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On extendability of invariant distributions*

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Abstract. In this paper sufficient conditions are given in order that every distribution invariant under a Lie group extend from the set of orbits of maximal dimension to the whole of the space. It is shown that these conditions are satisfied for the n-point action of the pure Lorentz group and for a standard action of the Lorentz group of arbitrary signature.

1. Notation and definitions. Let M be a p-dimensional Hausdorff analytic manifold and let $R : G \times M \to M$ be a smooth action of a connected Lie group G on M. We shall denote by M/G the orbit space of the action R and by π the natural projection $M \to M/G$. For every subset $A \subset M$, $\operatorname{Inv} A$ will stand for the set $\pi^{-1}(\pi(A))$. Orbits of maximal dimension will be called *non-singular*. The remaining orbits will be termed *singular*. An orbit θ is said to be *regular* if the submanifold topology on θ coincides with the topology induced from M (see [17], p. 68).

Two sets A_1 and A_2 are said to be *non-separable* iff any invariant neighbourhoods of A_1 and A_2 have a non-empty intersection. An orbit θ is called *separable* iff there is no orbit $\widetilde{\theta} \neq \theta$ such that θ and $\widetilde{\theta}$ are non-separable.

A set $E \subset \mathbb{R}^p$ is called *semianalytic* iff every point $x \in E$ possesses a neighbourhood U such that

$$E \cap U = \bigcup_{i=1}^{p} \Big(\bigcap_{j=1}^{q} \{g_{ij} > 0\} \cap \{f_i = 0\} \Big)$$

with g_{ij}, f_i analytic on U. A function f is called *semianalytic* iff its graph is a semianalytic set.

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The Sobolev space H_m, $m \in \mathbb{N}$, is the completion with respect to the norm $|f|_m = \sum_{|\alpha|\le m} \int |D^\alpha f(x)|\,dx$ of the space of all smooth functions f such that $|f|_m$ is finite.

All remaining symbols and definitions can be found in [17].

2. Hyperbolic sets and their properties

DEFINITION 1. Let Z be the set of singular orbits in M ([1]). We shall say that Z is *hyperbolic* in M if

(a) for every compact set $K \subset M$ there exists a compact set V_K, $V_K \cap Z \neq \emptyset$, such that for every non-singular orbit θ if $\theta \cap K \neq \emptyset$ then $\theta \cap V_K \neq \emptyset$,

(b) the orbits in $M \setminus Z$ are regular.

DEFINITION 2. We say that Z is *strongly hyperbolic* if Z is hyberbolic and the set B of all orbits in $M \setminus Z$ non-separable from Z has empty interior.

PROPOSITION 1. *Let Z be hyperbolic and let θ be an orbit such that $\theta \notin B$. Then every distribution u on $M \setminus Z$ with* $\operatorname{supp} u \subset \theta$ *extends to a disrtibution on M.*

Proof. Since $\theta \notin B$ there exist open invariant sets U_1 and U_2, $U_1 \cap U_2 = \emptyset$, such that $Z \subset U_1$, $\theta \subset U_2$. Let $\omega \in \Omega_0^p(M)$, the set of smooth compactly supported densities on M. Select a $\varphi \in C^\infty(M)$ such that $\varphi = 1$ in a neighbourhood of θ and $\operatorname{supp} \varphi \subset U_2$. Then $\varphi \cdot \omega \in \Omega_0^p(U_2)$ and we define

$$\widetilde{u}[\omega] = u[\varphi \cdot \omega].$$

$\widetilde{u}$ is the desired extension of u.

PROPOSITION 2. *If Z is hyperbolic then for every compact set $K \subset M$ and every open covering $\{U_\beta\}_{\beta \in \mathcal{B}}$ of the set $K \setminus Z$ by invariant sets U_β there exist a finite number of indices $\beta_1, \ldots, \beta_r$ such that*

$$K \setminus Z \subset \bigcup_{i=1}^{r} U_{\beta_i}.$$

Proof. By Definition 1 there exists a compact set $V_K \subset M \setminus Z$ with $\pi(K) \subset \pi(V_K)$. The set V_K being compact, there exist indices $\beta_1, \ldots, \beta_r$ such that $V_K \subset \bigcup_{i=1}^r U_{\beta_i}$. Since the set $\bigcup_{i=1}^r U_{\beta_i}$ is invariant the assertion follows.

REMARK 1. If we assume that $U \setminus Z$ consists of regular orbits then also the converse of Proposition 2 is true.

DEFINITION 3. Let Z be the set of singular orbits in M. Let $K \subset M$ be a compact subset of M with $\operatorname{Int} K \neq \emptyset$ and $K \subset U$, an open neighbourhood. Suppose $A_1, \ldots, A_r$ is an open covering of $K \setminus Z$. A family $\{\varphi_i\}_{i=1}^r$

([1]) Then Z is a closed subset of M.

of functions $\varphi_i \in C^\infty(U \setminus Z)$ will be called a *partition of unity* on $K \setminus Z$ subordinate to the covering $\{A_i\}_{i=1}^r$ iff

(a) $\varphi_i \geq 0$ for $i = 1, \ldots, r$,
(b) $K \cap \operatorname{supp} \varphi_i \subset A_i$, $i = 1, \ldots, r$,
(c) $\sum_{i=1}^r \varphi_i = 1$ on $K \setminus Z$,
(d) if $\psi_j \to \psi_0$ in $C_0^\infty(U)$ as $j \to \infty$, $\operatorname{supp} \psi_j \subset K \setminus Z$, then for every i, $\varphi_i \psi_j \to \varphi_i \psi_0$ in $C_0^\infty(U)$ as $j \to \infty$.

THEOREM 1. *Let $M = \mathbb{R}^p$. Let Z be the set of hyperbolic orbits in M. Fix a compact set K in M with non-empty interior. Put $U_1 = \{x \in M : \operatorname{dist}(x, K) < 2\}$. Suppose that there exists an open covering $\{A_i\}_{i=1}^r$ of the set $\overline{U}_1 \setminus Z$* (1) *consisting of sets whose complements A_i^c in M are semianalytic. Then there exists a partition of unity on $K \setminus Z$ subordinate to the covering $\{A_i\}_{i=1}^r$ of K.*

Proof. Define
$$\widetilde{\delta}(x) = \max(\operatorname{dist}(x, A_1^c), \ldots, \operatorname{dist}(x, A_r^c)), \qquad x \in U_1.$$
We observe that $\widetilde{\delta}$ is a semianalytic function since the distance from a semianalytic set and the maximum of semianalytic functions are also semianalytic.

The function $\widetilde{\delta}$ can vanish only on Z for if we take an arbitrary compact set $K_1 \subset U_1 \setminus Z$ then $\{A_i\}_{i=1}^r$ is an open covering of K_1 and there exists an $\varepsilon > 0$ such that for every $x \in K_1$ the ball centred at x with radius ε is contained in one of the A_i's and so we have $\widetilde{\delta}(x) \geq \varepsilon$ for $x \in K_1$.

Put $\delta(x) = \min(1, \widetilde{\delta}(x))$, $x \in U_1$, and define $U = \{x \in M : \operatorname{dist}(x, K) < 1\}$. It follows from the inequality of Łojasiewicz ([7], p. 85) that there exist positive constants $\widetilde{C}$ and $\widetilde{a}$ such that
$$\delta(x) \geq \widetilde{C}(\operatorname{dist}(x, Z))^{\widetilde{a}} \qquad \text{for } x \in U. \tag{1}$$

We shall now construct the required partition of unity on K. To this end we put
$$A_i^\delta = \{x \in A_i : \operatorname{dist}(x, A_i^c) > \delta(x)/2\}.$$
We assert that $\bigcup_{i=1}^r A_i = \bigcup_{i=1}^r A_i^\delta$. To prove this we have to show that $\bigcup_{i=1}^r A_i \subset \bigcup_{i=1}^r A_i^\delta$, the converse inclusion being obvious. So we take $x \in \bigcup_{i=1}^r A_i$. Let $\{i_1, \ldots, i_s\}$ be the set of all indices $1 \leq i \leq r$ such that $x \in A_{i_1} \cap \ldots \cap A_{i_s}$. The definition of $\widetilde{\delta}$ implies that there exists an $i_0 \in \{i_1, \ldots, i_s\}$ such that $\operatorname{dist}(x, A_{i_0}^c) = \widetilde{\delta}(x)$ and so $x \in A_{i_0}^\delta$, which was to be proved.

(1) The overbar ¯ denotes closure.

The partition is now easily constructed. Let χ_i be the characteristic function of the set A_i and let

$$\varphi(x) = \begin{cases} e^{-1/(1-|x|^2)}, & |x| < 1, \\ 0, & |x| \ge 1. \end{cases}$$

For $x \in U \setminus Z$ we define

$$\eta_i(x) = d \int_{\mathbb{R}^p} \left(\frac{\delta(y)}{4}\right)^{-2p} \varphi\left(\frac{4(y-x)}{\delta(y)}\right) \chi_i(y)\, dy \tag{2}$$

where

$$d = \Big(\int_{\mathbb{R}^p} \varphi(x)\, dx\Big)^{-1} \sup_{x,y\in U_1} \left(J\left(\frac{4(y-x)}{\delta(y)}\right) \cdot \left(\frac{\delta(y)}{4}\right)^{2p}\right).$$

The integral in (2) makes sense since for a fixed $x \in U \setminus Z$ the set $\{y : |y-x| \le \delta(x)\}$ is compact and does not intersect Z. We also observe that

$$d \int_{\mathbb{R}^d} \left(\frac{\delta(y)}{4}\right)^{-2p} \varphi\left(\frac{4(y-x)}{\delta(y)}\right) dy \ge 1 \quad \text{for } x \in U \setminus Z, \tag{3}$$

which follows by substitution $z(y) = 4(y-x)/\delta(y)$ from the fact that z is "onto" $\mathbb{R}^p$ and the integrand is non-negative.

From (2) we see that $\eta_i \in C^\infty(U \setminus Z)$ (since $\delta(y) > 0$ on $U \setminus Z$) and $(\operatorname{supp} \eta_i) \cap K \subset A_i$. It remains to normalize η_i so we put

$$\varphi_i = \frac{\eta_i}{\sum_{i=1}^r \eta_i}.$$

The above properties of η_i imply in view of (3) that the φ_i satisfy items (a)–(c) of Definition 3. To prove (d) we first show that for every multiindex α there exist positive constants C and a such that

$$\left|\frac{\partial^\alpha \varphi_i(x)}{\partial x^\alpha}\right| \le \frac{C}{(\operatorname{dist}(x,Z))^a} \quad \text{for } x \in K \setminus Z,\ i = 1, \ldots, r. \tag{4}$$

Inequality (4) is proved by induction on the length $|\alpha|$ of α. For $\alpha = 0$ we see from the definition of φ_i that

$$|\varphi_i(x)| \le 1, \quad x \in U \setminus Z.$$

We now prove (4) for $\alpha = (1, 0')$, $0' \in \mathbb{R}^{p-1}$. Set $h = \sum_{i=1}^r \eta_i$. Since $h \ge 1$ on $U \setminus Z$ by (3), we have

$$\left|\frac{\partial \varphi_i}{\partial x_1}\right| \le \left|\frac{\partial \eta_i}{\partial x_1}\right| + \sum_{j=1}^r \left|\frac{\partial \eta_j}{\partial x_1}\right|.$$

Thus it suffices to prove (4) for the functions η_i instead of φ_i.

By differentiating (2) we find

$$\frac{\partial \eta_i}{\partial x_1} = d \int\limits_U \left(\frac{4}{\delta(y)}\right)^{2p+1} \frac{\partial}{\partial z_1}\varphi\left(\frac{4(y-x)}{\delta(y)}\right)\chi_i(y)\,dy \qquad \text{for } x \in K \setminus Z.$$

Hence by (1) for $x \in K \setminus Z$ we get

$$\begin{aligned}\left|\frac{\partial \eta_i(x)}{\partial x_1}\right| &\le C_2 \int\limits_U \frac{1}{(\operatorname{dist}(y,Z))^{\widetilde{a}(2p+1)}}\left|\frac{\partial \varphi}{\partial z_1}\left(\frac{4(y-x)}{\delta y}\right)\right|\chi_i(y)\,dy \\ &\le C_3 \int\limits_{U\cap\{y:|x-y|\le\delta(y)/4\}} \frac{dy}{(\operatorname{dist}(y,Z))^{\widetilde{a}(2p+1)}} \\ &\le C_3 \int\limits_{U\cap\{y:\operatorname{dist}(y,Z)\ge\operatorname{dist}(x,Z)/2\}} \frac{dy}{(\operatorname{dist}(y,Z))^{\widetilde{a}(2p+1)}} \\ &\le C_4 \frac{1}{(\operatorname{dist}(x,Z))^{\widetilde{a}(2p+1)}} \int\limits_U dy\end{aligned}$$

(where C_2, C_3, C_4 are suitable positive constants).

To prove the penultimate inequality it is enough to show that

$$\{y : |x-y| \le \operatorname{dist}(y,Z)/4\} \subset \{y : \operatorname{dist}(y,Z) \ge \operatorname{dist}(x,Z)/2\},$$

which is equivalent, after passing to complements, to

$$\{y : \operatorname{dist}(y,Z) < \operatorname{dist}(x,Z)/2\} \subset \{y : |x-y| > \operatorname{dist}(y,Z)/4\}.$$

To prove the last inclusion suppose conversely that for a certain y such that $\operatorname{dist}(y,Z) < \operatorname{dist}(x,Z)/2$ we have $|x-y| \le \operatorname{dist}(y,Z)/4$. Let $w \in Z$ be such that $\operatorname{dist}(y,Z) = |y-w|$. Then $|y-w| < \operatorname{dist}(x,Z)/2$ and $|x-y| < \operatorname{dist}(x,Z)/8$, hence $|x-w| < \frac{5}{8}\operatorname{dist}(x,Z)$, which is impossible since $w \in Z$.

Finally, we state a well-known general lemma which shows how (4) implies (d) of Definition 3.

LEMMA 1. *Let Z be a closed subset of $\mathbb{R}^p$, K a compact subset of $\mathbb{R}^p$ and U an open neighbourhood of K. If a function $\varphi \in C^\infty(U \setminus Z)$ satisfies (4) for every α then for every function $\psi \in C_0^\infty(K)$ flat on Z, we have*

(i) $\psi\varphi \in C_0^\infty(\mathbb{R}^p)$ *and is flat on* Z,

(ii) $|\partial^\alpha(\psi\varphi)/\partial x^\alpha| \le C\|\psi\|_m$ *for certain constants* $C > 0$, $m \in \mathbb{N}$ *depending only on* α *where*

$$\|\psi\|_m = \sum_{|\beta|\le m} \sup_{x\in K}\left|\frac{\partial^\beta \psi}{\partial x^\beta}(x)\right|.$$

Proof. This follows from Taylor's formula (see [8], p. 154).

REMARK 2. Since Theorem 1 has a local character its proof generalizes easily to the case where M is an analytic manifold.

3. Theorems on extendability of invariant distributions. In what follows M will be a fixed G-manifold as in Section 1. We also retain all notation and definitions from [17] (e.g. K_α, N_α etc.)

THEOREM 2. *Let the set Z of singular orbits be hyperbolic in M. Let $\{N_\alpha\}_{\alpha\in\mathcal{A}}$ be the Hausdorff partition of the manifold $(M\setminus Z)/G$. Suppose that*

(a) *for every non-singular orbit θ there exists an invariant neighbourhood U_θ such that $\pi(U_\theta)\subset N_\alpha$ for some α, $\overline{\pi(U_\theta)}^{N_\alpha}$ is compact and the complement of U_θ in M is a semianalytic set*,

(b) *for every sequence $\omega_j\in\Omega_0^p(\pi^{-1}(N_\alpha))$ convergent to zero in $\Omega_0^p(M)$, $K_\alpha\omega_j$ is convergent to zero locally uniformly with all derivatives on N_α.*

Then every G-invariant distribution on $M\setminus Z$ extends to a distribution on M.

Proof. Let u be an invariant distribution on $M\setminus Z$. Then by Theorem 2 of [17] there exists a unique distribution $\{T_\alpha\}_{\alpha\in\mathcal{A}}$ on $N=(M\setminus Z)/G$ such that

$$u[\omega]=T_\alpha[K_\alpha\omega],\qquad \omega\in\Omega_0^p(\pi^{-1}(N_\alpha)).$$

Let $\omega_j\in\Omega_0^p(M\setminus Z)$, $\omega_j\to 0$ in $\Omega_0^p(M)$ as $j\to\infty$. Let K be a compact set containing all $\operatorname{supp}\omega_j$. Choose an open neighbourhood U_1 of K with compact closure U_1. The sets $\{U_\theta\}$ form an invariant covering of U_1 and by Proposition 2 we can select a finite subcovering $\{U_{\theta_i}\}_{i=1}^r$.

Next we apply the "manifold version" of Theorem 1 to obtain a partition of unity $\{\varphi_i\}_{i=1}^r$ on $K\setminus Z$ subordinate to the covering $\{U_{\theta_i}\}_{i=1}^r$. Let $\alpha_i\in\mathcal{A}$ correspond to U_{θ_i} as in (a). We observe that for every j,

$$u[\omega_j]=\sum_{i=1}^r T_{\alpha_i}[a_j^i]$$

where $a_j^i=K_{\alpha_i}(\varphi_i\omega_j)$.

Thus in order to prove that $u[\omega_j]\to 0$ it is enough to show that for every fixed i, $a_j^i\to 0$ in $\Omega_0^{p-n}(N_{\alpha_i})$ as $j\to\infty$. Since $\omega_j\to 0$ in $\Omega_0^p(M)$ we infer from condition (d) of Definition 3 that for every i, $\varphi_i\omega_j\to 0$ in $\Omega_0^p(M)$. From the definition of the operation K_α we see that $\operatorname{supp}a_j^i\subset\overline{\pi(U_{\theta_i})}^{N_{\alpha_i}}$, which is compact by (a). From this and (b) we conclude that $a_j^i\to 0$ in $\Omega_0^{p-n}(N_{\alpha_i})$ for every $i=1,\ldots,r$.

Now, we construct the required extension. To this end let $U\subset M$ be an open set with $\overline{U}$ compact. By the above, the sets $\Omega_0^p(U\setminus Z)\subset\Omega_0^p(U)$ satisfy the assumptions of the Hahn–Banach theorem. It follows that there exists a distribution $\widetilde{u}$ extending $u|_{U\setminus Z}$ to U. Taking successive open sets U and gluing the distributions thus obtained we get the required extension. This ends the proof of the theorem.

REMARK 2. The extension which exists by Theorem 2 need not be invariant. In fact an example given in [3] shows that there exist invariant distributions which are extendable but which have no invariant extensions.

THEOREM 3. *Let the set Z be strongly hyperbolic. If for every sequence $\omega_j \in \Omega_0^p(\pi^{-1}(N_\alpha))$ with $\omega_j \to \omega_0$ in $\Omega_0^p(M)$ as $j \to \infty$ the form $K_\alpha\omega_0$ is C^∞ on N_α then $K_\alpha\omega_j$ converges to $K_\alpha\omega_0$ locally uniformly together with all derivatives.*

Proof. Let $\omega_j \in \Omega_0^p(\pi^{-1}(N_\alpha))$, $\omega_j \to \omega_0$ in $\Omega_0^p(M)$. We have to show that $L^{\text{tr}}(K_\alpha\omega_j) \to L^{\text{tr}}(K_\alpha\omega_0)$ locally uniformly on N_α for every C^∞ linear differential operator L on N.

Let $x \in N_\alpha$ and let H be a coordinate system around x. We define a distribution $\delta_x^H = \delta_{H(x)} \circ H^{-1}$ where δ_a is the Dirac delta in $\mathbb{R}^{p-n}$ at the point a. Suppose that $\overline{a}_j = L^{\text{tr}}(K_\alpha\omega_j)$ is not locally uniformly convergent to $\overline{a}_0 = L^{\text{tr}}(K_\alpha\omega_0)$ on N_α. Then there exists an $\varepsilon_0 > 0$ such that for every $j = 1, 2, \ldots$ there exists an $x_j \in N_\alpha \setminus B$ (see Definition 2) such that

$$|\delta_{x_j}^{H_j}[L^{\text{tr}}(\overline{a}_j) - L^{\text{tr}}(\overline{a}_0)]| \geq \varepsilon_0 \tag{5}$$

(H_j a coordinate system around x_j). Put

$$\Delta_j[\eta] = e_j\delta_{x_j}^{H_j}[L^{\text{tr}}(K_\alpha\eta)] \quad \text{for } \eta \in \Omega_0^p(M \setminus Z)$$

where $e_j = \pm$ is such that $\Delta_j[\overline{a}_j - \overline{a}_0] \geq \varepsilon_0$.

We observe that Δ_j is a distribution on $M \setminus Z$ with support in $\pi^{-1}(x_j)$. In view of Proposition 1, Δ_j extends to a distribution $\widetilde{\Delta}_j$ on M (because $x_j \notin B$). By assumption if $\omega \in \overline{\Omega_0^p(\pi^{-1}(N_\alpha))}$ (the closure of $\Omega_0^p(\pi^{-1}(N_\alpha))$ in $\Omega_0^p(M)$) then $K_\alpha\omega$ is C^∞ on N_α, hence there is a constant C_ω such that

$$|\delta_{x_j}^{H_j}[L^{\text{tr}}(K_\alpha\omega)]| \leq C_\omega, \quad j = 1, 2, \ldots \tag{6}$$

From (6) we see that $\widetilde{\Delta}_j$, $j = 1, 2, \ldots$, satisfy the assumptions of the Banach–Steinhaus theorem on $\overline{\Omega_0^p(\pi^{-1}(N_\alpha))}$. It follows from that theorem that $\max_{i\in\mathbb{N}} \widetilde{\Delta}_i$ is a continuous operation on $\Omega_0^p(\pi^{-1}(N_\alpha))$ (not a distribution since max is not additive). Thus in particular $\max_{i\in\mathbb{N}} \Delta_i[\omega_j - \omega_0] \to 0$ as $j \to \infty$, which contradicts (5).

In an analogous way one proves the following "converse" of Theorem 2.

THEOREM 4. *Let Z be hyperbolic. If for every sequence $\omega_j \in \Omega_0^p(\pi^{-1}(N_\alpha))$ such that $\omega_j \to \omega_0$ in $\Omega_0^p(M)$, $K_\alpha\omega_0$ is of class C^∞ on N_α and if every invariant distribution on $M \setminus Z$ extends to a distribution on M then $K_\alpha\omega_j$ converges to $K_\alpha\omega_0$ locally uniformly with all derivatives.*

We now establish conditions under which (b) of Theorem 2 is satisfied.

THEOREM 5. *Let* $\{(H_k, A_k)\}$ *be a family of coordinate systems covering* $M \setminus Z$ *and such that*:

(a) *the coordinates* $H_k^1, \ldots, H_k^{p-n}$ *of the vector function* H_k *are constant along orbits in* M,

(b) H_k^i, $i = 1, \ldots, p$, *extend to invariant analytic functions on* M (*denoted by the same symbol*),

(c) *for each* k *there exists an* $\alpha_k \in \mathcal{A}$ *such that* $\pi(A_k) \subset N_{\alpha_k}$.

If $\omega_j \in \Omega_0^p(M \setminus Z)$, $\operatorname{supp} \omega_j \subset A_k$, $\omega_j \to 0$ *in* $\Omega_0^p(M)$ *then* $K_{\alpha_k}\omega_j \to 0$ *as* $j \to \infty$ *locally uniformly with all derivatives on* N_k.

Proof. Fix a coordinate system (H_k, A_k) and let α_k be such that $\pi(A_k) \subset N_{\alpha_k}$.

In view of the Sobolev lemma ([10], p. 197) it suffices to show that locally $K_{\alpha_k}\omega_j \to 0$ in H_m for all m (H_m is the Sobolev space). Let $(\Phi_k, \pi(A_k))$ be the coordinates on N_{α_k} induced by (H_k, A_k) (see [17], p. 69). Define $K = K_{\alpha_k}$, $H = H_k$, $A = A_k$, $s_i = H^i$, $i = 1, \ldots, p-n$, $y_i = H^{p-n+i}$ for $i = 1, \ldots, n$. Then by (a), $K\omega_j$ has the following form in the coordinate system Φ:

$$(K\omega_j)(s_1, \ldots, s_{p-n}) = \int_{\mathbb{R}^n} \left(\frac{\omega_j}{JH}\right) \circ H^{-1}(s_1, \ldots, s_{p-n}, y_1, \ldots, y_n)\, dy$$

where JH is the Jacobian of H. Let us compute

$$\frac{\partial}{\partial s_1}(K\omega_j)(s) = \int_{\mathbb{R}^n} \frac{\partial}{\partial s_1}\left(\left(\frac{\omega_j}{JH}\right) \circ H^{-1}(s, y)\right) dy.$$

We have

$$\left(\frac{\partial}{\partial s_1}\left(\left(\frac{\omega_j}{JH}\right) \circ H^{-1}(s, y)\right) \circ H = \frac{\omega_j w_0}{(JH)^2} + \sum_{i=1}^{p} \frac{\frac{\partial \omega_j}{\partial x_i} w_i}{(JH)^2}. \tag{7}$$

where in view of (b), w_i, $i = 0, 1, \ldots, p$, are analytic functions in a neighbourhood of A. Let Z_1 be the set of zeros of the function JH. Then Z_1 is disjoint from A and from the Łojasiewicz inequality

$$|JH(x)| \geq C(\operatorname{dist}(x, Z_1))^a, \quad x \in D,$$

where D is a compact subset of M containing all $\operatorname{supp} \omega_j$ and C, a are positive constants. Since ω_j are flat on Z_1 we know from Lemma 1 that there are constants $\widetilde{C}, \widetilde{\widetilde{C}}$ such that

$$\left|\frac{\omega_j}{JH}\right| \leq \widetilde{C}\|\omega_j\|_m, \quad \left|\frac{1}{JH}\frac{\partial \omega_j}{\partial x_i}\right| \leq \widetilde{\widetilde{C}}\|\omega_j\|_m \tag{8}$$

for some $m \in \mathbb{N}$.

Set

$$\overline{\omega}_j = \frac{\omega_j w_0}{JH} + \frac{1}{JH}\sum_{j=1}^{p}\frac{\partial \omega_j}{\partial x_i} w_i. \tag{9}$$

Then from (7) and (8) we see that $(\partial/\partial s_1)K\omega_j = K\overline{\omega}_j$ with $\overline{\omega}_j \in \Omega_0^p(M\setminus Z)$ and $\overline{\omega}_j \to 0$ in $\Omega_0^p(M)$.

Now it is easy to show that

$$\int\left|\frac{\partial}{\partial s_1}K\omega_j\right| ds \to 0.$$

Namely we have

$$\int\left|\frac{\partial}{\partial s_1}K\omega_j\right| ds = \int|K\overline{\omega}_j|\, ds \le \int K|\overline{\omega}_j|\, ds = \int|\overline{\omega}_j| \to 0.$$

Take a point $\theta \in N_{\alpha_k}$ and a coordinate system $\widetilde{\Phi}$ around θ in N_{α_k}. Suppose that $\widetilde{\Phi}$ is induced by one of the coordinate systems (H_k, A_k). Denote that system by $(\widetilde{H}, \widetilde{A})$. Let h be the characteristic function of an invariant open neighbourhood U of θ such that $\overline{\pi(U)}^{N_\alpha} \subset \pi(\widetilde{A})$. Then $K(h\omega_j) = K\omega_j$ in a neighbourhood of θ and $\operatorname{supp} K(h\omega_j) \subset \pi(\widetilde{A})$. We shall prove that $K(h\omega_j)$ is convergent to zero on $\pi(U)$ in H_m for every m.

Denote by $\partial/\partial\widetilde{s}_i$ the differentiations in the coordinate system $\widetilde{\Phi}$. Let $\Psi = \widetilde{H}\circ H^{-1}$. Then the transition mapping R for Φ and $\widetilde{\Phi}$ has the form

$$R(s) = (\Psi^1(s,y), \ldots, \Psi^{p-n}(s,y))$$

where y is an arbitrary point such that (s,y) belongs to the domain of Ψ. Thus the Jacobi matrix DR has the form

$$(DR)(s) = (b_{ij}(s))_{i,j=1,\ldots,p-n}$$

where $b_{ij}(s) = \frac{\partial \Psi^i}{\partial s_j}(s,y)$ and is independent of y. Let (a_{ij}) be the inverse matrix to (b_{ij}). Then to the differentiation $\partial/\partial\widetilde{s}_1$ in the coordinate system $\widetilde{\Phi}$ there corresponds in the coordinate system Φ the operator

$$L = \sum_{i=1}^{p-n} a_{i1}\frac{\partial}{\partial s_i}$$

$(\frac{\partial}{\partial \widetilde{s}_1}(\varphi\circ R^{-1}) = (L\varphi)\circ R^{-1})$. We want to show that $a_{ij}\circ H$ is of the form A_{ij}/B_{ij} where A_{ij} and B_{ij} are analytic functions of M and B_{ij} can be zero on the set on which all $h\omega_j$ are flat. To this end we observe that it follows from the forms of the mappings H and $\widetilde{H}$ and the formula for the inverse of a matrix that $b_{ij}\circ H = D_{ij}/JH$ where D_{ij} is an analytic function on M.

Hence

$$a_{ij} \circ H = \frac{\dfrac{C_{ij}}{(JH)^{p-n-1}}}{\dfrac{B_{ij}}{(JH)^{p-n}}}$$

where C_{ij} and B_{ij} are analytic functions on M.

To obtain the required representation observe that B_{ij} can vanish on the set on which all $h\omega_j$ are flat. This follows from the fact that $(JR) \circ H = B_{ij}/(JH)^{p-n}$ is constant along orbits and JH is not zero on A.

We compute

$$\begin{aligned} LK(h\omega_j)(s) &= \sum_{i=1}^{p-n} a_{i1} \frac{\partial}{\partial s_i} K(h\omega_j)(s) = \sum_{i=1}^{p-n} a_{i1} K(h\overline{\omega}_j)(s) \\ &= \sum_{i=1}^{p-n} \int_{\mathbb{R}^n} a_{i1}(s) \left(\frac{h\overline{\omega}_j}{JH} \right) \circ H^{-1}(s,y)\, dy \\ &= \sum_{i=1}^{p-n} \int_{\mathbb{R}^n} \left(\frac{A_{i1} h\overline{\omega}_j}{B_{i1} JH} \right) \circ H^{-1}(s,y)\, dy. \end{aligned}$$

Define

$$\widetilde{\omega}_j = \sum_{i=1}^{p-n} \frac{A_{i1} h\overline{\omega}_j}{B_{i1}}.$$

In the same way as in the case of $\widetilde{\omega}_j$ we show that $\widetilde{\omega}_j \to 0$ in $\Omega_0^p(M)$ so that

$$\int |LK(h\omega_j)|\, ds \to 0 \quad \text{as } j \to \infty.$$

The remaining part is proved by induction.

PROPOSITION 4 (on regularity of foliations). *Let S be an involutive C^∞ differential system on an analytic T_2-manifold. Let θ be a leaf of the foliation given by S and suppose that θ is an analytic set. Then θ is regular.*

Proof. Let $p \in \theta$ and let (H, U) be an arbitrary analytic coordinate system around p such that there exists an analytic function φ on U and $\theta \cap U$ is the set of zeros of φ on U. Suppose θ is not regular. Then the set $H(\theta \cap U)$ has infinitely many components which have a condensation point (the component of $H(p)$). Let W be the orthogonal complement of the affine subspace tangent to $H(\theta)$ at $H(p)$. Then W intersects transversally the components of the set $H(\theta \cap U)$ which are close enough to $H(p)$. Thus there is a neighbourhood V of the point $H(p)$ in W such that $W \cap H(\theta \cap U)$ is an infinite set with $H(p)$ as a condensation point and this set does not contain any one-dimensional submanifold. But this is impossible since this set is analytic (described by $h = \varphi \circ H^{-1}|_W$). To see this take an arbitrary sequence $x_n \to \widetilde{p} = H(p)$, $x_n \in W \cap H(\theta \cap U)$. We can assume that the sequence of vectors $(x_n - \widetilde{p})/|x_n - p|$ is convergent to a vector α. Then all derivatives of the function h at $\widetilde{p}$ in the direction of the vector α are zero. Since $h(\widetilde{p}) = 0$ and h is analytic it must be zero along the vector α so the set $W \cap H(\theta \cap U)$ contains an interval, which is impossible.

4. Examples (n-point Lorentz invariant distributions). Let $\mathrm{SO}_0(1,1)$ be the group of proper Lorentz rotations in $\mathbb{R}^2$, i.e. the group generated by the mappings

$$\sigma_\beta(x_1, x_2) = \left(\frac{x_1 + \beta x_2}{\sqrt{1-\beta^2}}, \frac{\beta x_1 + x_2}{\sqrt{1-\beta^2}}\right), \qquad |\beta| < 1.$$

Take the Cartesian product of n copies of $\mathbb{R}^2$,

$$\mathbb{R}^2 \times \ldots \times \mathbb{R}^2 = \mathbb{R}^{2n}.$$

Let $\xi \in \mathbb{R}^{2n}$, $\xi = (\xi_1, \ldots, \xi_n)$, $\xi_i \in \mathbb{R}^2$, $i = 1, \ldots, n$.

We define an action of the group $\mathrm{SO}_0(1,1)$ on $\mathbb{R}^{2n}$ by putting

$$g(\xi_1, \ldots, \xi_n) = (g\xi_1, \ldots, g\xi_n), \qquad g \in \mathrm{SO}_0(1,1).$$

This action is called the *n-point action* of the Lorentz group $\mathrm{SO}_0(1,1)$. A distribution invariant under this action is called an *n-point Lorentz invariant distribution.*

We recall certain facts concerning n-point Lorentz invariant polynomials. Namely Weyl's theorem [16] states that every n-point Lorentz invariant polynomial can be expressed as a polynomial in a finite number of fundamental invariants. These invariants form the matrix

$$((\xi_i \,|\, \xi_j))_{i,j=1,\ldots,n},$$

where $\xi_i = (\xi_{i,1}, \xi_{i,2}) \in \mathbb{R}^2$, $i = 1, \ldots, n$, and

$$(\xi_i \,|\, \xi_j) = \xi_{i,1}\xi_{j,1} - \xi_{i,2}\xi_{j,2}.$$

Since only proper Lorentz transformations are considered we have for $n \geq 2$ the additional invariants:

$$\det(\xi_{i_1}, \xi_{i_2}), \qquad \{i_1, i_2\} \subset \{1, \ldots, n\}.$$

We intend to prove that every n-point Lorentz invariant distribution defined outside the origin extends to the whole of the space $\mathbb{R}^{2n}$. To this end we will show that the assumptions of Theorem 5 and Theorem 2 are satisfied.

It is natural to form the coordinate systems in Theorem 5 from the above invariants. Unfortunately the coordinate systems formed in this way do not cover the whole of the space of non-singular orbits. Namely we lack coordinate systems around the points $\xi = (\xi_1, \ldots, \xi_n) \neq (0, \ldots, 0)$, $\xi_i \in \mathbb{R}^2$, which satisfy the equations

$$(\xi_i \mid \xi_j) = 0, \quad i, j = 1, \ldots, n.$$

Let $v = (\overline{\xi}_1, \ldots, \overline{\xi}_n)$ be such a point. Write

$$\langle v \mid \xi \rangle = \sum_{i=1}^{n} (\overline{\xi}_{i,1}\xi_{i,1} + \overline{\xi}_{i,2}\xi_{i,2}).$$

Then a coordinate system around v can be formed from the following family of invariant rational functions:

$$\begin{aligned} f_i(\xi) &= \frac{1}{2\langle v \mid \xi \rangle}(((\langle v \mid v \rangle^2 - \langle v \mid \xi \rangle^2)\xi_{i,1} + ((\langle v \mid \xi \rangle^2 + \langle v \mid v \rangle^2)\xi_{i,2}), \\ g_i(\xi) &= \frac{1}{2\langle v \mid \xi \rangle}(((\langle v \mid \xi \rangle^2 + \langle v \mid v \rangle^2)\xi_{i,1} + ((\langle v \mid v \rangle^2 - \langle v \mid \xi \rangle^2)\xi_{i,2}). \end{aligned} \tag{10}$$

Thus in Theorem 5 we have to admit functions of the form (10) and this is possible since such functions satisfy the Łojasiewicz inequality (apply the standard Łojasiewicz inequality to the numerator).

To prove that the set $Z = \{0\}$ is hyperbolic we note that every non-singular orbit is unbounded so that if it passes close to zero it must intersect an annulus around zero.

It is also easy to see that assumption (b) of Theorem 2 is satisfied, i.e. every non-singular orbit θ admits an invariant neighbourhood U_θ such that $\pi(U_\theta) \subset N_\alpha$ with $\overline{\pi(U_\theta)}$ compact in N_α and whose complement is semianalytic. For the proof suppose first that θ is a separable orbit. Take an arbitrary point $\xi_0 \in \theta$ and a coordinate system (H, A) around ξ_0 whose coordinates $(H^1, \ldots, H^{p-1})$ are formed by the fundamental invariants. Then there exists r_0 sufficiently small such that the set

$$\begin{aligned} U_\theta = {} & \mathrm{Inv}(A) \\ & \cap \{\xi \in \mathbb{R}^{2n} : (H^1(\xi) - H^1(\xi_0))^2 + \ldots + (H^{p-1}(\xi) - H^{p-1}(\xi_0))^2 < r_0\} \end{aligned}$$

has the required properties. To see that $\mathbb{R}^{2n} \setminus U_\theta$ is semianalytic observe that $(\mathbb{R}^{2n} \setminus U_\theta) \cap \mathrm{Inv}(A)$ is described by the condition

$$(H^1(\xi) - H^1(\xi_0))^2 + \ldots + (H^{p-1}(\xi) - H^{p-1}(\xi_0))^2 \ge r_0.$$

If θ is a non-separable orbit and $v \in \theta$ then for U_θ we take the set

$$\begin{aligned} & \{\xi \in \mathbb{R}^{2n} : \langle \xi \mid v \rangle > 0\} \\ & \cap \{\xi \in \mathbb{R}^{2n} : (H^1(\xi) - H^1(v))^2 + \ldots + (H^{p-1}(\xi) - H^{p-1}(v))^2 < r_0\}, \end{aligned}$$

where H^i, $i = 1, \ldots, p-1$, are functions of the form (10).

Finally, we remark that an analogous statement concerning extendability of distributions invariant under the n-point action of $\mathrm{SO}_0(p, q)$, $p \ge 1$, $q \ge 1$, $p + q > 2$, is not true for $n > 1$. E.g. for the 2-point action of $\mathrm{SO}_0(1, 2)$ the function

$$\exp\left(\frac{1}{(1-x_1^2+x_2^2+x_3^2)^2+(1-y_1^2+y_2^2+y_3^2)^2+(1-x_1y_1+x_2y_2+x_3y_3)^2}\right),$$

$(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6$, does not extend to $\mathbb{R}^6$.

The verification of the assumptions of Theorems 2 and 5 in the case of a natural action of the group $\mathrm{SO}_0(p, q)$, $p, q \geq 1$ $(n = 1)$ is immediate.

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