositions).

Let $\Gamma \subset \mathbb{R}_+^n$ be a proper cone. If $u \in SP(s, s')$, $s + s' \in \mathbb{N}_0$ and $\text{supp } u \subset \Gamma$ then

- (i) $\mathcal{M}u \in \mathcal{O}(\sum_{j=1}^n \text{Re } z_j < s - n/2)$.
- (ii) $\mathcal{M}u(\alpha + i \cdot) \in L^{2, s+s'}(\mathbb{R}^n) = L^2(\mathbb{R}^n, (1 + \|\beta\|)^{s+s'})$ for α such that $\sum \alpha_j \leq -n/2$.

This leads to the following equivalent definition of the second wave front set.

DEFINITION 2 ([8]). Let $(0, \tilde{\xi}) \in T_0^*(\mathbb{R}^n)$, $\delta\tilde{x} \in \mathbb{R}_+^n$, $u \in SP(s, -\infty)$. Then $u \in SP(s, s')$ 2-microlocally at $(0, \tilde{\xi}, \delta\tilde{x}, 0)$ if there exist functions

ϱ, κ, χ satisfying $1^\circ, 2^\circ, 3^\circ$ such that

$$\chi(z)\varrho(z)\mathcal{M}(\kappa u)(z)|_{z=\alpha+i} \in L^{2,s+s'}(\mathbb{R}^n) \quad \text{for } \alpha_1 + \dots + \alpha_n \leq s - n/2.$$

Now we can formulate the main result of [8].

THEOREM 5 (Propagation of singularities along outgoing bicharacteristics). *Let $\delta\tilde{x} = (1, 0)$, $\tilde{\xi} = (0, \tilde{\xi}')$ and let $v \in SP(s, -\infty)$ be such that $v \in SP(s, \sigma)$ 2-microlocally at the points $(0, \tilde{\xi}, \delta x, 0)$ where $\delta x \in \mathbb{R}_+^n$. Suppose $w = \frac{\partial}{\partial x_1} v$ and $w \in SP(s-1, \sigma+1)$ 2-microlocally at $(0, \tilde{\xi}, \delta\tilde{x}, 0)$. If $\sigma < -1/2$ then $v \in SP(s, \sigma)$ 2-microlocally at $(0, \tilde{\xi}, \delta\tilde{x}, 0)$.*

A similar theorem is true for incoming and second bicharacteristics. These theorems together give the well known Bony theorem on the propagation of 2-microlocal singularities (see [1]).

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