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The BV-algebra of a Jacobi manifold

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Abstract. We show that the Gerstenhaber algebra of the 1-jet Lie algebroid of a Jacobi manifold has a canonical exact generator, and discuss duality between its homology and the Lie algebroid cohomology. We also give new examples of Lie bialgebroids over Poisson manifolds.

1. Introduction. A *Gerstenhaber algebra* is a triple $(\mathcal{A} = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}^k, \wedge, [\ ,\ ])$ where $\wedge$ is an associative, graded commutative algebra structure (e.g., over $\mathbb{R}$), $[\ ,\ ]$ is a graded Lie algebra structure for the *shifted degree* $[k] := k+1$ (the sign $:=$ denotes a definition), and

$$[a, b\wedge c] = [a,b]\wedge c + (-1)^{kj} b\wedge [a,c] \tag{1.1}$$

for all $a \in \mathcal{A}^{k+1}$, $b\in\mathcal{A}^j$, $c\in\mathcal{A}$. If this structure is supplemented by an endomorphism $\delta : \mathcal{A}\to\mathcal{A}$, of degree -1, such that $\delta^2 = 0$ and

$$[a,b] = (-1)^k(\delta(a\wedge b) - \delta a\wedge b - (-1)^k a\wedge \delta b) \quad (a\in\mathcal{A}^k,\ b\in\mathcal{A}), \tag{1.2}$$

one gets an *exact Gerstenhaber algebra* or a *Batalin–Vilkovisky algebra* (*BV-algebra*) with the *exact generator* δ. If we also have a differential $d:\mathcal{A}^k\to\mathcal{A}^{k+1}$ ($d^2=0$) such that

$$d(a\wedge b) = (da)\wedge b + (-1)^k a\wedge (db) \quad (a\in\mathcal{A}^k,\ b\in\mathcal{A}), \tag{1.3}$$

we will say that we have a *differential BV-algebra*. Finally, if

$$d[a,b] = [da,b] + (-1)^k[a,db] \quad (a\in\mathcal{A}^k,\ b\in\mathcal{A}) \tag{1.4}$$

the differential BV-algebra is said to be *strong* [21].

On the other hand, a Jacobi manifold (see, e.g., [5]) is a smooth manifold M^m (everything is of class C^∞ in this paper) with a Lie algebra structure of local type on the space of functions $C^\infty(M)$ or, equivalently [5], with a

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bivector field Λ and a vector field E such that

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \quad [\Lambda, E] = 0. \tag{1.5}$$

In (1.5) one has the usual Schouten–Nijenhuis brackets. If $E = 0$, (M, Λ) is a *Poisson manifold*.

One of the most interesting examples of a BV-algebra is that of the Gerstenhaber algebra of the cotangent Lie algebroid of a Poisson manifold, described by many authors (see [9], [8], etc.). More generally, Xu [21] extends a result of Koszul [9] and proves that the exact generators of the Gerstenhaber algebra of a Lie algebroid $A \to M$ are provided by flat connections on $\bigwedge^r A$ ($r = \operatorname{rank} A$), and Huebschmann [6] proves a corresponding result for *Lie–Rinehart algebras*.

The main aim of this note is to show that a Jacobi manifold also has a canonically associated, differential BV-algebra (which, however, is not strong), namely, the Gerstenhaber algebra of the 1-jet Lie algebroid defined by Kerbrat and Souici-Benhammadi [7]. Then we apply results of Xu [21] and Evens–Lu–Weinstein [3] to discuss duality between the homology of this BV-algebra and the cohomology of the Lie algebroid. (The homology was also independently introduced and studied by de León, Marrero and Padron [11].)

In the final section, we come back to a Poisson manifold M with the Poisson bivector Q, and show that the infinitesimal automorphisms E of Q yield natural Poisson bivectors of the Lie algebroid $TM \oplus \mathbb{R}$. These bivectors lead to triangular Lie bialgebroids and BV-algebras in the usual way [8], [21].

Notice that BV-algebras play an important role in some recent research in theoretical physics (see, e.g., [4]).

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2. The Jacobi BV-algebra. For any Lie algebroid $A \to M$ with anchor $\alpha : A \to TM$ one has the Gerstenhaber algebra $\mathcal{A}(A)$ defined by

$$\mathcal{A}(A) := \Big(\bigoplus_{k\in\mathbb{N}} \Gamma \bigwedge^k A, \wedge, [\ ,\ ]_{\mathrm{SN}}\Big), \tag{2.1}$$

where Γ denotes spaces of global cross sections, and SN denotes the Schouten–Nijenhuis bracket (see, e.g., [8], [21]; on the other hand, we refer the reader to [15, 8, 3], for instance, for the basics of Lie algebroids and Lie algebroid calculus). The BV-algebra which we want to discuss is associated with the 1-jet Lie algebroid of a Jacobi manifold (M, Λ, E) defined in [7], which we present as follows.

We identify M with $M \times \{0\} \subseteq M \times \mathbb{R}$, where $M \times \mathbb{R}$ is endowed with the Poisson bivector [5]

$$P := e^{-t}\left(\Lambda + \frac{\partial}{\partial t} \wedge E\right) \qquad (t \in \mathbb{R}). \tag{2.2}$$

Let $J^1M = T^*M \oplus \mathbb{R}$ be the vector bundle of 1-jets of real functions on M, and notice that ΓJ^1M is isomorphic as a $C^\infty(M)$-module to

$$\Gamma_0(M) := \{e^t(\alpha + f dt) \ / \ \alpha \in \textstyle\bigwedge^1 M,\ f \in C^\infty(M)\} \subseteq \textstyle\bigwedge^1(M \times \mathbb{R}). \tag{2.3}$$

(For any differentiable manifold V we denote by $\bigwedge^k V$ the space $\Gamma \bigwedge^k T^*V$ of differential k-forms on V.) A straightforward computation shows that $\Gamma_0(M)$ is closed under the bracket of the cotangent Lie algebroid of $(M \times \mathbb{R}, P)$ (see, e.g., [19]), namely

$$\begin{aligned}\{e^t(\alpha + f dt), e^t(\beta + g dt)\}_P = e^t[&L_{\sharp_\Lambda \alpha}\beta - L_{\sharp_\Lambda \beta}\alpha - d(\Lambda(\alpha, \beta)) \\ &+ f L_E \beta - g L_E \alpha - \alpha(E)\beta + \beta(E)\alpha \\ &+ (\{f, g\} - \Lambda(df - \alpha, dg - \beta))dt],\end{aligned} \tag{2.4}$$

where $\langle \sharp_\Lambda \alpha, \beta\rangle := \Lambda(\alpha, \beta)$ $(\alpha, \beta \in \bigwedge^1 M)$, and

$$\{f, g\} = \Lambda(df, dg) + f(Eg) - g(Ef) \qquad (f, g \in C^\infty(M))$$

is the bracket which defines the Jacobi structure [5].

Therefore, (2.4) produces a Lie bracket on $\Gamma J^1 M$. Moreover, if $\sharp_P$ is defined similarly to $\sharp_\Lambda$, we get

$$\sharp_P(e^t(\alpha + f dt)) = \sharp_\Lambda \alpha + fE - \alpha(E)\frac{\partial}{\partial t}, \tag{2.5}$$

and

$$\varrho := (\mathrm{pr}_{TM} \circ \sharp_P)_{t=0} : J^1 M \to TM \tag{2.6}$$

has the properties of an anchor, since so does $\sharp_P$.

Formulas (2.4), (2.6) precisely yield the Lie algebroid structure on J^1M defined in [7]. In what follows we refer to it as the 1-*jet Lie algebroid*. The mapping $f \mapsto e^t(df + f dt)$ is a Lie algebra homomorphism from the Jacobi algebra of M to $\Gamma_0(M)$.

2.1. Proposition. *The Gerstenhaber algebra* $\mathcal{A}(J^1M)$ *is isomorphic to the subalgebra* $\mathcal{A}_0(M) := \bigoplus_{k\in\mathbb{N}} \bigwedge^k \Gamma_0(M)$ *of the Gerstenhaber algebra* $\mathcal{A}(T^*(M \times \mathbb{R}))$.

P r o o f. The elements of $\mathcal{A}_0^k(M) := \bigwedge^k \Gamma_0(M)$ are of the form

$$\lambda = e^{kt}(\lambda_1 + \lambda_2 \wedge dt) \quad (\lambda_1 \in \bigwedge^k M,\ \lambda_2 \in \bigwedge^{k-1} M), \tag{2.7}$$

and we see that $\mathcal{A}_0(M)$ is closed under the wedge product and under the bracket $\{\ ,\ \}_P$ of differential forms on the Poisson manifold $(M \times \mathbb{R}, P)$ (see, e.g., [19]). Accordingly, $(\mathcal{A}(J^1M), \wedge, \{\ ,\ \})$ and $(\mathcal{A}_0(M), \wedge, \{\ ,\ \}_P)$ are isomorphic Gerstenhaber algebras since they are isomorphic at the grade 1 level, and the brackets of terms of higher degree are spanned by those of degree 1. ∎

2.2. Remark. Since $\mathcal{A}_0(M)$ is a Gerstenhaber algebra, the pair $(\mathcal{A}_0^0 = C^\infty(M),\ \mathcal{A}_0^1 = \Gamma_0(M))$ is a Lie–Rinehart algebra [6].

Now, we can prove

2.3. Proposition. *The Gerstenhaber algebra $\mathcal{A}_0(M)$ has a canonical exact generator.*

P r o o f. It is known that $\mathcal{A}(T^*(M\times\mathbb{R}))$ has the exact generator of Koszul and Brylinski (see, e.g., [19])

$$\delta_P = i(P)d - di(P), \tag{2.8}$$

where P is the bivector (2.2). Hence, all we have to do is to check that $\delta_P\lambda \in \mathcal{A}_0^{k-1}(M)$ if λ is given by (2.7).

First, we notice that

$$i(P)(dt \wedge \mu) = e^{-t}(i(E)\mu + dt \wedge (i(\Lambda)\mu)) \quad (\mu \in \bigwedge^* M). \tag{2.9}$$

Then, if we also introduce the operator $\delta_\Lambda := i(\Lambda)d - di(\Lambda)$ (cf. [1]), and compute for λ of (2.7), we get

$$\begin{aligned}\delta_P\lambda = e^{(k-1)t}[&\delta_\Lambda\lambda_1 + (-1)^k L_E\lambda_2 + ki(E)\lambda_1 \\ &+ (\delta_\Lambda\lambda_2 + (-1)^k i(\Lambda)\lambda_1 + (k-1)i(E)\lambda_2) \wedge dt].\end{aligned} \tag{2.10}$$ ∎

It follows from (2.10) that δ_P restricts to an exact generator δ of the Gerstenhaber algebra $\mathcal{A}(J^1M)$, and the latter becomes a BV-algebra. This is the BV-algebra announced in Section 1, and we call it the *Jacobi BV-algebra of the Jacobi manifold* (M, Λ, E). We can look at it under the two isomorphic forms indicated by Proposition 2.1.

It is easy to see that the Jacobi BV-algebra above has the differential

$$\bar{d}\lambda := e^{(k+1)t}d(e^{-kt}\lambda), \tag{2.11}$$

where λ is given by (2.7). But a computation shows that $\bar{d}$ is not a derivation of the Lie bracket $\{\ ,\ \}$ of $\mathcal{A}(J^1M)$. Another difference from the Poisson case is the formula

$$\begin{aligned}(\delta_P\bar{d} + \bar{d}\delta_P)\lambda = e^{kt}[&(k+1)i(E)d\lambda_1 \\ &+ (L_E\lambda_2 + (k+1)i(E)d\lambda_2 - (-1)^k\delta_\Lambda\lambda_1) \wedge dt],\end{aligned} \tag{2.12}$$

where λ is given by (2.7) again. This formula is the result of technical computations which we omit.

2.4. Remark. If we refer to the Poisson case $E = 0$, we see that both T^*M and J^1M have natural structures of Lie algebroids. The Lie bracket and anchor map of J^1M are given by

$$\{e^t(\alpha + f dt), e^t(\beta + g dt)\} = e^t[\{\alpha, \beta\}_\Lambda + ((\sharp_\Lambda\alpha)g - (\sharp_\Lambda\beta)f - \Lambda(\alpha, \beta))dt] \tag{2.13}$$

and

$$\varrho(e^t(\alpha + f dt)) = \sharp_\Lambda\alpha, \tag{2.14}$$

and the mapping $\alpha \mapsto e^t(\alpha + 0dt)$ preserves the Lie bracket, hence T^*M is a Lie subalgebroid of J^1M, and the latter is an extension of the former by the trivial line bundle $M \times \mathbb{R}$. J^1M has not yet been used in Poisson geometry.

3. The homology of the Jacobi BV-algebra. We call the homology of the Jacobi BV-algebra of a Jacobi manifold (M, Λ, E), with boundary operator δ, the *Jacobi homology* $H_k^{\mathrm{J}}(M, \Lambda, E)$. (Another "Jacobi homology" was studied in [1].) Here, we look at this homology from the point of view of [21] and [3], and discuss a duality between the Jacobi homology and the Lie algebroid cohomology of J^1M, called *Jacobi cohomology*.

Jacobi cohomology coincides with the one studied by de León, Marrero and Padrón in [10]. If $C \in \Gamma \bigwedge^k (J^1M)^*$ is seen as a k-multilinear skew symmetric form on arguments (2.7) of degree 1, at $t = 0$, it may be written as

$$C = \widetilde{C}|_{t=0} := e^{-kt}\Big[\Big(C_1 + \frac{\partial}{\partial t} \wedge C_2\Big)\Big]_{t=0} \quad (C_1 \in \mathcal{V}^k M,\ C_2 \in \mathcal{V}^{k-1} M), \tag{3.1}$$

where $\mathcal{V}^k M := \Gamma \bigwedge^k TM$ is the space of k-vector fields on M. Furthermore, the coboundary, say σ, is given by the usual formula

$$(\sigma C)(s_0, \ldots, s_k) = \sum_{i=0}^{k} (-1)^i (\varrho s_i) C(s_0, \ldots, \widehat{s}_i, \ldots, s_k) + \sum_{i<j=1}^{k} (-1)^{i+j} C(\{s_i, s_j\}, s_0, \ldots, \widehat{s}_i, \ldots, \widehat{s}_j, \ldots, s_k), \tag{3.2}$$

where ϱ is given by (2.6), and $s_i \in \Gamma J^1M$. Again, if we view the arguments as forms (2.7) with $k = 1$, (3.2) becomes

$$\sigma C = [\sigma_P \widetilde{C}]_{t=0} = [P, \widetilde{C}]_{t=0}, \tag{3.3}$$

where σ_P is the Lichnerowicz coboundary (see, e.g., [19]). Up to sign, (3.3) is the coboundary defined in [10], namely

$$\begin{aligned}(3.4)\qquad \sigma C = {} & [\Lambda, C_1] - kE \wedge C_1 - \Lambda \wedge C_2 \\ & - \frac{\partial}{\partial t} \wedge ([\Lambda, C_2] - (k-1)E \wedge C_2 + [E, C_1]).\end{aligned}$$

We denote the Jacobi cohomology spaces by $H^k_{\mathrm{J}}(M, \Lambda, E)$.

3.1. Remark [10]. The anchor ϱ induces homomorphisms $\varrho^\sharp : H^k_{\mathrm{de\,R}}(M) \to H^k_{\mathrm{J}}(M, \Lambda, E)$ given by

$$(3.5)\quad (\varrho^\sharp \lambda)(s_1, \ldots, s_k) = (-1)^k \lambda(\varrho s_1, \ldots, \varrho s_k) \qquad (\lambda \in \bigwedge^k M,\ s_i \in \Gamma J^1 M).$$

Now, we need a recapitulation of several results of [21] and [3].

For a Lie algebroid $A \to M$ with anchor a, an *A-connection* ∇ on a vector bundle $E \to M$ consists of *derivatives* $\nabla_s e \in \Gamma E$ $(s \in \Gamma A,\ e \in \Gamma E)$ which are $\mathbb{R}$-bilinear and satisfy the conditions

$$\nabla_{fs} e = f \nabla_s e, \quad \nabla_s (fe) = (a(s)f)e + f \nabla_s e \qquad (f \in C^\infty(M)).$$

For an A-connection, curvature may be defined as for usual connections. Any flat A-connection ∇ on $\bigwedge^r A$ $(r = \operatorname{rank} A)$ produces a *Koszul operator* $D : \Gamma \bigwedge^k A \to \Gamma \bigwedge^{k-1} A$, locally given by

$$DU = (-1)^{r-k+1} \Big[i(d\omega)\Omega + \sum_{h=1}^r \alpha^h \wedge (i(\omega)\nabla_{s_h} \Omega) \Big],$$

where $\Omega \in \Gamma \bigwedge^r A$, $\omega \in \Gamma \bigwedge^{r-k} A^*$ is such that $i(\omega)\Omega = U$, s_h is a local basis of A, and α^h is the dual cobasis of A^*. Moreover, D is an exact generator of the Gerstenhaber algebra of A, and every exact generator is defined by a flat A-connection as above. The operator D is a boundary operator, and yields a corresponding homology, called the *homology of the Lie algebroid A with respect to the flat A-connection* ∇, $H_k(A, \nabla)$. For two flat connections ∇, $\overline{\nabla}$ such that $D - \overline{D} = i(\alpha)$, where $\alpha = d_A f$ $(f \in C^\infty(M)$; d_A is the differential of the Lie algebroid calculus of $A)$, one has $H_k(M, \nabla) = H_k(M, \overline{\nabla})$. If there exists $\Omega \in \Gamma \bigwedge^r A^*$ which is nowhere zero, and $\nabla^* \Omega = 0$ where ∇^* is the connection induced by ∇ in the dual bundle $\bigwedge^r A^*$ of $\bigwedge^r A$, one has the duality $H_k(A, \nabla) = H^{r-k}(A)$, defined by sending $Q \in \Gamma \bigwedge^k A$ to $*_\Omega Q := i(Q)\Omega$.

These results may be applied to the case where A is the cotangent Lie algebroid of an orientable Poisson manifold (N^n, Q). In this case, the flat connection $\nabla_\theta \Psi = \theta \wedge (di(Q)\Psi)$ $(\theta \in T^*N,\ \Psi \in \bigwedge^n N)$ precisely has the Koszul operator δ_Q and defines the known Poisson homology $H_k(N, Q)$ (see, e.g., [19]). Finally ([21], Proposition 4.6 and Theorem 4.7), if N has the volume form Ω, which defines a connection ∇_0 by $\nabla_0 \Omega = 0$, and if W^Q

is the *modular vector field* which acts on $f \in C^{\infty}(M)$ according to the equation

$$L_{X_f^Q}\Omega = (W^Q f)\Omega$$

(X_f^Q is the Hamiltonian field of f) [20], one has $\delta_Q - D_0 = i(W^Q)$, where D_0 is the Koszul operator defined by ∇_0. Accordingly, if the modular field W^Q is Hamiltonian (i.e., (N,Q) is a *unimodular Poisson manifold*), $H_k(N,Q) = H^{n-k}(T^*N)$.

The case of a general, possibly non-orientable, Poisson manifold is studied in [3]. The expression of $\nabla_\theta\Psi$ above can be viewed as the local equation of a connection on $\bigwedge^n T^*N$, and it still defines the Koszul operator δ_Q. The general duality Theorem 4.5 of [3] is

$$(*) \qquad H_k(N,Q) = H^{n-k}(T^*N, \textstyle\bigwedge^n T^*N),$$

where the right hand side is the cohomology of the Lie algebroid T^*N with values in the line bundle $\bigwedge^n T^*N$. This means that the k-cocycles are spanned by cross sections $V \otimes \Psi$, $V \in \mathcal{V}^k N$, $\Psi \in \Gamma \bigwedge^n T^*N$, and the coboundary is given by

$$\partial(V \otimes \Psi) = [Q,V] \otimes \Psi + (-1)^k V \wedge \nabla\Psi,$$

where $\nabla\Psi \in \mathcal{V}^1 N \otimes \Gamma \bigwedge^n T^*N = \mathrm{Hom}(\Gamma T^*N, \Gamma \bigwedge^n T^*N)$ is defined by $(\nabla\Psi)(\theta) = \nabla_\theta\Psi$, $\theta \in \Gamma T^*N$. The duality $(*)$ is again defined by the isomorphism which sends $V \otimes \Psi$ to $i(V)\Psi$.

With this recapitulation finished, we apply the results to Jacobi manifolds (M^m, Λ, E). Consider the Poisson manifold $(M \times \mathbb{R}, P)$ which we already used before. Then δ_P is the Koszul operator of the $(T^*M \times \mathbb{R})$-connection

$$(3.6) \qquad \nabla_\theta\Psi = \theta \wedge (di(P)\Psi) \qquad (\theta \in T^*(M \times \mathbb{R}),\ \Psi \in \textstyle\bigwedge^{m+1}(M \times \mathbb{R})).$$

In particular, if we take

$$(3.7) \qquad \theta = e^t(\alpha + f dt), \quad \Psi = e^{(m+1)t}\Phi \wedge dt \qquad (\alpha \in T^*M,\ \Phi \in \textstyle\bigwedge^m M),$$

then $\Psi \in \bigwedge^{m+1}(J^1M)$, (2.9) implies

$$(3.8) \qquad \nabla_\theta\Psi = e^{(m+1)t}[f di(E)\Phi - \alpha \wedge (di(\Lambda)\Phi + mi(E)\Phi)] \wedge dt,$$

and this formula may be viewed as defining a J^1M-connection on $\bigwedge^{m+1} J^1M$. Clearly, the Koszul operator of this connection must be the δ of (2.10). Therefore, we have

3.2. PROPOSITION. *The Jacobi homology of* (M, Λ, E) *is equal to the homology of the Lie algebroid* J^1M *with respect to the flat connection* (3.8), *i.e.*,

$$(3.9) \qquad H_k^{\mathrm{J}}(M, \Lambda, E) = H_k(J^1M, \nabla).$$

Now, assume that M has a volume form $\Phi \in \bigwedge^m M$. Then $\Omega := e^{(m+1)t}\Phi \wedge dt$ is a volume form on $M \times \mathbb{R}$, and one has a connection ∇_0 defined by $\nabla_0 \Omega = 0$ with a Koszul operator D_0 such that

$$\delta_P - D_0 = i(W^P), \tag{3.10}$$

where W^P is the corresponding modular vector field, i.e.

$$L_{X^P_\varphi}\Omega = (W^P\varphi)\Omega \quad (\varphi \in C^\infty(M \times \mathbb{R})). \tag{3.11}$$

We need the interpretation of (3.10) at $t = 0$. To get it, we take local coordinates (x^i) on M, and compute the local components of W^P by using (3.11) for $\varphi = x^i$ and $\varphi = t$. Generally, we have

$$X^P_\varphi = i(d\varphi)P = e^{-t}\bigg(\sharp_\Lambda d\varphi + \frac{\partial\varphi}{\partial t}E - (E\varphi)\frac{\partial}{\partial t}\bigg). \tag{3.12}$$

On the other hand, on M, define a vector field V and a function $\operatorname{div}_\Phi E$ by

$$L_{\sharp_\Lambda df}\Phi = (Vf)\Phi, \quad L_E\Phi = (\operatorname{div}_\Phi E)\Phi \quad (f \in C^\infty(M)). \tag{3.13}$$

(The fact that V is a derivation of $C^\infty(M)$ follows easily from the skew symmetry of Λ.) Then the calculation of the local components of W^P yields

$$W^P = e^{-t}\bigg[V - mE + (\operatorname{div}_\Phi E)\frac{\partial}{\partial t}\bigg]. \tag{3.14}$$

At $t = 0$, (3.14) defines a section of $TM \oplus \mathbb{R}$ which we denote by $V^{(\Lambda,E)}$ and call the *modular field* (not a vector field, of course) of the Jacobi manifold.

As in the Poisson case, if $\Phi \mapsto a\Phi$ $(a > 0)$, then $V^{(\Lambda,E)} \mapsto V^{(\Lambda,E)} + \sigma(\ln a)$, hence what is well defined is the Jacobi cohomology class $[V^{(\Lambda,E)}]$, to be called the *modular class*. If the modular class is zero, (M, Λ, E) is a *unimodular Jacobi manifold*.

It is also possible to get the modular class $[V^{(\Lambda,E)}]$ from the general definition of the modular class of a Lie algebroid [3]. In the case of the algebroid J^1M, the definition of [3] means computing the expression

$$\mathcal{E} := (L^{J^1M}_{e^t(df+f dt)}[(e^{mt}\Phi) \wedge (e^t dt)]) \otimes \Phi + (e^{(m+1)t}\Phi \wedge dt) \otimes (L_{\varrho(e^t(df+f dt))}\Phi),$$

where ϱ is given by (2.6), and

$$L^{J^1M}_{e^t(df+f dt)}[(e^{mt}\Phi) \wedge (e^t dt)] = \{e^t(df + f dt), (e^{mt}\Phi) \wedge (e^t dt)\}_P.$$

If we decompose $(e^{mt}\Phi) = \bigwedge_{i=1}^n (e^t\varphi_i)$, $\varphi_i \in \bigwedge^1 M$, the result of the required computation turns out to be

$$\mathcal{E} = (2(Vf) + 2f(\operatorname{div}_\Phi E) - Ef)(e^{(m+1)t}\Phi \wedge dt) \otimes \Phi.$$

By comparing with (3.14), we see that the modular class in the sense of [3] is the Jacobi cohomology class of the cross section of $TM \oplus \mathbb{R}$ defined by

$$A^{(\Lambda,E)} = 2V^{(\Lambda,E)} - (2m + 1)E.$$

With all this notation in place, the recalled results of [21], Proposition 4.6, and [3], Theorem 4.5 (see also [6]) yield

3.3. PROPOSITION. *If* (M, Λ, E) *is a unimodular Jacobi manifold one has duality between Jacobi homology and cohomology*:

$$H_k^{\mathrm{J}}(M, \Lambda, E) = H_{\mathrm{J}}^{m-k+1}(M, \Lambda, E). \tag{3.15}$$

If (M, Λ, E) *is an arbitrary Jacobi manifold*, *one has the duality*

$$H_k^{\mathrm{J}}(M, \Lambda, E) = H_{\mathrm{J}}^{m-k+1}(J^1M, \textstyle\bigwedge^{m+1} J^1M). \tag{3.15$'$}$$

Proof. The right hand side of (3.15′) is *Jacobi cohomology with values in* $\bigwedge^{m+1} J^1M$, similar to that in $(*)$. The homologies and cohomologies of (3.15) and (3.15′) are to be seen as given by subcomplexes of $\bigoplus_k \bigwedge^k(M\times\mathbb{R})$, $\bigoplus_k \mathcal{V}^k(M\times\mathbb{R})$ defined by (2.7) and (3.1). Then the result follows by the proofs of the theorems of [21], [3] quoted earlier, if we notice that

$$\begin{aligned} i\bigg[e^{-kt}\bigg(C_1 + \frac{\partial}{\partial t}\wedge C_2\bigg)\bigg](e^{(m+1)t}\Phi\wedge dt) \\ = e^{(m-k+1)t}[(-1)^m i(C_2)\Phi + (i(C_1)\Phi)\wedge dt]. \end{aligned}$$

The notation is that of (3.1) and (3.7). ■

In particular, let us consider the *transitive Jacobi manifolds* [5].

a) Let M^{2n} be a locally or globally conformal symplectic manifold with the global 2-form Ω such that $\Omega|_{U_\alpha} = e^{\sigma_\alpha}\Omega_\alpha$, where Ω_α are symplectic forms on the sets U_α of an open covering of M, and $\sigma_\alpha \in C^\infty(U_\alpha)$. Then (see, e.g., [18]) $\{d\sigma_\alpha\}$ glue up to a global closed 1-form ω, which is exact iff there exists α with $U_\alpha = M$, and $\sharp_\Lambda := \flat_\Omega^{-1}$, $E := \sharp_\Lambda\omega$ define a Jacobi structure on M (cf. [5]). It follows easily that $L_E\Omega = 0$, hence $\operatorname{div}_{\Omega^n} E = 0$. Furthermore,

$$L_{\sharp_\Lambda df}\Omega^n = -n(n-1)df\wedge\omega\wedge\Omega^{n-1}.$$

Using the *Lepage decomposition theorem* ([12], p. 46) we see that $df\wedge\omega = \xi + \varphi\Omega$, where

$$\xi\wedge\Omega^{n-1} = 0, \qquad \varphi = -\frac{1}{n}i(\Lambda)(df\wedge\omega) = Ef.$$

Hence, $V = -n(n-1)E$, and $V^{(\Lambda,E)} = -n(2n-1)E$. Then, for $f\in C^\infty(M)$, (3.4) yields $\sigma f = \sharp_\Lambda df - (Ef)(\partial/\partial t)$, and $\sigma f = E$ holds iff $\omega = df$. Thus, (3.15) holds on globally conformal symplectic manifolds. But (3.15) may not hold in the true locally conformal symplectic case. For instance, it follows from Corollary 3.15 of [11] that the result does not hold on a Hopf manifold with its natural locally conformal Kähler structure (private correspondence from J. C. Marrero).

b) Let M^{2n+1} be a contact manifold with contact 1-form θ such that $\Phi := \theta \wedge (d\theta)^n$ is nowhere zero. Then M has the Reeb vector field E where

$$i(E)\theta = 1, \quad i(E)d\theta = 0,$$

and for all $f \in C^\infty(M)$ there is a *Hamiltonian vector field* X^θ_f such that

$$i(X^\theta_f)\theta = f, \quad i(X^\theta_f)d\theta = -df + (Ef)\theta.$$

Furthermore, if

$$\Lambda(df, dg) := d\theta(X^\theta_f, X^\theta_g) \quad (f, g \in C^\infty(M)),$$

then (Λ, E) is a Jacobi structure [5].

Now, let (q^i, p_i, z) $(i = 1, \ldots, n)$ be local canonical coordinates such that $\theta = dz - \sum_i p_i dq^i$. Then

$$E = \frac{\partial}{\partial z}, \quad \Lambda = \sum_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{\partial}{\partial z} \wedge \Big(\sum_i p_i \frac{\partial}{\partial p_i} \Big).$$

This leads to $\operatorname{div}_\Phi E = 0$, $V^{(\Lambda,E)} = nE$, and it follows that there is no $f \in C^\infty(M)$ satisfying $\sigma f = (nE \oplus 0)$.

We close this section by the remark that the identification of a manifold M with $M \times \{0\} \subseteq M \times \mathbb{R}$ leads to other interesting structures as well. For instance, if we define the spaces

$$\textstyle\bigwedge_0^k M := \{e^t(\xi_1 + \xi_2 \wedge dt) \mid \xi_1 \in \bigwedge^k M,\ \xi_2 \in \bigwedge^{k-1} M\},$$

the triple $(\bigoplus_k \bigwedge_0^k M, d, i(X + f(\partial/\partial t))$ is a *Gelfand–Dorfman complex* [2], and a Jacobi structure on M is equivalent to a *Hamiltonian structure* [2] on this complex.

On the other hand, if we have a Jacobi manifold (M, Λ, E), and put

$$\mathcal{V}_0^k M := \Big\{ e^{-(k-1)t}\Big(Q_1 + \frac{\partial}{\partial t} \wedge Q_2\Big) \Big/ Q_1 \in \mathcal{V}^k M,\ Q_2 \in \mathcal{V}^{k-1} M \Big\},$$

then $(\bigoplus_k \mathcal{V}_0^k M, [\ ,\ ], \sigma_P)$ (P is defined by (2.2) and $[\ ,\ ]$ is the usual Schouten–Nijenhuis bracket) is a differential graded Lie algebra, whose cohomology is exactly the 1-*differentiable Chevalley–Eilenberg cohomology* $H^k_{1\text{-dif}}(M, \Lambda, E)$ of Lichnerowicz [13]. In particular, $H^1_{1\text{-dif}}(M, \Lambda, E)$ is the quotient of the space of conformal Jacobi infinitesimal automorphisms by the space of Jacobi Hamiltonian vector fields [13].

4. Lie bialgebroid structures on $TM \oplus \mathbb{R}$. In the Poisson case, T^*M is a Lie bialgebroid over M (see [8], [16]) with dual TM. This is not true for J^1M on Jacobi manifolds in spite of the fact that $(J^1M)^* = TM \oplus \mathbb{R}$ has a natural Lie algebroid structure, which extends the one of TM. Namely, if we view $\mathcal{X} \in \Gamma(TM \oplus \mathbb{R})$ as a vector field of $M \times \mathbb{R}$ given by

$$\mathcal{X} = \left(X + f\frac{\partial}{\partial t}\right)\Big|_{t=0} \quad (X \in \Gamma TM,\ f \in C^\infty(M)), \tag{4.1}$$

we have the Lie bracket

$$[\mathcal{X}, \mathcal{Y}]_0 := \left[X + f\frac{\partial}{\partial t}, Y + g\frac{\partial}{\partial t}\right] = [X, Y] + (Xg - Yf)\frac{\partial}{\partial t}, \tag{4.2}$$

and the anchor map $a(\mathcal{X}) := X$. If we were in the case of a Lie bialgebroid, the bracket $\{f, g\}_s := \langle df, d_* g\rangle$ $(f, g \in C^\infty(M))$, where d, d_* are the differentials of the Lie algebroids $TM \oplus \mathbb{R}$ and $J^1 M$, respectively, would be Poisson [8], [16]. This is not true since one gets $\{f, g\}_s = \Lambda(df, dg)$.

4.1. Remark. The differential $\bar{d}$ defined by (2.11) is the same as the differential d of the Lie algebroid $TM \oplus \mathbb{R}$ with bracket (4.2).

In Poisson geometry, the cotangent Lie bialgebroid structure is produced by a Poisson bivector Π of TM, i.e., $[\Pi, \Pi] = 0$. It is natural to ask what is the structure produced by a Poisson bivector Π of $TM \oplus \mathbb{R}$. As a matter of fact, we will ask this question in the more general situation where we fix a closed 2-form Ω on M, and take the Lie bracket

$$[\mathcal{X}, \mathcal{Y}]_\Omega := [\mathcal{X}, \mathcal{Y}]_0 + \Omega(X, Y)\frac{\partial}{\partial t}. \tag{4.2$'$}$$

The notation and the anchor map a are the same as for (4.2). It is known that (4.2′) defines all the transitive Lie algebroid structures over M such that the kernel of the anchor is a trivial line bundle, up to isomorphism [15]. A Poisson bivector Π on $TM \oplus \mathbb{R}$ with bracket (4.2′) will be called an *Ω-Poisson structure* on M.

4.2. Proposition. *An Ω-Poisson structure Π on M is equivalent to a pair (Q, E), where Q is a Poisson bivector on M (i.e., $[Q, Q] = 0$), and E is a vector field such that*

$$L_E Q = \sharp_Q \Omega. \tag{4.3}$$

Proof. Using the identification (4.1) of the cross sections of $TM \oplus \mathbb{R}$ with vector fields on $M \times \mathbb{R}$ for $t = 0$, and local coordinates (x^i) on M, we may write

$$\Pi = Q + \frac{\partial}{\partial t} \wedge E = \frac{1}{2}Q^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \frac{\partial}{\partial t} \wedge \left(E^k(x)\frac{\partial}{\partial x^k}\right), \tag{4.4}$$

where Q is a bivector field on M, E is a vector field, and the Einstein summation convention is used.

Now, $[\Pi, \Pi]_\Omega = 0$ can be expressed by the known formula for the Schouten–Nijenhuis bracket of decomposable multivectors (see, e.g., [19], formula (1.12)), and (4.2′). The result is equivalent to $[Q, Q] = 0$ and (4.3). ∎

4.3. Corollary. *If (M,Q) is a Poisson manifold, then Q extends to an Ω-Poisson structure for every closed 2-form Ω, where the de Rham class $[\Omega]$ has zero $\sharp_Q$-image in the Poisson cohomology of (M,Q), by taking E such that* (4.3) *holds.*

This is just a reformulation of Proposition 4.2.

It is well known that a Poisson bivector on a Lie algebroid A induces a bracket on ΓA^* such that (A,A^*) is a triangular Lie bialgebroid [8], [16]. Namely, the Poisson bivector Π of (4.4) yields the following bracket:

$$\{\alpha\oplus f,\beta\oplus g\}_\Omega := L^\Omega_{\sharp_\Pi(\alpha\oplus f)}(\beta\oplus g) - L^\Omega_{\sharp_\Pi(\beta\oplus g)}(\alpha\oplus f) - d_\Omega(\Pi(\alpha\oplus f,\beta\oplus g)), \tag{4.5}$$

where $\alpha\oplus f,\beta\oplus g\in\Gamma J^1M$, and the index Ω denotes the fact that the operators involved are those of the Lie algebroid calculus of (4.2′).

To make this formula explicit, notice that

$$\sharp_\Pi(\alpha+f\,dt)=\sharp_Q\alpha+fE-\alpha(E)\frac{\partial}{\partial t}, \tag{4.6}$$

whence

$$\Pi(\alpha+f\,dt,\beta+g\,dt)=Q(\alpha,\beta)+f\beta(E)-g\alpha(E). \tag{4.7}$$

Then, by evaluation on a field of the form (4.1), and with (4.2′), we obtain

$$L^\Omega_{\sharp_\Pi(\alpha+f\,dt)}(\beta+g\,dt)=L_{\sharp_\Pi(\alpha+f\,dt)}(\beta+g\,dt)-g(\flat_\Omega\sharp_Q\alpha)-fg\,i(E)\Omega, \tag{4.8}$$

where $\flat_\Omega X:=i(X)\Omega$. As a consequence, (4.5) becomes

$$\{\alpha\oplus f,\beta\oplus g\}_\Omega := [\{\alpha,\beta\}_Q+f(L_E\beta+\flat_\Omega\sharp_Q\beta)-g(L_E\alpha+\flat_\Omega\sharp_Q\alpha)] \oplus[(\sharp_Q\alpha)g-(\sharp_Q\beta)f+f(Eg)-g(Ef)], \tag{4.9}$$

where (Q,E) are associated with Π as in Proposition 4.2.

The anchor map of the Lie algebroid J^1M with (4.9) is $\varrho:=\mathrm{pr}_{TM}\circ\sharp_\Pi$, and it is provided by (4.6).

In particular, Proposition 4.2 tells us that a pair (Q,E) which consists of a Poisson bivector Q and an infinitesimal automorphism E of Q, to which we will refer as an *enriched Poisson structure*, provides a Poisson bivector Π on $TM\oplus\mathbb{R}$ with bracket (4.2), and a Lie bialgebroid $(TM\oplus\mathbb{R}, J^1M=T^*M\oplus\mathbb{R})$.

An example (suggested by [14]) can be obtained as follows. Let (M,Λ,E) be a Jacobi manifold. A *time function* is a function $\tau\in C^\infty(M)$ which satisfies $E\tau=1$. If such a function exists, then $(\Lambda_0:=\Lambda-(\sharp_\Lambda dt)\wedge E,E)$ is an enriched Poisson structure. Jacobi manifolds with time may be seen as generalized phase spaces of time-dependent Hamiltonian systems. Namely, if $H\in C^\infty(M)$ is the Hamiltonian function, the trajectories of the system are the integral lines of the vector field $X^0_H:=\sharp_{\Lambda_0}dH+E$.

Let us briefly indicate the important objects associated with the Lie algebroids $TM \oplus \mathbb{R}$ defined by the bracket (4.2′), and J^1M with the bracket (4.9).

The cohomology of $TM \oplus \mathbb{R}$ is that of the cochain spaces

$$\bigwedge\nolimits_{\Omega}^{k} M := \{\lambda = \lambda_1 + \lambda_2 \wedge dt \;/\; \lambda_1 \in \bigwedge\nolimits^{k} M,\ \lambda_2 \in \bigwedge\nolimits^{k-1} M\} \tag{4.10}$$

with the corresponding coboundary, say d_{Ω}. A straightforward evaluation of $d_{\Omega}\lambda$ on arguments $X_i + f_i(\partial/\partial t)$, in accordance with the Lie algebroid calculus [15], yields the formula

$$d_{\Omega}\lambda = d\lambda - (-1)^k \Omega \wedge \lambda_2. \tag{4.11}$$

The *Poisson cohomology* of $TM\oplus\mathbb{R}$ above, i.e. the cohomology of the Lie algebroid J^1M with (4.9), can be viewed (with (4.1)) as having the cochain spaces

$$\mathcal{C}^k(M) := \left\{ C = C_1 + \frac{\partial}{\partial t} \wedge C_2 \;\Big/\; C_1 \in \mathcal{V}^k M,\ C_2 \in \mathcal{V}^{k-1} M \right\}, \tag{4.12}$$

and the coboundary $\partial C = [\Pi, C]_{\Omega}$, with Π of (4.4) and the Ω-Schouten–Nijenhuis bracket. In order to write down a concrete expression of this coboundary, we define an operation $U \wedge_{\Omega} V \in \mathcal{V}^{k+h-2}$, for $U \in \mathcal{V}^k M$, $V \in \mathcal{V}^h M$, by the formula

$$\begin{aligned} U \wedge_{\Omega} V(\alpha_1, \ldots, \alpha_{k+h-2}) &= \frac{1}{(k-1)!(h-1)!} \sum_{\sigma \in S_{k+h-2}} [(\operatorname{sign} \sigma) \\ &\quad \cdot \sum_{i=1}^{m} U(\varepsilon^i, \alpha_{\sigma_1}, \ldots, \alpha_{\sigma_{k-1}}) V(\flat_{\Omega} e_i, \alpha_{\sigma_k}, \ldots, \alpha_{\sigma_{k+h-2}})], \end{aligned} \tag{4.13}$$

where S is the symmetric group, e_i is a local tangent basis of M, and ε^i is the corresponding dual cobasis. If U, V are vector fields, $U \wedge_{\Omega} V = \Omega(U, V)$. By computing, for decomposable multivectors C_1, C_2 we get

$$\partial C = [Q, C_1] + \frac{\partial}{\partial t} \wedge ([Q, C_2] + Q \wedge_{\Omega} C_1 - L_E C_1), \tag{4.14}$$

where the brackets are the usual Schouten–Nijenhuis brackets on M.

Furthermore, the exact generator of the BV-algebra of the Lie algebroid J^1M is $\delta_{\Omega} := [i(\Pi), d_{\Omega}]$, and using (4.11) we get

$$\begin{aligned} \delta_{\Omega}(\lambda_1 + \lambda_2 \wedge dt) &= \delta_Q \lambda_1 + (-1)^{k-1}([i(Q), e(\Omega)]\lambda_2 - di(E)\lambda_2) + (\delta_Q \lambda_2) \wedge dt, \end{aligned} \tag{4.15}$$

where $e(\Omega)$ is exterior product and $[\ ,\ ]$ is the commutator of the operators.

Finally, let us discuss the *modular class* of the Lie algebroid J^1M with bracket (4.9). For simplicity, we assume the manifold M to be orientable, with volume form $\Phi \in \Gamma \bigwedge^m M$. In the non-orientable case, the same computations hold if Φ is replaced by a density $\Phi \in \Gamma |\bigwedge^m M|$. Again, we denote by W^Q the modular vector field defined by $L_{\sharp_Q df}\Phi = (W^Q f)\Phi$ (see Section 3).

There are two natural possibilities to define a modular class for the algebroid J^1M. One is by computing the Lie derivative:

$$L_{\sharp_\Pi(\alpha+f dt)}(\Phi \wedge dt) = [L_{\sharp_Q\alpha}\Phi + fL_E\Phi + df \wedge i(E)\Phi] \wedge dt. \tag{4.16}$$

This result is obtained if the computation is done after Φ is decomposed into a product of m 1-forms, and by using (4.6). Since $i(E)(df \wedge \Phi) = 0$, the last term in (4.16) is $(Ef)\Phi$, and if we also use (3.13), then (4.16) yields

$$L_{\sharp_\Pi(df+f dt)}(\Phi \wedge dt) = (W^Q f + f \operatorname{div}_\Phi E + Ef)(\Phi \wedge dt). \tag{4.17}$$

Therefore, we get the *modular field*

$$W^\Pi := W^Q + E + (\operatorname{div}_\Phi E)\frac{\partial}{\partial t}. \tag{4.18}$$

If Φ is replaced by $h\Phi$ ($h \in C^\infty(M)$), it follows easily that the Π-Poisson cohomology class $[W^\Pi]$ is preserved. This will be the *modular class*.

The second possibility is to apply the general definition of [3]. Similar to what we had for Jacobi manifolds in Section 3, this requires computing the flat connection D on $(\bigwedge^{m+1} J^1M) \otimes (\bigwedge^m T^*M)$ given by

$$\begin{aligned} D_{(df+f dt)}[(\Phi \wedge dt) \otimes \Phi] = {} & L^{J^1M}_{(df+f dt)}(\Phi \wedge dt) \otimes \Phi \\ & + (\Phi \wedge dt) \otimes (L_{\varrho(df+f dt)}\Phi). \end{aligned} \tag{4.19}$$

From Lie algebroid calculus, we know that

$$L^{J^1M}_{(df+f dt)}(\Phi \wedge dt) = \{df + f dt, \Phi \wedge dt\}_\Omega, \tag{4.20}$$

where the bracket is the Schouten–Nijenhuis extension of (4.9). If we look at a decomposition $\Phi = \varphi_1 \wedge \ldots \wedge \varphi_n$ ($\varphi_i \in \bigwedge^1 M$), (4.9) yields

$$\{df + f dt, dt\}_\Omega = 0$$

and

$$\{df + f dt, \varphi_i\}_\Omega = L_{\varrho(df+f dt)}\varphi_i - \varphi_i(E)df + f\flat_\Omega\sharp_Q\beta - (\sharp_Q\beta(f))dt,$$

and we get

$$L^{J^1M}_{(df+f dt)}(\Phi \wedge dt) = [L_{(df+f dt)}\Phi - (Ef)\Phi + f \operatorname{tr}(\flat_\Omega \circ \sharp_Q)\Phi] \wedge dt. \tag{4.21}$$

But we also have

$$L_{\varrho(df+f dt)}\Phi = L_{\sharp_Q df + fE}\Phi = [W^Q f + f \operatorname{div}_\Phi E + Ef]\Phi. \tag{4.22}$$

By inserting (4.21), (4.22) into (4.19), we get another *modular field*, namely,

$$\begin{aligned} A_\Omega & := (2W^Q + E) \oplus (\operatorname{div}_\Phi E + \operatorname{tr}(\flat_\Omega \circ \sharp_Q)) \\ & = (2W^\Pi - E) \oplus \operatorname{tr}(\flat_\Omega \circ \sharp_Q). \end{aligned} \tag{4.23}$$

From the general results of [3], it is known that the Π-Poisson cohomology class of this field is independent of the choice of Φ, and it is a *modular class* of J^1M.

As for the modular class of $TM \oplus \mathbb{R}$ with bracket (4.2′), it vanishes for reasons similar to those for the class of the tangent algebroid TM (cf. [3]).

We finish by another interpretation of the enriched Poisson structures. If $\mathcal{F}$ is an arbitrary associative, commutative, real algebra, we may say that $f : M \to \mathcal{F}$ is differentiable if for any $\mathbb{R}$-linear mapping $\phi : \mathcal{F} \to \mathbb{R}$, $\phi \circ f \in C^\infty(M)$. Furthermore, an $\mathbb{R}$-linear operator v_x which acts on germs of $\mathcal{F}$-valued differentiable functions at $x \in M$, and satisfies the Leibniz rule, will be an *$\mathcal{F}$-tangent vector* of M at x. Then we have natural definitions of tangent spaces $T_x(M,\mathcal{F})$, differentiable $\mathcal{F}$-vector fields, etc. [17]. A bracket $\{\ ,\ \}$ which makes $C^\infty(M,\mathcal{F})$ a Poisson algebra will be called an *$\mathcal{F}$-Poisson structure* on M.

Now, take $\mathcal{F}$ to be the *Studi algebra of parabolic dual numbers* $\mathcal{S} := \mathbb{R}[s \mid s^2 = 0]$, where s is the generator. An $\mathcal{S}$-Poisson structure Π in the above-mentioned sense will be called a *Studi–Poisson structure*. The restriction of Π to real-valued functions is a Poisson bivector Q on M, and the Jacobi identity shows that the Hamiltonian vector field X_s^Π of the constant function s is an infinitesimal automorphism E of Q. Conversely, the pair (Q,E) defines the Studi–Poisson bracket

$$\begin{aligned}\{f_0 + f_1 s, g_0 + g_1 s\} := {} & \{f_0, g_0\}_Q + f_1(Eg_0) - g_1(Ef_0) \\ & + s(\{f_0, g_1\}_Q + \{f_1, g_0\}_Q + f_1(Eg_1) - g_1(Ef_1)).\end{aligned}$$

Notice that we cannot say that $v_x s = 0$ for all $v_x \in T_x(M,\mathcal{F})$ since v_x was linear over $\mathbb{R}$ only.

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