Harmonic functions in a cylinder with normal derivatives vanishing on the boundary
by Ikuko Miyamoto and Hidenobu Yoshida (Chiba)

Dedicated to the memory of Professor Bogdan Ziemian

Abstract. A harmonic function in a cylinder with the normal derivative vanishing on the boundary is expanded into an infinite sum of certain fundamental harmonic functions. The growth condition under which it is reduced to a finite sum of them is given.

1. Introduction. Let $\mathbb{R}^n$ ($n \geq 2$) denote the $n$-dimensional Euclidean space. The solution of the Neumann problem for an infinite cylinder

$$\Gamma_n(D) = \{(X, y) \in \mathbb{R}^n : X \in D, -\infty < y < \infty\},$$

with $D$ a bounded domain of $\mathbb{R}^{n-1}$, is not unique, because we can add to each solution harmonic functions in $\Gamma_n(D)$ with normal derivatives vanishing on the boundary. Hence, to classify general solutions we need to characterize such functions. If $D = (0, \pi)$ and $\Gamma_n(D)$ is the strip

$$H = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi, -\infty < y < \infty\},$$

then by applying a result of Widder [6, Theorem 2] which characterizes a harmonic function in $H$ vanishing continuously on the boundary $\partial H$ of $H$, we can obtain the following result:

**Theorem A.** Let $h(x, y)$ be a harmonic function in $H$ such that $\partial h/\partial x$ vanishes continuously on $\partial H$. Then

$$h(x, y) = A_0 y + B_0 + \sum_{k=1}^{\infty} (A_k e^{ky} + B_k e^{-ky}) \cos kx,$$

where the series converges for all $x$ and $y$, and $A_0, B_0, A_1, B_1, A_2, B_2, \ldots$ are

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constants such that

\[ A_k e^{ky} + B_k e^{-ky} = \frac{2}{\pi} \int_0^\pi h(x,y)kx \, dx \quad (k = 1, 2, \ldots). \]

Although this theorem is easily proved, we cannot proceed similarly in the case where \( \Gamma_n(D) \) is a cylinder in \( \mathbb{R}^n \) \( (n \geq 3) \). This kind of problem was originally treated by Bouligand [1] in 1914.

**Theorem B** (Bouligand [1, p. 195]). Let \( h(X, y) \) be a harmonic function in \( \Gamma_n(D) \) such that the normal derivative of \( h \) vanishes continuously on the boundary \( \partial \Gamma_n(D) \) of \( \Gamma_n(D) \). If \( h(X, y) \) tends to zero as \( |y| \to \infty \), then \( h(X, y) \) is identically zero in \( \Gamma_n(D) \).

In this paper we shall prove a cylindrical version of Theorem A (Theorem). As corollaries we shall obtain two results generalizing Theorem B (Corollaries 1 and 2).

2. **Preliminaries.** Let \( D \) be a bounded domain in \( \mathbb{R}^{n-1} \) \( (n \geq 3) \) having a sufficiently smooth boundary \( \partial D \). For example, \( D \) can be a \( C^{2,\alpha} \)-domain \( (0 < \alpha < 1) \) in \( \mathbb{R}^{n-1} \) bounded by a finite number of mutually disjoint closed hypersurfaces (see Gilbarg and Trudinger [3, pp. 88–89] for the definition of \( C^{2,\alpha} \)-domain). Consider the Neumann problem

\[ (\Delta_{n-1} + \mu)\varphi(X) = 0 \quad \text{for any } X = (x_1, \ldots, x_{n-1}) \in D, \]
\[ \lim_{X \to X', X \in D} (\nabla_{n-1}\varphi(X), \nu(X')) = 0 \]

for any \( X' \in \partial D \), where

\[ \Delta_{n-1} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}, \quad \nabla_{n-1} = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}} \right) \]

and \( \nu(X') \) is the outer unit normal vector at \( X' \in \partial D \).

Let \( \{\mu_k(D)\}_{k=0}^\infty \) be the non-decreasing sequence of non-negative eigenvalues of this Neumann problem. In this sequence we write \( \mu_k(D) \) the number of times equal to the dimension of the corresponding eigenspace. If the normalized eigenfunction corresponding to \( \mu_k(D) \) is denoted by \( \varphi_k(D)(X) \), the set of consecutive eigenfunctions corresponding to the same value of \( \mu_k(D) \) in the sequence \( \{\varphi_k(D)(X)\}_{k=0}^\infty \) forms an orthonormal basis for the eigenspace of the eigenvalue \( \mu_k(D) \). It is evident that \( \mu_0(D) = 0 \) and

\[ \varphi_0(D)(X) = |D|^{-1/2} \quad (X \in D), \quad |D| = \int_D dX. \]

In the following we shall denote \( \{\mu_k(D)\}_{k=0}^\infty \) and \( \{\varphi_k(D)(X)\}_{k=0}^\infty \) by \( \{\mu(k)\}_{k=0}^\infty \) and \( \{\varphi_k(X)\}_{k=0}^\infty \) respectively, without specifying \( D \). For each \( D \)
there is a sequence \( \{k_i\} \) of non-negative integers such that \( k_0 = 0, k_1 = 1, \mu(k_i) < \mu(k_{i+1}) \),
\[
\mu(k_i) = \mu(k_i + 1) = \mu(k_i + 2) = \ldots = \mu(k_{i+1} - 1)
\]
and \( \{\varphi_{k_i}, \varphi_{k_i + 1}, \ldots, \varphi_{k_{i+1}-1}\} \) is an orthonormal basis for the eigenspace of the eigenvalue \( \mu(k_i) \) \( (i = 0, 1, 2, \ldots) \). Since \( D \) has a sufficiently smooth boundary, we know that
\[
\mu(k) \sim A(D, n)k^{2/(n-1)} \quad (k \to \infty)
\]
and
\[
\sum_{\mu(k) \leq t} (\varphi_k(X))^2 \sim B(D, n)t^{(n-1)/2} \quad (t \to \infty)
\]
uniformly with respect to \( X \in D \), where \( A(D, n) \) and \( B(D, n) \) are constants depending on \( D \) and \( n \) (e.g. see Carleman [2], Minakshisundaram and Pleijel [4], Weyl [5]). Hence there exist positive constants \( M_1, M_2 \) such that
\[
M_1k^{2/(n-1)} \leq \mu(k) \quad (k = 1, 2, \ldots)
\]
and
\[
|\varphi_k(X)| \leq M_2k^{1/2} \quad (X \in D, k = 1, 2, \ldots).
\]

3. Statement of our results. The gradient of a function \( f(P) \) defined on \( \Gamma_n(D) \) is
\[
\nabla_n f(P) = \left( \frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_{n-1}}(P), \frac{\partial f}{\partial y}(P) \right)
\]
\( (P = (x_1, \ldots, x_{n-1}, y) \in \Gamma_n(D)) \). We first remark that
\[
I_k(P) = e^{\sqrt{\mu(k)}y}\varphi_k(X) \quad \text{and} \quad J_k(P) = e^{-\sqrt{\mu(k)}y}\varphi_k(X)
\]
\( (P = (X, y) \in \Gamma_n(D)) \) are harmonic functions on \( \Gamma_n(D) \) satisfying
\[
\lim_{P \to Q, P \in \Gamma_n(D)} (\nabla_n I_k(P), \nu(Q)) = 0
\]
and
\[
\lim_{P \to Q, P \in \Gamma_n(D)} (\nabla_n J_k(P), \nu(Q)) = 0,
\]
where \( \nu(Q) \) is the outer unit normal vector at \( Q \in \partial \Gamma_n(D) \).

Theorem. Let \( h(P) \) be a harmonic function on \( \Gamma_n(D) \) satisfying
\[
(3.1) \quad \lim_{P \to Q, P \in \Gamma_n(D)} (\nabla_n h(P), \nu(Q)) = 0
\]
for any \( Q \in \partial \Gamma_n(D) \). Then
\[
h(P) = A_0y + B_0 + \sum_{k=1}^{\infty} (A_kI_k(P) + B_kJ_k(P))
\]
for any \( P = (X, y) \in \Gamma_n(D) \), where the series converges uniformly and absolutely on any compact subset of the closure \( \Gamma_n(D) \) of \( \Gamma_n(D) \), and \( A_k, B_k \) \((k = 0, 1, 2, \ldots)\) are constants such that

\[
A_k e^{\sqrt{\mu(k)} y} + B_k e^{-\sqrt{\mu(k)} y} = \int_D h(X, y) \varphi_k(X) \, dX \quad (k = 1, 2, \ldots).
\]

**Corollary 1.** Let \( p \) and \( q \) be non-negative integers. If \( h(P) \) is a harmonic function on \( \Gamma_n(D) \) satisfying (3.1) and

\[
\lim_{y \to -\infty} e^{\sqrt{\mu(k_p+1)} y} M_h(y) = 0, \quad \lim_{y \to -\infty} e^{-\sqrt{\mu(k_q+1)} y} M_h(y) = 0,
\]

where

\[
M_h(y) = \sup_{X \in D} |h(X, y)| \quad (\infty < y < \infty),
\]

then

\[
h(P) = A_0 y + B_0 + \sum_{k=1}^{k_{p+1}-1} A_k I_k(P) + \sum_{k=1}^{k_{q+1}-1} B_k J_k(P)
\]

for any \( P = (X, y) \in \Gamma_n(D) \), where \( A_k \) \((k = 0, 1, \ldots, k_{p+1} - 1)\) and \( B_k \) \((k = 0, 1, \ldots, k_{q+1} - 1)\) are constants.

**Corollary 2.** Let \( h(P) \) be a harmonic function on \( \Gamma_n(D) \) satisfying (3.1) and

\[
M_h(y) = o(e^{\sqrt{\mu(1)} |y|}) \quad (|y| \to \infty).
\]

Then \( h(P) = A_0 y + B_0 \) for any \( P = (X, y) \in \Gamma_n(D) \), where \( A_0 \) and \( B_0 \) are constants.

**4. Proofs of Theorem and Corollaries 1, 2.** Let \( f(X, y) \) be a function on \( \Gamma_n(D) \). The function \( c_k(f, y) \) of \( y \) \((\infty < y < \infty)\) defined by

\[
c_k(f, y) = \int_D f(X, y) \varphi_k(X) \, dX
\]

is simply denoted by \( c_k(y) \) in the following, without specifying \( f \).

**Lemma 1.** Let \( h(P) \) be a harmonic function on \( \Gamma_n(D) \) satisfying (3.1). Then

\[
c_0(y) = A_0 y + B_0,
\]

\[
c_k(y) = A_k e^{\sqrt{\mu(k)} y} + B_k e^{-\sqrt{\mu(k)} y} \quad (k = 1, 2, \ldots)
\]

with constants \( A_k, B_k \) \((k \geq 0)\) and
from Green’s identity, (2.2) and (3.1), we have
\[ c_k(y) = \frac{\{e^{\sqrt{\mu(k)}(y-y_2)} - e^{\sqrt{\mu(k)}(y_2-y)}\}c_k(y_1)}{e^{\sqrt{\mu(k)}(y_1-y)} - e^{\sqrt{\mu(k)}(y_2-y)}} + \frac{\{e^{\sqrt{\mu(k)}(y_1-y)} - e^{\sqrt{\mu(k)}(y-y_1)}\}c_k(y_2)}{e^{\sqrt{\mu(k)}(y_1-y)} - e^{\sqrt{\mu(k)}(y_2-y)}}\]
for any \( y_1 \) and \( y_2, -\infty < y_1 < y_2 < \infty \) \((k = 1, 2, 3, \ldots)\).

**Proof.** First of all, we remark that \( h \in C^2(\overline{T_n(D)}) \) (Gilbarg and Trudinger [3, p. 124]). Since
\[
\int_D (\Delta_{n-1} h(X,y)) \varphi_k(X) \, dX = \int_D h(X,y)(\Delta_{n-1} \varphi_k(X)) \, dX \quad (-\infty < y < \infty),
\]
from Green’s identity, (2.2) and (3.1), we have
\[
\frac{\partial^2 c_k(y)}{\partial y^2} = \int_D \frac{\partial^2 h(X,y)}{\partial y^2} \varphi_k(X) \, dX = -\int_D \Delta_{n-1} h(X,y) \varphi_k(X) \, dX
\]
\[
= -\int_D h(X,y)(\Delta_{n-1} \varphi_k(X)) \, dX
\]
\[
= \mu(k) \int_D h(X,y) \varphi_k(X) \, dX = \mu(k)c_k(y)
\]
from (2.1) \((k = 0, 1, 2, \ldots)\). With constants \( A_k \) and \( B_k \) \((k = 0, 1, 2, \ldots)\) these give
\[
c_0(y) = A_0 y + B_0
\]
and
\[
c_k(y) = A_k e^{\sqrt{\mu(k)} y} + B_k e^{-\sqrt{\mu(k)} y} \quad (k = 1, 2, \ldots),
\]
which are (4.1) and (4.2). When we solve for \( A_k \) and \( B_k \) the equations
\[
c_k(y_i) = A_k e^{\sqrt{\mu(k)} y_i} + B_k e^{-\sqrt{\mu(k)} y_i} \quad (i = 1, 2),
\]
we immediately obtain (4.3).

**Remark.** From (4.2) we have, for \( k = 1, 2, \ldots \)
\[
\lim_{y \to \infty} c_k(y) e^{-\sqrt{\mu(k)} y} = A_k \quad \text{and} \quad \lim_{y \to -\infty} c_k(y) e^{\sqrt{\mu(k)} y} = B_k.
\]

**Lemma 2.** Let \( h(P) \) be a harmonic function on \( \Gamma_n(D) \) satisfying (3.1). Let \( y \) be any number and \( y_1, y_2 \) be two any numbers satisfying \(-\infty < y_1 < y - 1, y + 1 < y_2 < \infty\). For two non-negative integers \( p \) and \( q \),
\[
\sum_{k=k_p+q+1}^{\infty} |c_k(y)| \cdot |\varphi_k(X)| \leq L(p)M_n(y_1) + L(q)M_n(y_2),
\]
where
\[ L(j) = M_2 |D| \sum_{k=k_j+1}^{\infty} k \exp(-\sqrt{M_1 k^{1/(n-1)}}). \]

**Proof.** From Lemma 1, we see that
\[
\begin{align*}
c_k(y) &= \exp\{-\sqrt{\mu(k)}(y-y_1)\} \frac{1 - \exp\{2\sqrt{\mu(k)}(y-y_2)\}}{1 - \exp\{2\sqrt{\mu(k)}(y_1-y)\}} c_k(y) \\
&\quad + \exp\{\sqrt{\mu(k)}(y-y_2)\} \frac{1 - \exp\{2\sqrt{\mu(k)}(y_1-y)\}}{1 - \exp\{2\sqrt{\mu(k)}(y_1-y_2)\}} c_k(y_2).
\end{align*}
\]
Hence
\[ \sum_{k=k_{p+q+1}}^{\infty} |c_k(y)| \cdot |\varphi_k(X)| \leq I_1 + I_2, \tag{4.4} \]
where
\[
I_1 = \sum_{k=k_{p+1}}^{\infty} \exp\{-\sqrt{\mu(k)}(y-y_1)\} |c_k(y_1)| \cdot |\varphi_k(X)|
\]
\[
I_2 = \sum_{k=k_{q+1}}^{\infty} \exp\{-\sqrt{\mu(k)}(y_2-y)\} |c_k(y_2)| |\varphi_k(X)|.
\]
For \( I_1 \), we have
\[ I_1 \leq M_2^2 |D| M_h(y_1) \sum_{k=k_{p+1}}^{\infty} k \exp(-\sqrt{\mu(k)}) \]
\[ \leq M_2^2 |D| M_h(y_1) \sum_{k=k_{p+1}}^{\infty} k \exp(-\sqrt{M_1 k^{1/(n-1)}}), \tag{4.5} \]
because \( y - y_1 > 1 \).

For \( I_2 \), we also have
\[ I_2 \leq M_2^2 |D| M_h(y_2) \sum_{k=k_{q+1}}^{\infty} k \exp(-\sqrt{M_1 k^{1/(n-1)}}). \tag{4.6} \]
Finally (4.4)–(4.6) give the conclusion of the lemma.

**Proof of Theorem.** Take any compact set \( T \subset \overline{T_n(D)} \) and two numbers \( y_1, y_2 \) satisfying
\[
\max\{y : (X, y) \in T\} + 1 < y_2, \quad \min\{y : (X, y) \in T\} - 1 > y_1.
\]
Let \((X, y)\) be any point in \( T \). Since \( c_k(y) \) is the Fourier coefficient of the function \( h(X, y) \) of \( X \) with respect to the orthonormal sequence \( \{\varphi_k(X)\}_{k=0}^{\infty} \),
we have
\[ h(X, y) = \sum_{k=0}^{\infty} c_k(y) \varphi_k(X) \]
where the series converges uniformly and absolutely on \( T \) by Lemma 2. Further (4.1) and (4.2) of Lemma 1 give (3.2). The proof of the Theorem is complete.

**Proof of Corollaries 1 and 2.** From (3.3) and the Remark, it follows that \( A_k = 0 \) for any \( k \geq k_{p+1} \) and \( B_k = 0 \) for any \( k \geq k_{q+1} \). Hence the Theorem immediately gives the conclusion of Corollary 1. By putting \( p = q = 0 \) in Corollary 1, we obtain Corollary 2 at once.

**References**


Department of Mathematics and Informatics
Chiba University
Chiba 263-8522, Japan
E-mail: miyamoto@math.s.chiba-u.ac.jp
yoshida@math.s.chiba-u.ac.jp

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