ANNALES
POLONICI MATHEMATICI
LXXIV (2000)

Some constructive applications of Λ^2-representations to integration of PDEs

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Abstract. Two new applications of Λ^2-representations of PDEs are presented:

1. Geometric algorithms for numerical integration of PDEs by constructing planimetric discrete nets on the Lobachevsky plane Λ^2.

2. Employing Λ^2-representations for the spectral-evolutionary problem for nonlinear PDEs within the inverse scattering problem method.

The geometric interpretation of PDEs linking their investigation to analysis of certain objects on the hyperbolic plane (Lobachevsky plane Λ^2) was introduced in [3–5]. According to this interpretation to each differential equation $f[u(x,t)] = 0$ from a certain class under investigation (Λ^2-class or Lobachevsky class) one can associate a pseudospherical metric (of Gaussian curvature $K \equiv -1$) defined on any regular solution $u(x,t)$:

$$\{f[u(x,t)] = 0,\ u \in C^n(\mathbb{R}^2)\} \leftrightarrow \{ds^2[u(x,t)],\ K \equiv -1\}.$$

The set of coefficients $\{E[u(x,t)], F[u(x,t)], G[u(x,t)]\}$ of the pseudospherical metric $ds^2[u(x,t)]$ is called the *Λ^2-representation* for the given equation and is denoted by

$$\Lambda^2[f[u(x,t)] = 0] = \{E[u(x,t)], F[u(x,t)], G[u(x,t)]\}.$$

The existence of the Λ^2-representation for PDEs leads to the possibility of using the Lobachevsky geometry methods for their investigation. In this paper two different applications of Λ^2-representations for the study of PDEs are presented:

2000 *Mathematics Subject Classification*: Primary 35Q53.

Key words and phrases: Λ^2-representations of PDEs, Lobachevsky (hyperbolic) geometry, discrete net, pseudospherical metric.

1. Geometrical algorithms for numerical integration of PDEs (alternative to typical difference methods) by construction of special nets on the Lobachevsky plane Λ^2.

2. Employing Λ^2-representations for the spectral-evolutionary problem corresponding to a given nonlinear PDE in the frame of the inverse scattering problem (ISP) method.

1. Discrete rhombic Chebyshev nets and geometrical method of numerical integration of the sine-Gordon equation. In this section on the basis of Λ^2-representation theory a geometrical method of numerical integration of the sine-Gordon equation is presented.

In general the geometric approach to integration of Λ^2-equations amounts to the following scheme:

1. Construction of the metric $ds^2[u(x,t)]$, $K \equiv -1$, for a given PDE (finding the Λ^2-representation).

2. Selection of *key*-characteristics (k-characteristics) of the coordinate net $T(x,t)$ associated with the pseudospherical metric: k-characteristics determine a solution of the PDE.

3. Introduction of a discrete net $T^d \subset \Lambda^2$ (a discrete analog of the net $T(x,t) \subset \Lambda^2$) inheriting the k-characteristics of the smooth net $T(x,t)$.

4. Finding algorithmic (recurrent) intrinsic relations in the net T^d and consequently calculation of the discrete analog u^d of the solution $u(x,t)$ at the nodes of the net T^d.

5. Investigation of the convergence of the resulting discrete algorithms: the proof of the convergence of the discrete solution $\{u^d\}$ to the solution u of the original Λ^2-equation as the typical linear size a of the cell of the discrete net tends to zero:

$$u^d \to u, \qquad T^d \to T \qquad \text{as } a \to 0.$$

1.1. *Darboux problem for the sine-Gordon equation.* Consider the Darboux problem for the sine-Gordon equation:

$$u_{xt} = \sin u, \tag{1.1}$$
$$u(x,0) = \phi(x), \qquad u(0,t) = \psi(t).$$

Equation (1.1) is generated by the pseudospherical metric

$$ds^2 = dx^2 + 2\cos u\, dx dt + dt^2, \tag{1.2}$$

i.e.

$$\Lambda^2[u_{xt} = \sin u] \equiv \{1, \cos u, 1\}.$$

The metric (1.2) defines [3, 6] the Chebyshev coordinate net $\mathrm{Ch}(x,t)$ on the hyperbolic plane Λ^2. The characteristic property of the Chebyshev net is equality of lengths of opposite sides in an arbitrary coordinate quadrangle.

This property will be chosen as a k-characteristic of $\mathrm{Ch}(x,t)$. The solution $u(x,t)$ of equation (1.1) is interpreted as the net angle of the Chebyshev net $\mathrm{Ch}(x,t)$.

The Darboux problem (1.1) is originally considered on the usual parametric plane $E^2(x,t)$ (Euclidean plane) with the uniform coordinate net $T_0(x,t)$. The functions $\phi(x)$ and $\psi(t)$ should be considered as the initial data prescribed on the characteristics of equation (1.1). Geometrically these functions are the initial values of the net angles of $\mathrm{Ch}(x,t)$ on its elements $(x:0)$ and $(0:t)$. The functions satisfy the conjugation conditions $\phi^{(k)}(0)=\psi^{(k)}(0)$, $k\in\mathbb{N}_0$, which guarantee the desired smoothness of the solutions of (1.1).

By virtue of the geometrical interpretation of (1.1) in the theory of isometric imbeddings of certain domains on the Lobachevsky plane Λ^2 into Euclidean space E^3 [7, 8], problem (1.1) is transformed from $E^2(x,t)$ into Λ^2:

$$\{T_0(x,t),E^2(x,t)\}\to\{\mathrm{Ch}(x,t),\Lambda^2(x,t),ds^2[u(x,t)]\}.$$

In addition the uniform coordinate net $T_0(x,t)\subset E^2$ is mapped onto the Chebyshev net $\mathrm{Ch}(x,t)\subset\Lambda^2$.

Consider now the discrete counterpart $T^d(a)\subset\Lambda^2$ of the net $\mathrm{Ch}(x,t)$ consisting of rhombuses $R(a)$ with side a (the sides of the rhombuses are segments of geodesics on Λ^2). The net $T^d(a)$ can be geometrically reconstructed by using the initial data as follows: at every step, a standard planimetric procedure (in the sense of Λ^2) is applied to construct a point on Λ^2 (the vertex of a rhombus) that is equally distant from two vertices of the already existing adjoining sides of $R(a)$. (On the planimetric constructions on the hyperbolic plane see, for example, [11].) Thus, the present analysis of the Darboux problem (1.1) on Λ^2 is focused on the discrete net $T^d(a)$ defined by two families of piecewise geodesic polygonal curves with elements $l^x(a)$ and $l^t(a)$.

The geometric algorithm for obtaining a solution of the Darboux problem (1.1) defines the net angle $u(x,t)$ of $\mathrm{Ch}(x,t)$ as the limit of a discrete function $z_{m,n}$ as $a\to 0$. The function $z_{m,n}$ is determined at nodes of type (m,n) of the discrete rhombic net T^d, which is originally specified in terms of the polygonal (piecewise geodesic) elements $l^x(a)$ and $l^t(a)$ (the angles of a discrete net are denoted by z). Therefore, the construction of a solution of Darboux problem (1.1) for the sine-Gordon equation is reduced to an analysis of a purely planimetric problem on the hyperbolic plane Λ^2. Accordingly, problem (1.1) for the net angles of a regular Chebyshev net is reformulated in terms of the discrete net T^d by setting

$$T^d(a):\qquad \begin{aligned} z_{m,0}&=\phi(ma),\\ z_{0,n}&=\psi(na),\qquad m,n=0,\pm1,\pm2,\dots,\\ z_{0,0}&=\phi(0)=\psi(0).\end{aligned}$$

1.2. *Basic recursive net relation for the discrete rhombic Chebyshev net on Λ^2.* Consider the problem of recursive calculation of the angles $z_{k,l}$ of the net $T^d(a)$ basing on their initial values on the polygonal elements $l^x(a)$ and $l^t(a)$. For this purpose choose a fragment of the discrete (piecewise geodesic) net $T^d(a)$ consisting of four rhombuses of type $R_{k,l}(a)$ (here, the indices denoting a rhombus correspond to the minimal values of the indices of its vertices). Thus, these rhombuses adjoin the $(m+1, n+1)$th node $A_{m+1,n+1}$ (see Figure 1).

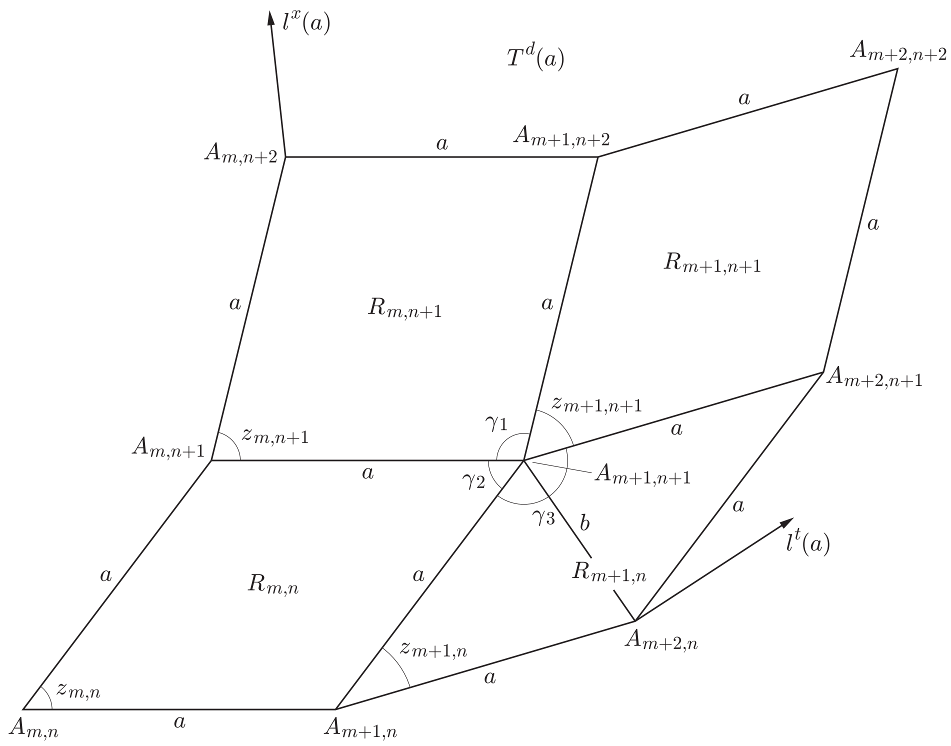

Fig. 1

Since on Λ^2 the total angle is 2π, we have

$$z_{m+1,n+1} = 2\pi - (\gamma_1 + \gamma_2 + \gamma_3), \tag{1.3}$$

where $\gamma_1 = \angle A_{m+1,n+2}A_{m+1,n+1}A_{m,n+1}$, $\gamma_2 = \angle A_{m,n+1}A_{m+1,n+1}A_{m+1,n}$ and $\gamma_3 = \angle A_{m+1,n}A_{m+1,n+1}A_{m+2,n+1}$.

We now calculate the angles γ_1 and γ_3 which correspond to the rhombuses $R_{m,n+1}$ and $R_{m+1,n}$ respectively. To do this, the cosine and sine laws must be formulated for an arbitrary geodesic triangle on the hyperbolic plane Λ^2 with sides a, b, c and angles α, β, γ:

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha, \tag{1.4}$$

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}. \tag{1.5}$$

From (1.4) and (1.5), it follows that opposite interior angles in an arbitrary rhombus $R_{k,l}(a)$ are equal (for example, $\gamma_2 = z_{m,n}$) and the diagonals of a rhombus are the bisectors of its interior angles (in particular, $b \equiv A_{m+1,n+1}A_{m+2,n}$ is the bisector of γ_3). Using (1.4) and then (1.5) for the triangle $\triangle A_{m+1,n}A_{m+1,n+1}A_{m+2,n} \subset R_{m+1,n}$ one obtains

$$\cosh b = \cosh^2 a - \sinh^2 a \cos z_{m+1,n}, \tag{1.6}$$

$$\frac{\sin(\angle A_{m+1,n}A_{m+1,n+1}A_{m+2,n})}{\sinh a} = \frac{\sin z_{m+1,n}}{\sinh b}. \tag{1.7}$$

From (1.6) and (1.7) it follows that

$$\sin(\angle A_{m+1,n}A_{m+1,n+1}A_{m+2,n}) = \frac{\sin z_{m+1,n} \sinh a}{[(\cosh^2 a - \sinh^2 a \cos z_{m+1,n})^2 - 1]^{1/2}}.$$

Introducing the symbol $\Omega_{i,j}$ determined by the relation

$$\sin \Omega_{i,j}(z_{i,j}, a) = \frac{\sin z_{i,j} \ \sinh a}{[(\cosh^2 a - \sinh^2 a \cos z_{i,j})^2 - 1]^{1/2}},$$

one obtains

$$\gamma_3 = 2\Omega_{m+1,n},$$

and in a similar manner,

$$\gamma_1 = 2\Omega_{m,n+1}.$$

Returning to (1.3), one obtains a recursion formula for the calculation of net angles in $T^d(a)$:

$$z_{m+1,n+1} = 2\pi - (z_{m,n} + 2\Omega_{m+1,n} + 2\Omega_{m,n+1}). \tag{1.8}$$

Formula (1.8) is the basic recursion relation for the net $T^d(a)$. In the next subsection, (1.8) will be used for proving that the presented algorithm for integration of the Darboux problem (1.1) is convergent.

1.3. *Convergence of the algorithm.* We prove that the algorithm for constructing a solution of (1.1) in accordance with (1.3) is convergent, i.e. $z_{m,n}(a)$ tends to the exact solution u of the sine-Gordon equation as $a \to 0$. Note that the recursion formula (1.8), which gives the values of $z_{m,n}(a)$ at the nodes of $T^d(a)$, being defined entirely on the two-dimensional manifold Λ^2 of Gaussian curvature $K \equiv -1$, can be interpreted as a difference scheme. Therefore, according to the general theory of difference schemes [9], the convergence of the algorithm can be proved by analyzing the order of approximation and stability of (1.8).

1.3.1. *The order of approximation of the algorithm.* To calculate the order of approximation of the difference scheme (1.8) defined on Λ^2, the discrete function $z_{m,n}(a)$ is replaced by the exact solution $u(x,t)$ of the sine-Gordon equation calculated at the (m,n)th node, and the residue is evaluated.

We substitute into the left-hand side of (1.8) the value of the solution $u(x,t)$ of (1.1) at the $(m+1,n+1)$th node represented as a Taylor expansion in powers of a at the (m,n)th node up to order $O(a^5)$:

$$\begin{aligned}(1.9)\quad u_{m+1,n+1} &= u^* + u_x^* a + u_t^* a + \tfrac{1}{2}u_{xx}^* a^2 + \tfrac{1}{2}u_{tt}^* a^2 + u_{xt}^* a^2 + \tfrac{1}{6}u_{xxx}^* a^3 \\ &+ \tfrac{1}{6}u_{ttt}^* a^3 + \tfrac{1}{2}u_{xxt}^* a^3 + \tfrac{1}{2}u_{xtt}^* a^3 + \tfrac{1}{24}u_{xxxx}^* a^4 + \tfrac{1}{24}u_{tttt}^* a^4 \\ &+ \tfrac{1}{6}u_{xxxt}^* a^4 + \tfrac{1}{4}u_{xxtt}^* a^4 + \tfrac{1}{6}u_{xttt}^* a^4 + O(a^5)\end{aligned}$$

(here the value of u at (m,n) is denoted by $u^* = u_{m,n}$).

To analyze the right-hand side of (1.9) one needs some properties of the functions $\Omega_{m+1,n}$ and $\Omega_{m,n+1}$; denote them by Ω for the moment.

Since $\Omega(z,a)$ is an even function of a we have

$$\left.\frac{\partial^{2p+1}\Omega}{\partial a^{2p+1}}\right|_{a=0} = 0, \qquad p = 1,2,\dots$$

In addition,

$$\frac{\partial\Omega(z,0)}{\partial z} = -\frac{1}{2}, \qquad \frac{\partial^l\Omega(z,0)}{\partial z^l} \equiv 0, \qquad l = 2,3,\dots$$

Taking these properties into account and substituting the exact solution of the sine-Gordon equation into (1.8), one obtains the following representations:

$$\begin{aligned}(1.10)\quad \Omega(u_{m+1,n},a) &= \Omega(u^*,0) + \Omega_z(u^*,0)h_1 + \tfrac{1}{2}\Omega_{aa}(u^*,0)a^2 \\ &+ \tfrac{1}{2}\Omega_{zaa}(u^*,0)h_1a^2 + \tfrac{1}{4}\Omega_{zzaa}(u^*,0)h_1^2a^2 \\ &+ \tfrac{1}{24}\Omega_{aaaa}(u^*,0)a^4 + O(a^5),\end{aligned}$$

where

$$h_1 = u_x^* a + \tfrac{1}{2}u_{xx}^* a^2 + \tfrac{1}{6}u_{xxx}^* a^3 + \tfrac{1}{24}u_{xxxx}^* a^4 + O(a^5),$$

and

$$\begin{aligned}(1.11)\quad \Omega(u_{m,n+1},a) &= \Omega(u^*,0) + \Omega_z(u^*,0)h_2 + \tfrac{1}{2}\Omega_{aa}(u^*,0)a^2 \\ &+ \tfrac{1}{2}\Omega_{zaa}(u^*,0)h_2a^2 + \tfrac{1}{4}\Omega_{zzaa}(u^*,0)h_2^2a^2 \\ &+ \tfrac{1}{24}\Omega_{aaaa}(u^*,0)a^4 + O(a^5),\end{aligned}$$

where

$$h_2 = u_t^* a + \tfrac{1}{2}u_{tt}^* a^2 + \tfrac{1}{6}u_{ttt}^* a^3 + \tfrac{1}{24}u_{tttt}^* a^4 + O(a^5).$$

Calculation of the derivatives of Ω contained in (1.10), (1.11) gives

$$\begin{aligned}&\Omega_z|_{a=0} = -\tfrac{1}{2}, \qquad \Omega_a|_{a=0} = 0, \qquad \Omega_{aa}|_{a=0} = -\tfrac{1}{2}\sin z,\\ (1.12)\quad &\Omega_{zaa}|_{a=0} = \tfrac{1}{2}\cos z, \qquad \Omega_{zzaa}|_{a=0} = \tfrac{1}{2}\sin z,\\ &\Omega_{aaa}|_{a=0} = 0, \qquad \Omega_{aaaa}|_{a=0} = -\tfrac{1}{2}\sin z\,(1 - 6\sin^2(z/2)).\end{aligned}$$

Substituting (1.10)–(1.12) into the left-hand and right-hand sides of (1.8) and invoking the differential consequences of (1.1):

$$u_{xxt} = u_x \cos u, \qquad u_{xxxt} = u_{xx} \cos u - u_x^2 \sin u,$$
$$u_{ttx} = u_t \cos u, \qquad u_{tttx} = u_{tt} \cos u - u_t^2 \sin u,$$

one obtains

$$\begin{aligned} u^*_{xt} - \sin u^* = \big[&\tfrac{1}{4} \cos u^* \, (u^*_{xx} + u^*_{tt}) - \tfrac{1}{4} \sin u^* \, (u^{*2}_x + u^{*2}_t) \\ &+ \tfrac{1}{48} \sin z \, (1 - 6 \sin^2(u^*/2)) \\ &- \tfrac{1}{6} (\cos u^* \, (u^*_{xx} + u^*_{tt}) - \sin u^* \, (u^{*2}_x + u^{*2}_t)) \\ &- \tfrac{1}{4} (\sin u^* \cos u^* - u^*_x u^*_t \sin u^*)\big] a^2 + O(a^3). \end{aligned} \tag{1.13}$$

We point out that the $O(a^0)$ terms arising in the process of derivation of (1.13) vanish:

$$2u^* - 2\pi + 4\Omega(u^*, 0) \equiv 0.$$

Thus, the order of approximation (accuracy) of the difference scheme (1.8) is determined by the estimate

$$u^*_{xt} - \sin u^* = O(a^2), \tag{1.14}$$

which implies the second order of approximation of the Darboux problem (1.1) for the sine-Gordon equation by its difference counterpart (1.8).

1.3.2. *Stability of the difference counterpart of the Darboux problem.* To prove the stability of the difference counterpart of the Darboux problem (1.1),

$$\begin{aligned} &z_{m+1,n+1} = 2\pi - (z_{m,n} + 2\Omega_{m+1,n} + 2\Omega_{m,n+1}), \\ &z_{m,0} = \phi(ma), \qquad z_{0,n} = \psi(na), \qquad \phi(0) = \psi(0), \end{aligned} \tag{1.15}$$

consider a perturbed problem that corresponds to (1.15):

$$\begin{aligned} &\overline{z}_{m+1,n+1} = 2\pi - (\overline{z}_{m,n} + 2\overline{\Omega}_{m+1,n} + 2\overline{\Omega}_{m,n+1}) + a^2 Y_{m,n}(a), \\ &\overline{z}_{m,0} = \phi(ma), \qquad \overline{z}_{0,n} = \psi(na), \qquad \phi(0) = \psi(0), \end{aligned} \tag{1.16}$$

where $\overline{\Omega}_{i,j} = \Omega_{i,j}(\overline{z}_{i,j}, a)$.

Problems (1.15), (1.16) are analyzed in the domain

$$D = l^x[0, B_1] \times l^t[0, B_2] \subset \Lambda^2,$$

defined by the polygonal lines $l^x[0, B_1]$ and $l^t[0, B_2]$ of length B_1 and B_2 respectively. (The length of each polygonal line is evaluated by summing the lengths of its components which are segments of geodesics on the plane Λ^2.)

To prove the stability of the difference problem one has to establish the existence of constants M_1, M_2 satisfying the following condition: for a sufficiently small typical linear size a of the discrete net $T^d(a)$ any perturbation

$Y_{m,n}(a)$ with $\|Y(a)\| \le M_1$ satisfies the estimate

$$\|\overline{z} - z\| \le M_2 \|Y(a)\|. \tag{1.17}$$

Here, the norm of a discrete function $q_{m,n}(a)$ is defined on $T^d(a)$ in a standard manner as the uniform Chebyshev norm:

$$\|q(a)\| = \max_{A_{m,n} \in T^d(a) \subset D} |q_{m,n}(a)|.$$

By introducing a function $Q(z_{i,j}, a)$:

$$Q(z_{i,j}, a) = \pi - 2\Omega_{i,j}(z_{i,j}, a) - z_{i,j},$$

it is easily verified that the recursion formulas in (1.15) and (1.16) can be rewritten as

$$\begin{aligned} z_{m+1,n+1} &= Q(z_{m+1,n}, a) + Q(z_{m,n+1}, a) \\ &\quad + z_{m+1,n} + z_{m,n+1} - z_{m,n}, \end{aligned} \tag{1.18}$$

$$\begin{aligned} \overline{z}_{m+1,n+1} &= Q(\overline{z}_{m+1,n}, a) + Q(\overline{z}_{m,n+1}, a) \\ &\quad + \overline{z}_{m+1,n} + \overline{z}_{m,n+1} - \overline{z}_{m,n} + a^2 Y_{m,n}(a). \end{aligned} \tag{1.19}$$

Now, let $\delta z_{m,n}$ be the difference of solutions of the perturbed problem (1.16) and the original problem (1.15):

$$\delta z_{m,n} = \overline{z}_{m,n} - z_{m,n}.$$

Then, subtracting equations (1.15) from the corresponding equations in (1.16), one obtains

$$\begin{aligned} \delta z_{m+1,n+1} &= Q(\overline{z}_{m+1,n}, a) - Q(z_{m+1,n}, a) \\ &\quad + Q(\overline{z}_{m,n+1}, a) - Q(z_{m,n+1}, a) \\ &\quad + \delta z_{m+1,n} + \delta z_{m,n+1} - \delta z_{m,n} + a^2 Y_{m,n}(a), \end{aligned} \tag{1.20}$$

$$\delta z_{m,0} = 0, \qquad \delta z_{0,n} = 0. \tag{1.21}$$

In (1.20) the differences of the type

$$\Delta Q = Q(\overline{z}_{i,j}, a) - Q(z_{i,j}, a)$$

are expressed by applying the Lagrange theorem as follows:

$$\Delta Q = \frac{\partial Q}{\partial z}(z^0, a)(\overline{z} - z) = -\left[1 + \frac{\partial \Omega}{\partial z}(z^0, a)\right](\overline{z} - z), \qquad z^0 \in [\overline{z}, z]. \tag{1.22}$$

By taking into account the form of Ω (see Subsection 1.2) we can rewrite (1.22) as

$$\Delta Q = R(z^0, a)(\overline{z} - z),$$

where

$$R(z^0, a) = -\frac{1 - \cosh a + \sinh^2 a \, \sin^2(z^0/2)}{1 + \sinh^2 a \, \sin^2(z^0/2)}. \tag{1.23}$$

Hence

$$\cosh a - (1+\sinh^2 a) \le R(z^0,a) \le \cosh a - 1. \tag{1.24}$$

Using the asymptotic representations holding for sufficiently small a:

$$\cosh a = 1 + \frac{a^2}{2} + O(a^4), \qquad \sinh a = a + O(a^3), \qquad \sinh^2 a = a^2 + O(a^4),$$

one obtains

$$|R(z^0,a)| \le \frac{a^2}{2} + \frac{\nu}{2}a^4, \qquad \nu = \mathrm{const} \ge 0. \tag{1.25}$$

Relation (1.20) is now rewritten as

$$\begin{aligned}[]&[\delta z_{m+1,n+1} - \delta z_{m+1,n}] - [\delta z_{m,n+1} - \delta z_{m,n}] \\ &\quad = R(z^0_{m+1,n},a)\delta z_{m+1,n} + R(z^0_{m,n+1},a)\delta z_{m,n+1} + a^2 Y_{m,n}(a).\end{aligned} \tag{1.26}$$

For the initial conditions we get

$$\delta z_{0,n+1} - \delta z_{0,n} = 0.$$

Set $n = n^* \in \{0,\dots,[B_2/a]\}$ ($[\cdot]$ stands for integral part) and sum over the first index in (1.26). Here, m runs from 0 to m^*, where $m^* \in \{0,\dots,[B_1/a]\}$. Then

$$\begin{aligned}&\delta z_{m^*+1,n^*+1} - \delta z_{m^*+1,n^*} \\ &= \sum_{m=0}^{m^*} [R(z^0_{m+1,n^*},a)\delta z_{m+1,n^*} + R(z^0_{m,n^*+1},a)\delta z_{m,n^*+1} + a^2 Y_{m,n^*}(a)].\end{aligned} \tag{1.27}$$

In view of (1.25),

$$\begin{aligned}&|\delta z_{m^*+1,n^*+1} - \delta z_{m^*+1,n^*}| \\ &\quad\le (m^*+1)(a^2+\nu a^4)\|\delta z\|_{m^*+n^*+1} + a^2(m^*+1)\|Y(a)\|.\end{aligned} \tag{1.28}$$

This estimate involves the auxiliary norm

$$\|q(a)\|_N = \max_{A_{m,n}\in T^d(a)\subset D:m+n\le N} |q_{m,n}(a)|.$$

Using the estimate $|\cdot| \le \|\cdot\|_N$ we rearrange the left-hand side of (1.28) as

$$\begin{aligned}&|\delta z_{m^*+1,n^*+1}| \\ &\le (m^*+1)(a^2+\nu a^4)\|\delta z\|_{m^*+n^*+1} + \|\delta z\|_{m^*+n^*+1} + a^2(m^*+1)\|Y(a)\|.\end{aligned} \tag{1.29}$$

Introduce the numerical parameter $N = m^*+n^*+1$. Note that inequalities of type (1.29) hold for every pair of indices such that $m+n+1 \le N$. Furthermore, either the norm $\|\delta z\|_{N+1}$ is attained by $|\delta z_{m+1,n+1}|$ when $m+n+2 = N+1$ (in this case, the left-hand side of (1.29) can be replaced by $\|\delta z\|_{N+1}$), or there exist $m = \overline{m}$ and $n = \overline{n}$ satisfying $\overline{m}+\overline{n}+1 \le N$ such that

$$|\delta z_{\overline{m}+1,\overline{n}+1}| = \|\delta z\|_{N+1},$$

and, therefore,

$$\begin{aligned}(1.30)\qquad &|\delta z_{\overline{m}+1,\overline{n}+1}|\\ &\le (\overline{m}+1)(a^2+\nu a^4)\|\delta z\|_{\overline{m}+\overline{n}+1} + \|\delta z\|_{\overline{m}+\overline{n}+1} + a^2(\overline{m}+1)\|Y(a)\|.\end{aligned}$$

Since $\|\cdot\|_{\overline{m}+\overline{n}+1} \le \|\cdot\|_N$ and $(\overline{m}+1)a \le B_1$ the foregoing analysis performed for (1.30) leads to the final recursive estimate

$$(1.31)\qquad \|\delta z\|_{N+1} \le B_1(a+\nu a^3)\|\delta z\|_N + \|\delta z\|_N + aB_1\|Y(a)\|,$$

where $N = 1,\dots,m+n+1$ or $N \in \{1,\dots,[B_1/a]+[B_2/a]-1\}$.

To analyze (1.31) rewrite it as

$$(1.32)\qquad \|\delta z\|_{N+1} \le \varrho_{N+1},$$

where

$$(1.33)\qquad \varrho_{N+1} = B_1(a+\nu a^3)\varrho_N + \varrho_N + aB_1\|Y(a)\|.$$

Note that, since $\|\delta z\|_0 = \|\delta z\|_1 = 0$ the initial value is

$$(1.34)\qquad \varrho_0 = 0.$$

Using (1.34) we calculate ϱ_{N+1} from (1.33):

$$(1.35)\qquad \varrho_{N+1} = B_1 a\|Y(a)\| \sum_{p=0}^{N} [1+B_1(a+\nu a^3)]^{N-p}.$$

The power terms in (1.35) can be estimated from above by the corresponding exponents (i.e. $(1+t)^k \le e^{kt},\ k \ge 0,\ t > 0$). Therefore,

$$\begin{aligned}(1.36)\qquad \varrho_{N+1} &\le B_1 a\|Y(a)\| \sum_{p=0}^{N} \exp\{(N-p)aB_1(1+\nu a^2)\}\\ &\le \Big(aB_1 \exp\{B_1(B_1+B_2)(1+\nu a^2)\}\\ &\quad\times \sum_{p=0}^{N} \exp\{-pa(1+\nu a^2)B_1\}\Big)\|Y(a)\|.\end{aligned}$$

The sum on the right-hand side of (1.36) can be interpreted (when the elementary segment of a partition of $[0,N]$ is of unit length) as a lower Darboux integral sum and estimated from above by the corresponding definite integral:

$$\sum_{p=0}^{N} \exp\{-pa(1+\nu a^2)B_1\} \le 1 + \int_0^N \exp\{-aB_1(1+\nu a^2)p\}\,dp,$$

Calculating this integral, one refines estimate (1.36):

$$(1.37)\qquad \varrho_{N+1} \le M\|Y(a)\|,$$

where (since $a \le B_1$)

$$M = (1 + B_1^2)\exp\{B_1(B_1 + B_2)(1 + \nu B_1^2)\} = \text{const}.$$

From (1.32) and (1.37), it follows that

$$\|\delta z\|_{N+1} \le M\|Y(a)\|, \qquad M = \text{const}. \tag{1.38}$$

Inequality (1.38) holds for any value of N from the range $\{1, \dots, [B_1/a] + [B_2/a] - 1\}$, including $N = \overline{N} = [B_1/a] + [B_2/a] - 1\}$. At the same time, note that

$$\|\delta z\|_{\overline{N}+1} = \|\delta z\|.$$

By (1.38) this yields the final estimate

$$\|\delta z\| \le M\|Y(a)\|, \qquad M = \text{const}, \tag{1.39}$$

which proves the stability of the difference (discrete) counterpart (1.15) of the Darboux problem (1.1) for the sine-Gordon equation.

1.3.3. *Convergence.* In Sections 1.3.1 and 1.3.2 the a priori estimates (1.14) and (1.39) were proved for the approximation error and stability of the difference scheme (1.8). According to the general theory of difference schemes [9] these estimates establish the convergence of the presented algorithm: the solution $z_{m,n}(a)$ of problem (1.15) tends to the exact solution u of the Darboux problem (1.1) for the sine-Gordon equation as $a \to 0$. Since the difference problem under consideration is well-posed, the result obtained also proves that the presented geometric approach to numerical integration, based on the concept of Λ^2-representations of nonlinear equations [3–5], is correct. From the geometric point of view, the convergence of the algorithm is associated with a "process of smoothing" of the discrete net $T^d(a)$ as $a \to 0$. In the limit one obtains a regular Chebyshev net that generates (according to the theory of Λ^2-representations) the sine-Gordon equation.

2. Λ^2-representations and inverse scattering problem (ISP). In this section we announce a result associating the existence of a Λ^2-representation for some PDE with the possibility of its integration by the method of ISP [1, 12, 14].

2.1. *General remarks on ISP.* The ISP method of integration assumes that with the PDE

$$f[u] = 0 \tag{2.1}$$

there is associated a certain linear system of differential equations of the type

$$\psi_x = U\psi, \qquad \psi_t = V\psi, \tag{2.2}$$

where U, V are 2×2-matrix operators, and ψ is a vector-valued function with components ψ_1 and ψ_2.

The system (2.2) is called a *spectral-evolutionary problem* for (2.1). Namely, the consistency condition for the system (2.2),

$$U_t - V_x + [U,V] = 0 \qquad (\text{where } [U,V] = UV - VU) \tag{2.3}$$

is equivalent [12] to equation (2.1).

Constructing a system of type (2.2) for (2.1) is an important starting point for the possible integration of (2.1) by the ISP method. No general methods to construct U and V are available.

In the next subsection we present exact formulas for U and V from the Λ^2-representation of the PDE considered.

2.2. *Λ^2-representations and the spectral-evolutionary problem.* The following theorem links the pseudospherical metrics, generating a given PDE (or Λ^2-representation) with the operators of problem (2.2).

THEOREM. *Suppose that the PDE* (2.1) *allows some Λ^2-representation*:

$$\Lambda^2[f[u(x,t)]=0] = \{E[u(x,t)], F[u(x,t)], G[u(x,t)]\}.$$

Then the operators U and V from the corresponding spectral-evolutionary problem (2.2) *are given by the following expressions* (*up to gauge transformations*):

$$\begin{aligned} U &= \begin{pmatrix} \frac{i}{2}\widetilde{a} & \frac{1}{2}E^{1/2}e^{i\Theta^+} \\ \frac{1}{2}E^{1/2}e^{-i\Theta^+} & -\frac{i}{2}\widetilde{a} \end{pmatrix}, \\ V &= \begin{pmatrix} \frac{i}{2}\widetilde{b} & \frac{1}{2}G^{1/2}e^{i\Theta^-} \\ \frac{1}{2}G^{1/2}e^{-i\Theta^-} & -\frac{i}{2}\widetilde{b} \end{pmatrix}. \end{aligned} \tag{2.4}$$

Here, the functions $\widetilde{a},\widetilde{b},\Theta^+,\Theta^-$ are defined from the Λ^2-representation as

$$\begin{aligned} \widetilde{a} &= \frac{1}{2W^{1/2}}\left[\frac{1}{2}\frac{FG_x}{G} - \frac{1}{2}\frac{FE_x}{E} + F_x - E_t\right] + \Theta_x, \\ \widetilde{b} &= \frac{1}{2W^{1/2}}\left[\frac{1}{2}\frac{FG_t}{G} - \frac{1}{2}\frac{FE_t}{E} - F_t + G_x\right] + \Theta_t \qquad (W = EG - F^2), \\ \Theta^{\pm} &= \Theta \pm \frac{1}{2}\arccos\left[\frac{F}{(EG)^{1/2}}\right], \end{aligned}$$

where $\Theta = \Theta(x,t)$ is an arbitrary C^2 function.

Note that the arbitrary function $\Theta(x,t)$ in $\Theta^{\pm}$ has the meaning of a gauge transformation. In fact, as a result of the transform $(\psi_1,\psi_2)\mapsto(\overline{\psi}_1,\overline{\psi}_2)$ determined by the matrix

$$\begin{pmatrix} e^{-i\Theta/2} & 0 \\ 0 & e^{i\Theta/2} \end{pmatrix}$$

new operators U, V independent of $\Theta(x,t)$ appear.

The validity of this theorem can be proved, for example, by the direct substitution of the operators U and V of (2.4) into the consistency equation (2.3).

It should be pointed out that to obtain (2.4) (see [13]), we apply the general analogy of the basic structural equations of pseudospherical surfaces (with metric $ds^2 = (\omega^1)^2 + (\omega^2)^2$) in Euclidean space E^3 [2, 10]:

$$\begin{aligned} d\omega^1 &= \omega^2 \wedge \omega_{21}, \\ d\omega^2 &= \omega_{21} \wedge \omega^1, \\ d\omega_{21} &= -\omega^1 \wedge \omega^2, \end{aligned}$$

where

$$\begin{aligned} \omega^1 &= E^{1/2} \cos \Theta^+ dx + G^{1/2} \cos \Theta^- dt, \\ \omega^2 &= E^{1/2} \sin \Theta^+ dx + G^{1/2} \sin \Theta^- dt, \\ \omega_{21} &= \widetilde{a} dx + \widetilde{b} dt, \end{aligned}$$

with equations (2.1) (and respectively (2.3)).

Thus for the operators U, V presented in the Theorem the relation (2.3) is equivalent to the Gauss equation $\widetilde{b}_x - \widetilde{a}_t = W^{1/2}$ (or $d\omega_{21} = -\omega^1 \wedge \omega^2$) generating the Λ^2-representation for (2.1).

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Reçu par la Rédaction le 20.7.1999
Révisé le 26.10.2000