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On 1-regular ordinary differential operators

by Grzegorz Łysik (Warszawa)

To the memory of Bogdan Ziemian,
my great friend and adviser

Abstract. Solutions to singular linear ordinary differential equations with analytic coefficients are found in the form of Laplace type integrals.

Introduction. Let

$$P\left(x, \frac{d}{dx}\right) = \sum_{i=0}^{n} a^i(x) \frac{d^i}{dx^i} \tag{1}$$

be a linear differential operator of order $n \in \mathbb{N}$ with coefficients $a^i(x) = \sum_{j=0}^{\infty} a^i_j x^j$ convergent for $|x|<r$, $i = 0, \ldots, n-1$, and $a^n(x) = x^m$ with some $m \in \mathbb{N}_0$. Let κ_P be the *Katz invariant* for P, i.e. the smallest $\kappa \in \mathbb{R}$ such that there are no points of $\mathcal{N}_P$ below the line $\{(i,j) \in \mathbb{N}_0 \times \mathbb{Z} : j = \kappa(i-n)+m-n\}$ where

$$\mathcal{N}_P = \{(i,j) \in \mathbb{N}_0 \times \mathbb{Z} : a^i_{i+j} \neq 0\}$$

is the Newton diagram for P. If $\kappa_P \leq 0$ then zero is a regular or regular singular point for P, and the well known Fuchs theorem states that the fundamental system of solutions of $Pu = 0$ consists of convergent series of Taylor type, whose coefficients can be easily determined (cf. [CL]). On the other hand, in the case $\kappa_P > 0$, zero is an irregular singular point for P, and there exist power series solutions to $Pu = 0$ but they need not be convergent. During the last several years a special method called *multisummability* was worked out to deal with divergent solutions of differential equations. By this method, starting from a formal power series solution, one constructs a holomorphic solution in a sector in $\widetilde{\mathbb{C} \setminus \{0\}}$ having the formal one as its

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asymptotic expansion (cf. [B], [E], [M]). Unfortunately, the method cannot be applied directly to the study of the Cauchy problem. We shall describe how the Cauchy problem can be treated by a method based on the Mellin transformation. We shall concentrate on the study of the Cauchy problem for the homogeneous equation $Pu = 0$ where P is a 1-*regular operator*, i.e. an operator with $\kappa_P \le 1$. Observe that any operator P with $\kappa_P > 0$ can be reduced to a 1-regular operator $\widetilde{P}$ by the change of variable $\widetilde{x} = x^{\kappa_P}$. The coefficients of $\widetilde{P}$ are analytic functions in the variable $\widetilde{x}^{1/\kappa_P}$, but this should not cause any essential difficulties.

Our method of treatment of the Cauchy problem for $Pu = 0$ with the Cauchy data at a non-singular point $0 < t < r$ can be described as follows. Firstly, we note that any 1-regular operator P given by (1) with $a^n(x) = x^{2n}$ can be written in the form

$$P\left(x, \frac{d}{dx}\right) = Q\left(x^2 \frac{d}{dx}\right) + \sum_{i=0}^{n-1} g^i(x)\left(x^2 \frac{d}{dx}\right)^i, \tag{2}$$

where Q is a polynomial of degree n and g^i, $i = 0, \ldots, n-1$, are functions analytic in the disc $B(r)$ and vanishing at zero. Next, after the change of variable $s(x) = \exp\{1/t - 1/x\}$ the original Cauchy problem is transformed into the one for the equation $R(s, sd/ds)w = 0$ with the Cauchy data at 1, where

$$R\left(s, s\frac{d}{ds}\right) = Q\left(s\frac{d}{ds}\right) + \sum_{i=0}^{n-1} \widetilde{g}^i(s)\left(s\frac{d}{ds}\right)^i$$

and $\widetilde{g}^i(s) = g^i((1/t - \log s)^{-1})$, $i = 0, \ldots, n-1$. The operator R has a regular singular point at zero and its coefficients $\widetilde{g}^i$, $i = 0, \ldots, n-1$, are generalized analytic functions, i.e. they can be represented in the form $\widetilde{g}^i(s) = \int_0^\infty \psi^i(\alpha) s^\alpha e^{-\alpha/t}\, d\alpha$ with some entire functions ψ^i, $i = 0, \ldots, n-1$, of exponential growth. Now, applying the Mellin transformation we obtain a convolution equation for the function $G(z) := \int_0^t w(s) s^{-z-1}\, ds$,

$$Q(z)G(z) + \int_0^\infty A(\alpha, z) G(z-\alpha) e^{-\alpha/t}\, d\alpha = \Phi(z),$$

where $A(\alpha, z) = \sum_{i=0}^{n-1} \psi^i(\alpha)(z-\alpha)^i$ and Φ is a polynomial determined by the Cauchy data. We solve the convolution equation by the method of successive approximations. Its solution G is a holomorphic function on $\mathbb{C} \setminus \bigcup_{\mu=1}^m (\varrho_\mu + \overline{\mathbb{R}}_+)$, where $\varrho_1, \ldots, \varrho_m$ are the roots of Q. Furthermore, assuming that $\arg(\varrho_\nu - \varrho_\mu) \ne 0$ for any $1 \le \nu < \mu \le m$, the jump of G across the half-line $\varrho_\mu + \overline{\mathbb{R}}_+$ is a Laplace ultradistribution S^μ, $\mu = 1, \ldots, m$, on the half-line. Finally, the solution to the Cauchy problem for $Rw = 0$ is given by $w(s) = \sum_{\mu=1}^m S^\mu[s^\cdot]$ and putting $u(x) = w(s(x))$ we get the solution to the original Cauchy problem. A closer examination of the ultradistributions S^μ allows representing the solution u in the form of Laplace integrals. This type of representation can be viewed as parallel to the one obtained by the multisummability method. The author believes that it can give a new insight into the Stokes phenomenon.

0. Notation. The open disc with centre at $z_0 \in \mathbb{C}$ and radius $r > 0$ is denoted by $B(z_0; r)$ or simply by $B(r)$ if $z_0 = 0$.

By $\widetilde{B}(r)$ (resp. $\widetilde{\mathbb{C}}$) we denote the universal covering space of the punctured disc $B(r) \setminus \{0\}$ (resp. $\mathbb{C} \setminus \{0\}$). A point $z \in \widetilde{B}(r)$ is written as $z = |z|e^{i\arg z}$ with $0 < |z| < r$ and $\arg z \in \mathbb{R}$.

For $\theta \in \mathbb{R}$ we set $l_\theta = (0, e^{i\infty\theta}) = \{z \in \mathbb{C} \setminus \{0\} : \arg z = \theta\}$. If $\theta = 0$ then $l_\theta = \mathbb{R}_+$.

By a *left* (resp. *right*) *tubular neighbourhood* of a ray $\varrho + l_\theta$, $\varrho \in \mathbb{C}$, $\theta \in \mathbb{R}$, we mean a set $\{z : \operatorname{dist}(z, \varrho + l_\theta) < b,\ \theta < \arg(z - \varrho) < \theta + \pi/2\}$ with some $b > 0$ (resp. $\{z : \operatorname{dist}(z, \varrho + l_\theta) < b,\ \theta - \pi/2 < \arg(z - \varrho) < \theta\}$).

For $\varrho \in \mathbb{C}$ and $\theta^- < \theta^+$ with $\theta^+ - \theta^- < 2\pi$ we set

$$S(\varrho; (\theta^-, \theta^+)) = \{z \in \mathbb{C} \setminus \{\varrho\} : \theta^- < \arg(z - \varrho) < \theta^+\}.$$

If $\theta^+ - \theta^- \geq 2\pi$ the set $S(\varrho; (\theta^-, \theta^+))$ is interpreted as a subset of $\widetilde{\mathbb{C} \setminus \{\varrho\}}$.

For $\theta \in \mathbb{R}$ and $\omega \in \mathbb{R}$ we set

$$\Omega^\theta_\omega = \{z \in \widetilde{\mathbb{C}} : \cos\theta \log|z| - \sin\theta \arg z < \omega\}.$$

For $z \in \mathbb{C}$ we put $\langle z \rangle = 1 + |z|$.

1. Generalized analytic functions, the Laplace and Mellin transformations. To fit our purposes we slightly modify the theory of generalized analytic functions given in [Z], and the definitions of the Laplace and Mellin transformations. We do not give the proofs of the stated facts since the proofs follow the ones given in [Z], [Ł2] and [Ł3].

Fix $\varrho_1, \ldots, \varrho_m \in \mathbb{C}$ and $\theta \in \mathbb{R}$. Set $\Gamma_\theta = \bigcup_{\mu=1}^m (\varrho_\mu + \bar{l}_\theta)$. For $a \in \mathbb{R}e^{-i\theta}$ and $\omega \in \mathbb{R}e^{-i\theta}$ define

$$L_a(\Gamma_\theta) = \{\varphi \in C^\infty(\Gamma_\theta) : \|\varphi\|_{a,h} = \sup_{0\leq\alpha\leq h} \sup_{y\in\Gamma_\theta} |e^{-ay} D^\alpha \varphi(y)| < \infty \quad \text{for any } h \in \mathbb{N}\},$$

$$L_{(\omega)}(\Gamma_\theta) = \varinjlim_{a <_\theta \omega} L_a(\Gamma_\theta),$$

where $a <_\theta \omega$ means that $ae^{i\theta} < \omega e^{i\theta}$. The dual space $L'_{(\omega)}(\Gamma_\theta)$ of $L_{(\omega)}(\Gamma_\theta)$ is called the space of *Laplace distributions* on Γ_θ. Replacing the norms $\|\varphi\|_{a,h}$

by

$$\|\varphi\|_{a,h}^{(M_p)} = \sup_{\alpha\in\mathbb{N}_0}\sup_{y\in\Gamma_\theta}\frac{|e^{-ay}h^\alpha D^\alpha\varphi(y)|}{M_\alpha},$$

where $(M_p)_{p=0}^\infty$ is a sequence of positive numbers satisfying conditions (M.1), (M.2) and (M.3) of [K], we obtain the space of *Laplace ultradistributions* $L_{(\omega)}^{(M_p)\prime}(\Gamma_\theta)$ (see also [Ł2]).

Observe that the function $\Gamma_\theta \ni y \mapsto \exp_z(y) := e^{yz}$ belongs to $L^*_{(\omega)}(\Gamma_\theta)$ where $* = \emptyset$ or (M_p) if and only if $\mathrm{Re}(e^{i\theta}z) < \omega e^{i\theta}$. Thus, we can define the *Laplace transform* of $S \in L^{*\prime}_{(\omega)}(\Gamma_\theta)$ by

$$\mathcal{L}S(z) = S[\exp_z] \quad \text{for } \mathrm{Re}(e^{i\theta}z) < \omega e^{i\theta}.$$

Note that $\mathcal{L}S(-1/x)$ is defined in the disc $(2\omega e^{i\theta})^{-1}B(e^{i\theta};1)$ if $\omega e^{i\theta} < 0$, in the half-plane $\mathrm{Re}(xe^{-i\theta}) > 0$ if $\omega = 0$, and outside $(2\omega e^{i\theta})^{-1}B(-e^{i\theta};1)$ if $0 < \omega e^{i\theta}$.

Analogously, the function $\Gamma_\theta \ni y \mapsto \varphi_s(y) := s^y$ belongs to $L^*_{(\omega)}(\Gamma_\theta)$ iff $s \in \Omega^\theta_{\omega e^{i\theta}} := \{s \in \widetilde{\mathbb{C}} : \cos\theta\log|s| - \sin\theta\arg s < \omega e^{i\theta}\}$. So, we can define the *Taylor transform* of $S \in L^{*\prime}_{(\omega)}(\Gamma_\theta)$ by

$$\mathcal{T}S(s) = S[\varphi_s] \quad \text{for } s \in \Omega^\theta_{\omega e^{i\theta}}.$$

We call the image of $L^{*\prime}_{(\omega)}(\Gamma_\theta)$ under the Taylor transformation the space of *generalized analytic functions* determined by $L^{*\prime}_{(\omega)}(\Gamma_\theta)$ and denote it by $\mathrm{GAF}(L^{*\prime}_{(\omega)}(\Gamma_\theta))$. If $(\varrho_\nu + l_\theta) \cap (\varrho_\mu + l_\theta) = \emptyset$ for $1 \le \nu < \mu \le m$, we have a natural decomposition

$$\mathrm{GAF}(L^{*\prime}_{(\omega)}(\Gamma_\theta)) = \bigoplus_{\mu=1}^m \mathrm{GAF}(L^{*\prime}_{(\omega)}(\varrho_\mu + \bar{l}_\theta)) = \bigoplus_{\mu=1}^m s^{\varrho_\mu}\cdot\mathrm{GAF}(L^{*\prime}_{(\omega)}(\bar{l}_\theta)). \tag{3}$$

The space $\mathrm{GAF}(L^{*\prime}_{(\omega)}(\bar{l}_\theta))$ can be characterized (cf. [Ł2], Th. 6) as the set of $w \in \mathcal{O}(\Omega^\theta_{\omega e^{i\theta}})$ such that for any $a <_\theta \omega$ one can find $k < \infty$ such that

$$|w(s)| \le \begin{cases} C(1+|\log s|)^k & \text{for } s \in \overline{\Omega}^\theta_{ae^{i\theta}} \text{ if } * = \emptyset,\\ C\exp\{M(k|\log s|)\} & \text{for } s \in \overline{\Omega}^\theta_{ae^{i\theta}} \text{ if } * = (M_p),\end{cases}$$

where M is the *associated function* of the sequence (M_p) defined by

$$M(\varrho) = \sup_{p\in\mathbb{N}_0}\log\frac{\varrho^p M_0}{M_p} \quad \text{for } \varrho > 0.$$

Fix $t \in \Omega^\theta_{\omega e^{i\theta}}$ and $\varrho \in \mathbb{C}$. We define the *Mellin transform* of $w \in \mathrm{GAF}(L^{*\prime}_{(\omega)}(\varrho + \bar{l}_\theta))$ by

$$\mathcal{M}^\theta_t w(z) = \int_{\gamma^\theta_t} w(s)s^{-z-1}\,ds, \tag{4}$$

where $\gamma_t^\theta = \{s \in \widetilde{\mathbb{C}} : s = t\exp\{-e^{-i\theta}r\},\ 0 \le r < \infty\}$ with the orientation reverse to that induced by the above parametrization. Then $\mathcal{M}_t^\theta w$ is holomorphic on $\{\operatorname{Re}((z-\varrho)e^{-i\theta}) < 0\}$. Since the integration curve γ_t^θ in (4) can be replaced by $\gamma_t^{\theta'}$ for any $|\theta - \theta'| \le \pi/2$ we conclude that $\mathcal{M}_t^\theta w \in \mathcal{O}(\mathbb{C} \setminus (\varrho + \bar{l}_\theta))$. Furthermore (cf. [Ł3], Th. 4, [Z], Th. 10.1), there exists $C < \infty$ such that for $0 < \operatorname{dist}(z, \varrho + \bar{l}_\theta) \le 1$,

$$(5)\qquad |\mathcal{M}_t^\theta w(z)| \le \begin{cases} C|t^{\varrho - z}|(\operatorname{dist}(z, \varrho + \bar{l}_\theta))^{-C} & \text{if } * = \emptyset, \\ C|t^{\varrho - z}| \exp\left\{M^*\left(\dfrac{C}{\operatorname{dist}(z, \varrho + \bar{l}_\theta)}\right)\right\} & \text{if } * = (M_p), \end{cases}$$

where M^* is the *growth function* of the sequence (M_p) given by

$$(6)\qquad M^*(\varrho) = \sup_{p \in \mathbb{N}_0} \log \frac{\varrho^p p! M_0}{M_p} \quad \text{for } \varrho > 0.$$

Moreover, the boundary value of $\mathcal{M}_t^\theta w$, $S = b(\mathcal{M}_t^\theta w)$, belongs to $L^{*\prime}_{(\omega)}(\varrho + \bar{l}_\theta)$ and $w = (2\pi i)^{-1}\mathcal{T}S$.

Conversely (cf. [Ł3], Th. 5), if $G \in \mathcal{O}(\mathbb{C} \setminus (\varrho + \bar{l}_\theta))$ satisfies (5) (with G in place of $\mathcal{M}_t^\theta w$) for $0 < \operatorname{dist}(z, \varrho + \bar{l}_\theta) \le 1$ and $|G(z)| \le C|t^{\varrho - z}|/\langle z\rangle$ for $\operatorname{dist}(z, \varrho + \bar{l}_\theta) \ge 1$ then $G = \mathcal{M}_t^\theta w$ with a unique w given by $w = (2\pi i)^{-1}\mathcal{T}b(G)$.

Analogously, using the decomposition (3), we define the Mellin transform of $w \in \mathrm{GAF}(L^{*\prime}_{(\omega)}(\Gamma_\theta))$, which is a holomorphic function on $\mathbb{C} \setminus \Gamma_\theta$ and satisfies appropriate estimates.

The Mellin transformation has the following operational property, which makes it useful in the study of the Cauchy problem.

If $w \in \mathrm{GAF}(L^{*\prime}_{(\omega)}(\Gamma_\theta))$ and $t \in \Omega^\theta_{\omega e^{i\theta}}$ then for $i \in \mathbb{N}_0$,

$$(7)\qquad \mathcal{M}_t^\theta\left(\left(s\frac{d}{ds}\right)^i w\right)(z) = z^i \mathcal{M}_t^\theta w(z) + W_i(z) \quad \text{for } z \in \mathbb{C} \setminus \Gamma_\theta,$$

where W_i is a polynomial of degree $\le i-1$ depending on $w(t), \ldots, w^{(i-1)}(t)$.

2. The main result. Let P be a differential operator (1) with coefficients analytic in $B(r)$, $r > 0$. Assume that P is 1-regular and $a^n(x) = x^{2n}$. Then for $i = 0, \ldots, n-1$, $a^i(x) = \sum_{j=2i}^\infty a^i_j x^j$ for $|x| < r$. Furthermore, it follows by Lemma 1.3 of Chapter 4 of [T] that P can be written in the form (2), where $Q(z) = z^n + \sum_{i=0}^{n-1} a^i_{2i} z^i$ and $g^i(x) = \sum_{j=1}^\infty g^i_j x^j$ for $|x| < r$, $i = 0, \ldots, n-1$. Fix $0 < t < r$ and consider the Cauchy problem

$$(8)\qquad \begin{cases} Pu = 0, \\ u(t) = u_0, \ldots, u^{(n-1)}(t) = u_{n-1}. \end{cases}$$

It is well known that the solution u of (8) is unique and it extends holomorphically to a function on $\widetilde{B}(r)$. Our aim is to represent the u in the form of Laplace type integrals. To formulate the main result denote by $\varrho_1, \ldots, \varrho_m$ the roots of Q with multiplicities $k_1, \ldots, k_m$, respectively. Define the set Θ_{s} of *singular directions* by

$$\Theta_{\mathrm{s}} = \{\theta \in \mathbb{R} : \theta \bmod (2\pi) = \arg(\varrho_\nu - \varrho_\mu) \text{ for some } 1 \le \nu \ne \mu \le m\}$$

($\Theta_{\mathrm{s}} = \emptyset$ if $m = 1$). Choose $\theta \notin \Theta_{\mathrm{s}}$ such that $t \in (r/2)B(e^{i\theta}; 1)$ and denote by θ^- (resp. θ^+) the greatest (resp. smallest) singular direction less (resp. greater) than θ ($\theta^\pm = \pm\infty$ if $\Theta_{\mathrm{s}} = \emptyset$).

MAIN THEOREM. *Let P be a* 1*-regular operator* (2). *Fix $0 < t < r$ and retain the preceding notations. Then the unique solution u of the Cauchy problem* (8) *is given by*

$$u(x) = \sum_{\mu=1}^{m} \mathcal{L}S^\mu(1/t - 1/x) \quad \text{for } x \in (r/2)B(e^{i\theta}; 1), \tag{9}$$

with a unique $S^\mu \in L^{*\prime}_{(\omega)}(\varrho_\mu + \bar{l}_\theta)$ ($\mu = 1, \ldots, m$), where $\omega = (\cos(\theta/t) - 1/r)e^{-i\theta}$ and

$$* = \begin{cases} \emptyset & \text{if } k_\mu = 1, \\ p!(p/\log p)^{p/(k_\mu - 1)} & \text{if } k_\mu > 1. \end{cases} \tag{10}$$

Furthermore, S^μ, $\mu = 1, \ldots, m$, restricted to $\varrho_\mu + l_\theta$ extends holomorphically to a function $\Psi^\mu \in \mathcal{O}(S(\varrho_\mu; (\theta^-, \theta^+)))$ such that for any $r' < r$ and $\theta^- < \widetilde{\theta}^- < \widetilde{\theta}^+ < \theta^+$,

$$|\Psi^\mu(\varrho_\mu + \gamma)e^{\gamma/t}| \le \begin{cases} C|\gamma|^{-C} \exp\{|\gamma|/r'\} & \text{if } k_\mu = 1, \\ C \exp\left\{\dfrac{C}{|\gamma|^{k_\mu - 1}} \log \dfrac{C}{|\gamma|} + \dfrac{|\gamma|}{r'}\right\} & \text{if } k_\mu > 1, \end{cases} \tag{11}$$

for $\widetilde{\theta}^- \le \arg\gamma \le \widetilde{\theta}^+$ with some $C < \infty$.

Thus, for any $\theta^- < \theta' < \theta^+$, u can be written in the form

$$u(x) = \sum_{\mu=1}^{m} e^{-\varrho_\mu(1/t - 1/x)} \operatorname{reg} \int_{l_{\theta'}} \Psi^\mu(\varrho_\mu + \gamma) e^{\gamma/t - \gamma/x}\, d\gamma \quad \text{for } x \in (r/2)B(e^{i\theta'}; 1), \tag{12}$$

where the regularization of the integral is distributional if $k_\mu = 1$ and ultradistributional of class $p!(p/\log p)^{p/(k_\mu - 1)}$ if $k_\mu > 1$.

REMARK. We conjecture that Ψ^μ is a multivalued holomorphic function on $\mathbb{C}$ with the set of branching points $\{\varrho_1, \ldots, \varrho_\mu\}$.

3. Auxiliary lemmas. In the proof of the main theorem we shall use the following lemmas.

LEMMA 1. *For* $\nu \in \mathbb{N}$ *put*

$$I^\nu(\gamma, z) = \int\limits_{T^\nu(\gamma)} \frac{d\alpha}{\langle z - \alpha_1 \rangle \ldots \langle z - \alpha_1 - \ldots - \alpha_\nu \rangle} \quad \text{for } \gamma \in \mathbb{R}_+,\ z \in \mathbb{C}$$

with $T^\nu(\gamma) = \{\alpha \in (\mathbb{R}_+)^\nu : \alpha_1 + \ldots + \alpha_\nu \le \gamma\}$. *Then*

$$|I^\nu(\gamma, z)| \le \frac{2^\nu}{\nu!} \log^\nu(1 + |\gamma|) \quad \text{for } \gamma \in \mathbb{R}_+,\ z \in \mathbb{C}.$$

Proof. We can consider only the case $z = x \in \mathbb{R}_+$. Let $0 < \gamma \le x$. By induction we show that

$$I^\nu(\gamma, x) = \frac{1}{\nu!} \log^\nu \left(\frac{1 + x}{1 + x - \gamma} \right),$$

which is bounded by $\frac{1}{\nu!} \log^\nu(1 + \gamma)$. In fact, $I^1(\gamma, x) = \log \frac{1+x}{1+x-\gamma}$ and for $\nu \ge 2$ we derive

$$\begin{aligned} I^\nu(\gamma, x) &= \int\limits_0^\gamma \frac{1}{1 + x - \alpha_1} I^{\nu-1}(\gamma - \alpha_1, x - \alpha_1)\, d\alpha_1 \\ &= \frac{1}{(\nu - 1)!} \int\limits_0^\gamma \frac{\log^{\nu-1}\left(\frac{1+x-\alpha_1}{1+x-\gamma}\right)}{1 + x - \alpha_1}\, d\alpha_1 = \frac{1}{\nu!} \log^\nu \left(\frac{1 + x}{1 + x - \gamma} \right). \end{aligned}$$

Now let $0 < x \le \gamma$. We observe that $T^\nu(\gamma) = \bigcup_{k=0}^\nu T_k^\nu(\gamma)$ with $T_k^\nu(\gamma) = \{\alpha \in \mathbb{R}_+^\nu : \alpha_1 \le x, \ldots, \alpha_1 + \ldots + \alpha_{\nu-k} \le x,\ x \le \alpha_1 + \ldots + \alpha_{\nu-k+1},\ \alpha_1 + \ldots + \alpha_\nu \le \gamma\}$. Now for $k \in \{0, 1, \ldots, \nu\}$ we compute

$$\begin{aligned} &\int\limits_{T_k^\nu(\gamma)} \frac{1}{1 + x - \alpha_1} \cdots \frac{1}{1 + x - \alpha_1 - \ldots - \alpha_{\nu-k}} \\ &\qquad \times \frac{1}{1 + \alpha_1 + \ldots + \alpha_{\nu-k+1} - x} \cdots \frac{1}{1 + \alpha_1 + \ldots + \alpha_\nu - x}\, d\alpha \\ &\quad = \int\limits_{T^{\nu-k}(x)} \frac{1}{1 + x - \alpha_1} \cdots \frac{1}{1 + x - \alpha_1 - \ldots - \alpha_{\nu-k}}\, d\alpha \\ &\qquad \times \int\limits_{T^k(\gamma - x)} \frac{1}{1 + \beta_1} \cdots \frac{1}{1 + \beta_1 + \ldots + \beta_k}\, d\beta \\ &\quad = \frac{1}{(\nu - k)!} \log^{\nu-k}(1 + x) \cdot \frac{1}{k!} \log^k(1 + \gamma - x). \end{aligned}$$

So

$$I^\nu(\gamma,x)=\sum_{k=0}^{\nu}\frac{1}{(\nu-k)!k!}\log^{\nu-k}(1+x)\cdot\log^k(1+\gamma-x)$$
$$=\frac{1}{\nu!}(\log(1+x)+\log(1+\gamma-x))^\nu,$$

which is bounded by $\frac{2^\nu}{\nu!}\log^\nu(1+\gamma)$.

LEMMA 2. *Let* $|\Psi(\gamma)|\le Ce^{|\gamma|/r}$ *for* $\gamma\in e^{i\theta}+l_\theta$ *with* $r>0$. *Then the integral*

$$w_\theta(s)=\int_{e^{i\theta}+l_\theta}\Psi(\gamma)s^\gamma e^{-\gamma}\,d\gamma$$

converges on the set of $s\in\widetilde{\mathbb{C}}$ *such that* $s/e\in\Omega^\theta_{-1/r}$ *and* $u_\theta(x)=w_\theta(\exp\{1-1/x\})$ *is defined in the disc* $(r/2)B(e^{i\theta};1)$.

Proof. Indeed

$$w_\theta(s)=\int_1^\infty\Psi(te^{i\theta})(s/e)^{te^{i\theta}}e^{i\theta}\,dt$$

and the integral converges if

$$1/r+\operatorname{Re}(e^{i\theta}\log(s/e))=1/r+\cos\theta\log(|s|/e)-\sin\theta\arg s<0.$$

To prove the second statement observe that for $s=\exp\{1-1/x\}$ we have $\log(|s|/e)=-\operatorname{Re}(1/x)=-\operatorname{Re}x/|x|^2$ and $\arg s=-\operatorname{Im}(1/x)=\operatorname{Im}x/|x|^2$. So, if $s/e\in\Omega^\theta_{-1/r}$ then x satisfies $(\cos\theta\operatorname{Re}x+\sin\theta\operatorname{Im}x)|x|^{-2}=\operatorname{Re}(e^{-i\theta}x)|x|^{-2}>r^{-1}$ and hence $x\in(r/2)B(e^{i\theta};1)$.

LEMMA 3. *Let* $s>0$, $M_0=M_1=1$ *and* $M_p=p!(p/\log p)^{ps}$ *for* $p\in\mathbb{N}$, $p\ge 2$. *Then* $M^*(\varrho)\sim\varrho^{1/s}\log\varrho$ *as* $\varrho\to\infty$.

Proof. By (8) we have, for $\varrho>1$,

$$M^*(\varrho)=\max(\log\varrho,\sup_{p\in\mathbb{N},\,p\ge2}p(\log\varrho+s\log\log p-s\log p)).$$

To compute the supremum define

$$g(\varrho,x)=x(\log\varrho+s\log\log x-s\log x)\quad\text{ for } x>1,\ \varrho>0.$$

Since $g'_x(\varrho,x)=\log\varrho+s\log\log x-s\log x+s/\log x-s$, for $\varrho\ge e^s$ there exists a unique $x(\varrho)\ge e$ such that $g'_x(\varrho,x(\varrho))=0$. Put $g(\varrho)=g(\varrho,x(\varrho))$ for $\varrho\ge e^s$. Then $g(\varrho)=sx(\varrho)(1-1/\log x(\varrho))$ and so $g(\varrho)\sim x(\varrho)$ as $\varrho\to\infty$. Put

$$f(x)=\left(\frac{ex}{\log x}\exp\left\{-\frac{1}{\log x}\right\}\right)^s\quad\text{ and }\quad h(\varrho)=\varrho^{1/s}\log\varrho^{1/s}.$$

Then for $\varrho>e^s$, $e^{-s}f(h(\varrho))\le\varrho\le2^{s-1}f(h(2\varrho))$. Since $x(\varrho)=f^{-1}(\varrho)$ this implies $h(\varrho)\sim x(\varrho)$ as $\varrho\to\infty$. Finally, in a standard way (see [Ł1]) we show that $M^*(\varrho)\sim g(\varrho)$ as $\varrho\to\infty$.

4. Proof of the main theorem. Let P be given by (2), where Q is a polynomial of degree n and $g^i(x) = \sum_{j=1}^{\infty} g_j^i x^j$, $i = 0, \ldots, n-1$, are functions analytic in the disc $B(r)$, $r > 0$. Consider the Cauchy problem (8). Putting, if necessary, $x' = x/t$ we can assume that $t = 1$ and $r > 1$. Observe that by the change of independent variable $s(x) = \exp\{1 - 1/x\}$, (8) is transformed into

$$\begin{cases} Q\left(s\frac{d}{ds}\right)w + \sum_{i=0}^{n-1} \widetilde{g}^i(s)\left(s\frac{d}{ds}\right)^i w = 0, \\ w(1) = w_0, \ldots, w^{(n-1)}(1) = w_{n-1}, \end{cases} \tag{13}$$

where $\widetilde{g}^i(s) = g^i((-\log(s/e))^{-1})$, $i = 0, \ldots, n-1$, $w(s) = u((-\log(s/e))^{-1})$, $w_0 = u_0, w_1 = u_1$, $w_2 = u_1 + u_2$ and so on.

Since $\lim_{s\to 0} \widetilde{g}^i(s) = 0$ for $i = 0, \ldots, n-1$, we have obtained an equation with a regular singular point at zero, but with coefficients which are generalized analytic functions of the form (cf. [Z], Th. 14.1)

$$\widetilde{g}^i(s) = \int_{l_\theta} \psi^i(\alpha) s^\alpha e^{-\alpha}\, d\alpha \quad \text{for } s/e \in \Omega^\theta_{-1/r},\ \theta \in \mathbb{R}, \tag{14}$$

where $\psi^i(\alpha) = \sum_{j=1}^{\infty} g_j^i \alpha^{j-1}/(j-1)!$ for $\alpha \in \mathbb{C}$ is the Borel transform of g^i, which is an entire function satisfying $|\psi^i(\alpha)| \le C_{r'} \exp\{|\alpha|/r'\}$ for any $r' < r$ $(i = 0, \ldots, n-1)$.

The equations of this type were studied by Bogdan Ziemian in [Z]. Under suitable conditions he proved the existence of generalized analytic solutions with positive radii of convergence. However his theorem ([Z], Theorem 16.2) cannot be applied here without additional assumptions on the functions g^i and does not guarantee that the radius of convergence of a solution is greater than 1.

We shall solve (13) by applying the Mellin transformation. Fix a non-singular direction $\theta \notin \Theta_{\mathrm{s}}$ such that $\cos\theta > 1/r$ (this assumption ensures that $1 \in \Omega^\theta_{-1/r}$). Observe that by (14) and (7),

$$\begin{aligned} \mathcal{M}_1^\theta&\left(\widetilde{g}^i(s)\left(s\frac{d}{ds}\right)^i w\right)(z) \\ &= \int_{l_\theta} \psi^i(\alpha)((z-\alpha)^i \mathcal{M}_1^\theta w(z-\alpha) + W_i(z-\alpha))e^{-\alpha}\, d\alpha \\ &= \int_{l_\theta} \psi^i(\alpha)(z-\alpha)^i \mathcal{M}_1^\theta w(z-\alpha) e^{-\alpha}\, d\alpha + \widetilde{W}_i(z), \end{aligned}$$

where $\widetilde{W}_i$ is a polynomial of degree $\le i-1$, $i = 0, \ldots, n-1$. Thus applying the Mellin transformation to (13) we get the convolution equation

(15) $$Q(z)G_\theta(z)+\int_{l_\theta} A^0(\alpha,z)G_\theta(z-\alpha)e^{-\alpha}\,d\alpha=\Phi(z),$$

where

$$G_\theta(z)=\mathcal{M}_1^\theta w(z),\qquad A^0(\alpha,z)=\sum_{i=0}^{n-1}\psi^i(\alpha)(z-\alpha)^i$$

and Φ is a polynomial of degree $\le n-1$ depending on $w_0,\dots,w_{n-1}$.

We solve (15) by the approximation scheme

$$G_\theta^0(z)=\frac{\Phi(z)}{Q(z)},$$
$$G_\theta^{\nu+1}(z)=\frac{1}{Q(z)}\Big\{\Phi(z)-\int_{l_\theta}A^0(\alpha,z)G_\theta^\nu(z-\alpha)e^{-\alpha}\,d\alpha\Big\},\qquad \nu\in\mathbb{N}.$$

Put $\widetilde{G}_\theta^{\nu+1}=G_\theta^{\nu+1}-G_\theta^\nu$ for $\nu\in\mathbb{N}_0$. Then we find

$$\widetilde{G}_\theta^{\nu+1}(z)=\frac{(-1)^{\nu+1}}{Q(z)}\int_{l_\theta}A_\theta^\nu(\gamma,z)\frac{\Phi(z-\gamma)}{Q(z-\gamma)}e^{-\gamma}\,d\gamma$$

where for $\gamma\in l_\theta,\ \nu\in\mathbb{N}$,

$$A_\theta^\nu(\gamma,z)=\int_{\substack{\alpha_1\in l_\theta\\ |\alpha_1|\le|\gamma|}}\frac{A^0(\alpha_1,z)A^{\nu-1}(\gamma-\alpha_1,z-\alpha_1)}{Q(z-\alpha_1)}\,d\alpha_1$$
$$=\int_{T_\theta^\nu(\gamma)}\frac{A^0(\alpha_1,z)}{Q(z-\alpha_1)}\cdots\frac{A^0(\alpha_\nu,z-\alpha_1-\dots-\alpha_{\nu-1})}{Q(z-\alpha_1-\dots-\alpha_\nu)}$$
$$\times A^0(\gamma-\alpha_1-\dots-\alpha_\nu,z-\alpha_1-\dots-\alpha_\nu)\,d\alpha$$

with $T_\theta^\nu(\gamma)=\{\alpha\in(l_\theta)^\nu:|\alpha_1+\dots+\alpha_\nu|\le|\gamma|\}$, $\gamma\in l_\theta$.

Assume that $\operatorname{dist}(z,\bigcup_{\mu=1}^m(\varrho_\mu+l_\theta))\ge b$ with some $b>0$. Then we can find C_b such that $\langle z\rangle^n\le C_b|Q(z)|$. Since $|A^0(\alpha,z)|\le Ce^{|\alpha|/r'}\langle z\rangle^{n-1}$ we derive

$$\frac{|A^0(\alpha_1,z)|}{|Q(z)|}\le\frac{CC_be^{|\alpha_1|/r'}}{\langle z\rangle},\qquad \frac{|A^0(\alpha_2,z-\alpha_1)|}{|Q(z-\alpha_1)|}\le\frac{CC_be^{|\alpha_2|/r'}}{\langle z-\alpha_1\rangle},\ \dots,$$
$$\frac{|A^0(\alpha_\nu,z-\alpha_1-\dots-\alpha_{\nu-1})|}{|Q(z-\alpha_1-\dots-\alpha_{\nu-1})|}\le\frac{CC_be^{|\alpha_\nu|/r'}}{\langle z-\alpha_1-\dots-\alpha_{\nu-1}\rangle},$$
$$\frac{|A^0(\gamma-\alpha_1-\dots-\alpha_\nu,z-\alpha_1-\dots-\alpha_\nu)|}{|Q(z-\alpha_1-\dots-\alpha_\nu)|}\le\frac{CC_be^{|\gamma-\alpha_1-\dots-\alpha_\nu|/r'}}{\langle z-\alpha_1-\dots-\alpha_\nu\rangle}.$$

So by Lemma 1,

$$\frac{|A_\theta^\nu(\gamma,z)|}{|Q(z)|}\le\frac{(CC_b)^{\nu+1}}{\langle z\rangle}e^{|\gamma|/r'}\frac{2^\nu}{\nu!}\log^\nu(1+|\gamma|).$$

Thus

$$G_\theta(z) = \frac{\Phi(z)}{Q(z)} + \int_{l_\theta} \frac{A_\theta(\gamma, z)}{Q(z)} \cdot \frac{\Phi(z-\gamma)}{Q(z-\gamma)} e^{-\gamma}\, d\gamma, \tag{16}$$

where $A_\theta(\gamma, z) = \sum_{\nu=0}^{\infty} (-1)^{\nu+1} A_\theta^\nu(\gamma, z)$ satisfies, with $K = 2CC_b$,

$$\frac{|A_\theta(\gamma, z)|}{|Q(z)|} \le \frac{CC_b}{\langle z\rangle} e^{|\gamma|/r'} (1+|\gamma|)^K \tag{17}$$

$$\text{for } \gamma \in l_\theta, \operatorname{dist}\Big(z, \bigcup_{\mu=1}^m (\varrho_\mu + l_\theta)\Big) \ge b.$$

Finally, since $|\Phi(z)|\langle z\rangle \le C_b |Q(z)|$ for $\operatorname{dist}(z, \{\varrho_1, \ldots, \varrho_m\}) \ge b$ we get, with some $C < \infty$,

$$|G_\theta(z)| \le \frac{C}{\langle z\rangle} \quad \text{for } \operatorname{dist}\Big(z, \bigcup_{\mu=1}^m (\varrho_\mu + \bar{l}_\theta)\Big) \ge b. \tag{18}$$

Now assume that z is close to ϱ_μ with a fixed $\mu \in \{1, \ldots, m\}$. To shorten notation put $k = k_\mu$. Assume $d \le |z - \varrho_\mu| \le b$, $|\arg(z - \varrho_\mu - \theta)| \ge \beta$ with some $\beta > 0$ and $0 < d < b \le 1$ with $\operatorname{dist}(\varrho_\mu + \bar{l}_\theta, \bigcup_{\nu\ne\mu} (\varrho_\nu + \bar{l}_\theta)) \ge 2b$. Since for $\alpha \in l_\theta$ we have $\langle z-\alpha\rangle^{n-k} |z - \varrho_\mu - \alpha|^k \le C|Q(z-\alpha)|$, $\langle z-\alpha\rangle^{k-1} \le C_1 \langle \alpha\rangle^{k-1}$ and $(d + |\alpha|)^k \le C_2 |z - \varrho_\mu - \alpha|^k$ we get, with a constant C independent of d,

$$\frac{|A^0(\alpha_1, z)|}{|Q(z)|} \le \frac{Ce^{|\alpha_1|/r'}}{d^k},$$

$$\frac{|A^0(\alpha_2, z-\alpha_1)|}{|Q(z-\alpha_1)|} \le Ce^{|\alpha_2|/r'} \frac{\langle z-\alpha_1\rangle^{k-1}}{|z-\varrho_\mu-\alpha_1|^k} \le Ce^{|\alpha_2|/r'} \frac{C_1C_2\langle \alpha_1\rangle^{k-1}}{(d+|\alpha_1|)^k}, \ \ldots,$$

$$\frac{|A^0(\alpha_\nu, z-\alpha_1-\ldots-\alpha_{\nu-1})|}{|Q(z-\alpha_1-\ldots-\alpha_{\nu-1})|} \le Ce^{|\alpha_\nu|/r'} \frac{C_1C_2\langle \alpha_1+\ldots+\alpha_{\nu-1}\rangle^{k-1}}{(d+|\alpha_1+\ldots+\alpha_{\nu-1}|)^k},$$

$$\frac{|A^0(\gamma-\alpha_1-\ldots-\alpha_\nu, z-\alpha_1-\ldots-\alpha_\nu)|}{|Q(z-\alpha_1-\ldots-\alpha_\nu)|} \le Ce^{|\gamma-\alpha_1-\ldots-\alpha_\nu|/r'} \frac{C_1C_2\langle \alpha_1+\ldots+\alpha_\nu\rangle^{k-1}}{(d+|\alpha_1+\ldots+\alpha_\nu|)^k}.$$

So for $\gamma \in l_\theta$,

$$\frac{|A_\theta^\nu(\gamma, z)|}{|Q(z)|} \le \frac{C}{d^k} (CC_1C_2)^\nu e^{|\gamma|/r'} \times \int_{T_\theta^\nu(\gamma)} \frac{\langle \alpha_1\rangle^{k-1} \ldots \langle \alpha_1+\ldots+\alpha_\nu\rangle^{k-1}}{(d+|\alpha_1|)^k \ldots (d+|\alpha_1+\ldots+\alpha_\nu|)^k}\, d\alpha \le \frac{C}{d^k} e^{|\gamma|/r'} \frac{1}{\nu!} \Big(\frac{L}{d^{k-1}} \log \frac{d+|\gamma|}{d}\Big)^\nu,$$

where $L = CC_1C_2$ (for $k > 1$ we use $\langle\alpha\rangle^{k-1}d^{k-1} \le (d + |\alpha|)^{k-1}$). Thus

$$\frac{|A_\theta(\gamma, z)|}{|Q(z)|} \le \frac{C}{d^k} e^{|\gamma|/r'} \left(\frac{d + |\gamma|}{d}\right)^{L/d^{k-1}}.$$

Finally, since

$$\int_{l_\theta} \left(1 + \frac{|\gamma|}{d}\right)^{L/d^{k-1}} e^{|\gamma|/r'} e^{-|\gamma|} \, |d\gamma| \le C \left(\frac{C}{d}\right)^{L/d^{k-1}} \Gamma\left(\frac{L}{d^{k-1}} + 1\right)$$

(here $C = 2r'/(r' - 1)$ and Γ is the Euler function), we obtain, with C independent of d,

$$(19) \qquad |G_\theta(z)| \le \begin{cases} Cd^{-L-1} & \text{if } k = 1, \\ \dfrac{C}{d^k} \exp\left\{\dfrac{L}{d^{k-1}} \log \dfrac{CL}{d^k}\right\} & \text{if } k > 1 \end{cases}$$

for $d \le |z - \varrho_\mu| \le b$, $|\arg(z - \varrho_\mu - \theta)| \ge \beta$.

Now, observe that θ can be changed within the interval (θ^-, θ^+), where θ^- (resp. θ^+) is the greatest (resp. smallest) singular direction less (resp. greater) than θ. Also β can be chosen arbitrarily small positive. Thus, restriction of G_θ to a small left (resp. right) tubular neighbourhood of $\varrho_\mu + l_\theta$ extends to a holomorphic function defined on $\theta^- < \arg(z - \varrho_\mu) \le 0$ (resp. $0 \le \arg(z - \varrho_1) < \theta^+$). The extension of G_θ obtained this way also satisfies (19) for $d \le |z - \varrho_\mu| \le b$ and (18) for $|z - \varrho_\mu| \ge b$.

Thus, by Lemma 3 and the results of Section 1, we get

$$\frac{1}{2\pi i} b(G_\theta) = \sum_{\mu=1}^{m} S_\theta^\mu,$$

where $S_\theta^\mu \in L_{(0)}^{*\prime}(\varrho_\mu + \bar{l}_\theta)$ with $*$ given by (10). So, the solution w of (13) is given by $w(s) = \sum_{\mu=1}^m \mathcal{T} S_\theta^\mu(s)$ for $s \in \Omega_0^\theta$, and $u(x) = w(e^{1-1/x})$ is defined only for $x \in \frac{1}{2}B(e^{i\theta}; 1)$. However, the estimate (17) gives an additional information about S_θ^μ, $\mu = 1, \ldots, m$. Namely, changing θ within (θ^-, θ^+), we note that the restriction of S_θ^μ to an open ray $\varrho_\mu + l_\theta$ is analytic, and extends holomorphically to a function Ψ^μ defined in a sector $S(\varrho_\mu; (\theta^-, \theta^+))$. To estimate Ψ^μ put (with $k = k_\mu$)

$$F^\mu(\alpha, z) = \frac{A(\alpha, z)\Phi(z - \alpha)(z - \varrho_\mu - \alpha)^k}{Q(z)Q(z - \alpha)}$$

$$\text{for } \alpha \in l_{\theta'}, \ z \notin \bigcup_{\nu=1}^{m} (\varrho_\nu + l_{\theta'}), \ \theta^- < \theta' < \theta^+,$$

where $A(\alpha, z) = A_{\theta'}(\alpha, z)$ for (α, z) as above, $\theta^- < \theta' < \theta^+$. Since

$b(\Phi/Q)|_{\varrho_\mu + l_\theta} = 0$, (16) implies that for $\gamma \in S(0;(\theta^-,\theta^+))$,

$$\begin{aligned}\Psi^\mu(\varrho_\mu+\gamma) &= \frac{1}{2\pi i}(G_{\widetilde{\theta}^-}(\varrho_\mu+\gamma) - G_{\widetilde{\theta}^+}(\varrho_\mu+\gamma)) \\ &= \frac{1}{2\pi i}\int\limits_{l_{\widetilde{\theta}^-} - l_{\widetilde{\theta}^+}} \frac{F^\mu(\alpha,\varrho_\mu+\gamma)}{(\gamma-\alpha)^k} e^{-\alpha}\,d\alpha\end{aligned}$$

where $\theta^- < \widetilde{\theta}^- < \arg\gamma < \widetilde{\theta}^+ < \theta^+$. So for γ with $\theta^- < \arg\gamma < \theta^+$,

$$\Psi^\mu(\varrho_\mu+\gamma) = \widetilde{\Psi}^\mu(\gamma)e^{-\gamma}, \quad \text{where} \quad \widetilde{\Psi}^\mu(\gamma) = \sum_{l=0}^{k_\mu-1} C_l \frac{\partial^l}{\partial\alpha^l} F^\mu(\alpha,\varrho_\mu+\gamma)\bigg|_{\alpha=\gamma}$$

with some constants C_l, $l = 0,\ldots,k_\mu-1$ ($\mu = 1,\ldots,m$). Observe that (17), (19), and the Cauchy formula imply that for any $\theta^- < \widetilde{\theta}^- < \widetilde{\theta}^+ < \theta^+$ and $r' < r$, $|\widetilde{\Psi}^\mu(\gamma)|$ can be estimated by the right hand side of (11) for $\widetilde{\theta}^- \le \arg\gamma \le \widetilde{\theta}^+$ with some $C < \infty$. Since the above holds for any $r' < r$, we conclude that $e^\gamma S^\mu_\theta \in L^{*\prime}_{(-1/re^{-i\theta})}(\varrho_\mu + \bar{l}_\theta)$ and so $S^\mu_\theta \in L^{*\prime}_{(\omega)}(\varrho_\mu + \bar{l}_\theta)$ with $\omega = (\cos\theta - 1/r)e^{-i\theta}$. Now, Lemma 2 implies that $w(s)$ is defined for $s \in \widetilde{\mathbb{C}}$ with $s/e \in \Omega^\theta_{-1/r}$. Finally, $u(x) = w(e^{1-1/x}) = \sum_{\mu=1}^m \mathcal{L}(e^\gamma S^\mu_\theta)(-1/x)$ is defined for $x \in (r/2)B(e^{i\theta};1)$, and a direct computation shows that u can be written in the form (12) (with $t = 1$).

5. An example. Let us solve the Cauchy problem for the Euler equation $x^2u' = u - x$, $u(1) = u_0$. Putting $s(x) = \exp\{1 - 1/x\}$ and $w(s) = u(1/(-\log(s/e)))$ we get $sw' - w = 1/\log(s/e)$, $w(1) = u_0$. Applying the Mellin transformation (4) with $0 < |\theta| < \pi/2$ and $t = 1$ we obtain the equation for $G_\theta = \mathcal{M}^\theta_1 w$,

$$(z-1)G_\theta(z) = -u_0 + \int\limits_{l_\theta} \frac{e^{-\alpha}}{z-\alpha}\,d\alpha.$$

Its solution is given by

$$G_\theta(z) = \frac{-u_0}{z-1} + \frac{1}{z-1}\int\limits_{l_\theta} \frac{e^{-\alpha}}{z-\alpha}\,d\alpha.$$

Now, we compute the boundary value $S = (2\pi i)^{-1}b(G_\theta)$:

$$S = (u_0 + A)\delta_{(1)} + \int\limits_{l_\theta} \log(\alpha-1)\frac{d}{d\alpha}(e^{-\alpha}\delta_{(\alpha)})\,d\alpha$$

with $A = \big(\Gamma'(1) - \sum_{j=1}^\infty \frac{1}{j!j}\big)/e$. Thus, the solution $w = \mathcal{T}S$ is given by

$$w(s) = (u_0 + A)s + \log(s/e)\int\limits_{l_\theta} \log(\alpha-1)s^\alpha e^{-\alpha}\,d\alpha \qquad \text{for } s/e \in \Omega^\theta_0.$$

Finally, $u(x) = w(e^{1-1/x})$ is given by

$$u(x) = (u_0 + A)e^{1-1/x} - \frac{1}{x}\int_{l_\theta} \log(\alpha - 1)e^{-\alpha/x}\,d\alpha \quad \text{for } \operatorname{Re}(e^{i\theta}/x) > 0,$$

which gives

$$u(x) = \left[u_0 e - \sum_{j=1}^{\infty}\frac{1}{j!j} + \sum_{j=1}^{\infty}\frac{1}{j!j}\frac{1}{x^j} - \log x\right]e^{-1/x} \quad \text{for } x \in \widetilde{\mathbb{C}}.$$

References

[B] B. L. J. Braaksma, *Multisummability and Stokes multipliers of linear meromorphic differential equations*, J. Differential Equations 92 (1991), 45–75.

[CL] A. E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[E] J. Ecalle, *Introduction à l'accélération et à ses applications*, Travaux en Cours, Hermann, 1993.

[K] H. Komatsu, *Ultradistributions, I. Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo 20 (1973), 25–105.

[Ł1] G. Łysik, *On extendible ultradistributions*, Bull. Polish Acad. Sci. Math. 43 (1995), 29–40.

[Ł2] —, *Laplace ultradistributions on a half line and a strong quasi-analyticity principle*, Ann. Polon. Math. 63 (1996), 13–33.

[Ł3] —, *The Mellin transformation of strongly increasing functions*, J. Math. Sci. Univ. Tokyo 6 (1999), 49–86.

[M] B. Malgrange, *Sommation des séries divergentes*, Exposition. Math. 13 (1995), 163–222.

[T] J. C. Tougeron, *Gevrey expansions and applications*, preprint, Univ. of Toronto, 1991.

[Z] B. Ziemian, *Generalized analytic functions with applications to singular ordinary and partial differential equations*, Dissertationes Math. 354 (1996).

Institute of Mathematics
Polish Academy of Sciences
P.O. Box 137
Śniadeckich 8
00-950 Warszawa, Poland
E-mail: lysik@impan.gov.pl

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