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Harmonic functions in a cylinder with normal derivatives vanishing on the boundary

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Dedicated to the memory of Professor Bogdan Ziemian

Abstract. A harmonic function in a cylinder with the normal derivative vanishing on the boundary is expanded into an infinite sum of certain fundamental harmonic functions. The growth condition under which it is reduced to a finite sum of them is given.

1. Introduction. Let $\mathbb{R}^n$ $(n \geq 2)$ denote the n-dimensional Euclidean space. The solution of the Neumann problem for an infinite cylinder

$$\Gamma_n(D) = \{(X, y) \in \mathbb{R}^n : X \in D,\ -\infty < y < \infty\},$$

with D a bounded domain of $\mathbb{R}^{n-1}$, is not unique, because we can add to each solution harmonic functions in $\Gamma_n(D)$ with normal derivatives vanishing on the boundary. Hence, to classify general solutions we need to characterize such functions. If $D = (0, \pi)$ and $\Gamma_n(D)$ is the strip

$$H = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi,\ -\infty < y < \infty\},$$

then by applying a result of Widder [6, Theorem 2] which characterizes a harmonic function in H vanishing continuously on the boundary ∂H of H, we can obtain the following result:

Theorem A. *Let $h(x, y)$ be a harmonic function in H such that $\partial h/\partial x$ vanishes continuously on ∂H. Then*

$$h(x, y) = A_0 y + B_0 + \sum_{k=1}^{\infty} (A_k e^{ky} + B_k e^{-ky}) \cos kx,$$

where the series converges for all x and y, and $A_0, B_0, A_1, B_1, A_2, B_2, \ldots$ are

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constants such that

$$A_k e^{ky} + B_k e^{-ky} = \frac{2}{\pi}\int_0^{\pi} h(x,y) kx\,dx \qquad (k = 1, 2, \ldots).$$

Although this theorem is easily proved, we cannot proceed similarly in the case where $\Gamma_n(D)$ is a cylinder in $\mathbb{R}^n$ ($n \geq 3$). This kind of problem was originally treated by Bouligand [1] in 1914.

THEOREM B (Bouligand [1, p. 195]). *Let $h(X, y)$ be a harmonic function in $\Gamma_n(D)$ such that the normal derivative of h vanishes continuously on the boundary $\partial\Gamma_n(D)$ of $\Gamma_n(D)$. If $h(X, y)$ tends to zero as $|y| \to \infty$, then $h(X, y)$ is identically zero in $\Gamma_n(D)$.*

In this paper we shall prove a cylindrical version of Theorem A (Theorem). As corollaries we shall obtain two results generalizing Theorem B (Corollaries 1 and 2).

2. Preliminaries. Let D be a bounded domain in $\mathbb{R}^{n-1}$ ($n \geq 3$) having a sufficiently smooth boundary ∂D. For example, D can be a $C^{2,\alpha}$-domain ($0 < \alpha < 1$) in $\mathbb{R}^{n-1}$ bounded by a finite number of mutually disjoint closed hypersurfaces (see Gilbarg and Trudinger [3, pp. 88–89] for the definition of $C^{2,\alpha}$-domain). Consider the Neumann problem

$$(\Delta_{n-1} + \mu)\varphi(X) = 0 \tag{2.1}$$

for any $X = (x_1, \ldots, x_{n-1}) \in D$,

$$\lim_{X \to X',\, X \in D} (\nabla_{n-1}\varphi(X), \nu(X')) = 0 \tag{2.2}$$

for any $X' \in \partial D$, where

$$\Delta_{n-1} = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_{n-1}^2}, \qquad \nabla_{n-1} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}}\right)$$

and $\nu(X')$ is the outer unit normal vector at $X' \in \partial D$.

Let $\{\mu_k(D)\}_{k=0}^{\infty}$ be the non-decreasing sequence of non-negative eigenvalues of this Neumann problem. In this sequence we write $\mu_k(D)$ the number of times equal to the dimension of the corresponding eigenspace. If the normalized eigenfunction corresponding to $\mu_k(D)$ is denoted by $\varphi_k(D)(X)$, the set of consecutive eigenfunctions corresponding to the same value of $\mu_k(D)$ in the sequence $\{\varphi_k(D)(X)\}_{k=0}^{\infty}$ forms an orthonormal basis for the eigenspace of the eigenvalue $\mu_k(D)$. It is evident that $\mu_0(D) = 0$ and

$$\varphi_0(D)(X) = |D|^{-1/2} \quad (X \in D), \qquad |D| = \int_D dX.$$

In the following we shall denote $\{\mu_k(D)\}_{k=0}^{\infty}$ and $\{\varphi_k(D)(X)\}_{k=0}^{\infty}$ by $\{\mu(k)\}_{k=0}^{\infty}$ and $\{\varphi_k(X)\}_{k=0}^{\infty}$ respectively, without specifying D. For each D

there is a sequence $\{k_i\}$ of non-negative integers such that $k_0 = 0$, $k_1 = 1$, $\mu(k_i) < \mu(k_{i+1})$,

$$\mu(k_i) = \mu(k_i + 1) = \mu(k_i + 2) = \ldots = \mu(k_{i+1} - 1)$$

and $\{\varphi_{k_i}, \varphi_{k_i+1}, \ldots, \varphi_{k_{i+1}-1}\}$ is an orthonormal basis for the eigenspace of the eigenvalue $\mu(k_i)$ $(i = 0, 1, 2, \ldots)$. Since D has a sufficiently smooth boundary, we know that

$$\mu(k) \sim A(D, n)k^{2/(n-1)} \quad (k \to \infty)$$

and

$$\sum_{\mu(k)\le t} \{\varphi_k(X)\}^2 \sim B(D, n)t^{(n-1)/2} \quad (t \to \infty)$$

uniformly with respect to $X \in D$, where $A(D, n)$ and $B(D, n)$ are constants depending on D and n (e.g. see Carleman [2], Minakshisundaram and Pleijel [4], Weyl [5]). Hence there exist positive constants M_1, M_2 such that

$$M_1 k^{2/(n-1)} \le \mu(k) \quad (k = 1, 2, \ldots)$$

and

$$|\varphi_k(X)| \le M_2 k^{1/2} \quad (X \in D,\ k = 1, 2, \ldots).$$

3. Statement of our results. The gradient of a function $f(P)$ defined on $\Gamma_n(D)$ is

$$\nabla_n f(P) = \left(\frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_{n-1}}(P), \frac{\partial f}{\partial y}(P)\right)$$

$(P = (x_1, \ldots, x_{n-1}, y) \in \Gamma_n(D))$. We first remark that

$$I_k(P) = e^{\sqrt{\mu(k)}y}\varphi_k(X) \quad \text{and} \quad J_k(P) = e^{-\sqrt{\mu(k)}y}\varphi_k(X)$$

$(P = (X, y) \in \Gamma_n(D))$ are harmonic functions on $\Gamma_n(D)$ satisfying

$$\lim_{P\to Q,\, P\in\Gamma_n(D)} (\nabla_n I_k(P), \nu(Q)) = 0$$

and

$$\lim_{P\to Q,\, P\in\Gamma_n(D)} (\nabla_n J_k(P), \nu(Q)) = 0,$$

where $\nu(Q)$ is the outer unit normal vector at $Q \in \partial\Gamma_n(D)$.

THEOREM. *Let $h(P)$ be a harmonic function on $\Gamma_n(D)$ satisfying*

$$\lim_{P\to Q,\, P\in\Gamma_n(D)} (\nabla_n h(P), \nu(Q)) = 0 \tag{3.1}$$

for any $Q \in \partial\Gamma_n(D)$. Then

$$h(P) = A_0 y + B_0 + \sum_{k=1}^{\infty} (A_k I_k(P) + B_k J_k(P))$$

for any $P = (X, y) \in \Gamma_n(D)$, *where the series converges uniformly and absolutely on any compact subset of the closure* $\overline{\Gamma_n(D)}$ *of* $\Gamma_n(D)$, *and* A_k, B_k $(k = 0, 1, 2, \ldots)$ *are constants such that*

$$A_k e^{\sqrt{\mu(k)}\, y} + B_k e^{-\sqrt{\mu(k)}\, y} = \int_D h(X, y)\varphi_k(X)\, dX \qquad (k = 1, 2, \ldots). \tag{3.2}$$

Corollary 1. *Let p and q be non-negative integers. If $h(P)$ is a harmonic function on $\Gamma_n(D)$ satisfying* (3.1) *and*

$$\varliminf_{y\to\infty} e^{-\sqrt{\mu(k_{p+1})}\, y} M_h(y) = 0, \qquad \varliminf_{y\to-\infty} e^{\sqrt{\mu(k_{q+1})}\, y} M_h(y) = 0, \tag{3.3}$$

where

$$M_h(y) = \sup_{X\in D} |h(X, y)| \qquad (-\infty < y < \infty),$$

then

$$h(P) = A_0 y + B_0 + \sum_{k=1}^{k_{p+1}-1} A_k I_k(P) + \sum_{k=1}^{k_{q+1}-1} B_k J_k(P)$$

for any $P = (X, y) \in \Gamma_n(D)$, *where* A_k $(k = 0, 1, \ldots, k_{p+1} - 1)$ *and* B_k $(k = 0, 1, \ldots, k_{q+1} - 1)$ *are constants.*

Corollary 2. *Let $h(P)$ be a harmonic function on $\Gamma_n(D)$ satisfying* (3.1) *and*

$$M_h(y) = o(e^{\sqrt{\mu(1)}|y|}) \qquad (|y| \to \infty).$$

Then $h(P) = A_0 y + B_0$ *for any* $P = (X, y) \in \Gamma_n(D)$, *where* A_0 *and* B_0 *are constants.*

4. Proofs of Theorem and Corollaries 1, 2. Let $f(X, y)$ be a function on $\Gamma_n(D)$. The function $c_k(f, y)$ of y $(-\infty < y < \infty)$ defined by

$$c_k(f, y) = \int_D f(X, y)\varphi_k(X)\, dX$$

is simply denoted by $c_k(y)$ in the following, without specifying f.

Lemma 1. *Let $h(P)$ be a harmonic function on $\Gamma_n(D)$ satisfying* (3.1). *Then*

$$c_0(y) = A_0 y + B_0, \tag{4.1}$$

$$c_k(y) = A_k e^{\sqrt{\mu(k)}\, y} + B_k e^{-\sqrt{\mu(k)}\, y} \qquad (k = 1, 2, \ldots) \tag{4.2}$$

with constants A_k, B_k $(k \geq 0)$ *and*

$$c_k(y) = \frac{\{e^{\sqrt{\mu(k)}\,(y-y_2)} - e^{\sqrt{\mu(k)}\,(y_2-y)}\}c_k(y_1)}{e^{\sqrt{\mu(k)}\,(y_1-y_2)} - e^{\sqrt{\mu(k)}\,(y_2-y_1)}} + \frac{\{e^{\sqrt{\mu(k)}\,(y_1-y)} - e^{\sqrt{\mu(k)}\,(y-y_1)}\}c_k(y_2)}{e^{\sqrt{\mu(k)}\,(y_1-y_2)} - e^{\sqrt{\mu(k)}\,(y_2-y_1)}} \tag{4.3}$$

for any y_1 *and* $y_2, -\infty < y_1 < y_2 < \infty$ $(k = 1, 2, 3, \ldots)$.

Proof. First of all, we remark that $h \in C^2(\overline{\Gamma_n(D)})$ (Gilbarg and Trudinger [3, p. 124]). Since

$$\int_D (\Delta_{n-1}h(X,y))\varphi_k(X)\,dX = \int_D h(X,y)(\Delta_{n-1}\varphi_k(X))\,dX \qquad (-\infty < y < \infty),$$

from Green's identity, (2.2) and (3.1), we have

$$\begin{aligned}\frac{\partial^2 c_k(y)}{\partial y^2} &= \int_D \frac{\partial^2 h(X,y)}{\partial y^2}\varphi_k(X)\,dX = -\int_D \Delta_{n-1}h(X,y)\varphi_k(X)\,dX \\ &= -\int_D h(X,y)(\Delta_{n-1}\varphi_k(X))\,dX \\ &= \mu(k)\int_D h(X,y)\varphi_k(X)\,dX = \mu(k)c_k(y)\end{aligned}$$

from (2.1) $(k = 0, 1, 2, \ldots)$. With constants A_k and B_k $(k = 0, 1, 2, \ldots)$ these give

$$c_0(y) = A_0 y + B_0$$

and

$$c_k(y) = A_k e^{\sqrt{\mu(k)}\,y} + B_k e^{-\sqrt{\mu(k)}\,y} \qquad (k = 1, 2, \ldots),$$

which are (4.1) and (4.2). When we solve for A_k and B_k the equations

$$c_k(y_i) = A_k e^{\sqrt{\mu(k)}\,y_i} + B_k e^{-\sqrt{\mu(k)}\,y_i} \qquad (i = 1, 2),$$

we immediately obtain (4.3).

Remark. From (4.2) we have, for $k = 1, 2, \ldots$

$$\lim_{y\to\infty} c_k(y)e^{-\sqrt{\mu(k)}\,y} = A_k \quad \text{and} \quad \lim_{y\to-\infty} c_k(y)e^{\sqrt{\mu(k)}\,y} = B_k.$$

Lemma 2. *Let* $h(P)$ *be a harmonic function on* $\Gamma_n(D)$ *satisfying* (3.1). *Let* y *be any number and* y_1, y_2 *be two any numbers satisfying* $-\infty < y_1 < y - 1$, $y + 1 < y_2 < \infty$. *For two non-negative integers* p *and* q,

$$\sum_{k=k_{p+q+1}}^{\infty} |c_k(y)| \cdot |\varphi_k(X)| \le L(p)M_h(y_1) + L(q)M_h(y_2),$$

where

$$L(j) = M_2^2|D| \sum_{k=k_{j+1}}^{\infty} k \exp(-\sqrt{M_1} k^{1/(n-1)}).$$

P r o o f. From Lemma 1, we see that

$$c_k(y) = \exp\{-\sqrt{\mu(k)}(y-y_1)\} \frac{1-\exp\{2\sqrt{\mu(k)}(y-y_2)\}}{1-\exp\{2\sqrt{\mu(k)}(y_1-y_2)\}} c_k(y_1)$$
$$+ \exp\{\sqrt{\mu(k)}(y-y_2)\} \frac{1-\exp\{2\sqrt{\mu(k)}(y_1-y)\}}{1-\exp\{2\sqrt{\mu(k)}(y_1-y_2)\}} c_k(y_2).$$

Hence

$$\sum_{k=k_{p+q+1}}^{\infty} |c_k(y)| \cdot |\varphi_k(X)| \le I_1 + I_2, \tag{4.4}$$

where

$$I_1 = \sum_{k=k_{p+1}}^{\infty} \exp\{-\sqrt{\mu(k)}(y-y_1)\}|c_k(y_1)| \cdot |\varphi_k(X)|$$
$$I_2 = \sum_{k=k_{q+1}}^{\infty} \exp\{-\sqrt{\mu(k)}(y_2-y)\}|c_k(y_2)||\varphi_k(X)|.$$

For I_1, we have

$$\begin{aligned} I_1 &\le M_2^2|D|M_h(y_1) \sum_{k=k_{p+1}}^{\infty} k \exp(-\sqrt{\mu(k)}) \\ &\le M_2^2|D|M_h(y_1) \sum_{k=k_{p+1}}^{\infty} k \exp(-\sqrt{M_1} k^{1/(n-1)}), \end{aligned} \tag{4.5}$$

because $y - y_1 > 1$.

For I_2, we also have

$$I_2 \le M_2^2|D|M_h(y_2) \sum_{k=k_{q+1}}^{\infty} k \exp(-\sqrt{M_1} k^{1/(n-1)}). \tag{4.6}$$

Finally (4.4)–(4.6) give the conclusion of the lemma.

Proof of Theorem. Take any compact set $T \subset \overline{\Gamma_n(D)}$ and two numbers y_1, y_2 satisfying

$$\max\{y : (X,y) \in T\} + 1 < y_2, \quad \min\{y : (X,y) \in T\} - 1 > y_1.$$

Let (X,y) be any point in T. Since $c_k(y)$ is the Fourier coefficient of the function $h(X,y)$ of X with respect to the orthonormal sequence $\{\varphi_k(X)\}_{k=0}^{\infty}$, we have

$$h(X,y) = \sum_{k=0}^{\infty} c_k(y)\varphi_k(X)$$

where the series converges uniformly and absolutely on T by Lemma 2. Further (4.1) and (4.2) of Lemma 1 give (3.2). The proof of the Theorem is complete.

Proof of Corollaries 1 and 2. From (3.3) and the Remark, it follows that $A_k = 0$ for any $k \geq k_{p+1}$ and $B_k = 0$ for any $k \geq k_{q+1}$. Hence the Theorem immediately gives the conclusion of Corollary 1. By putting $p = q = 0$ in Corollary 1, we obtain Corollary 2 at once.

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