ANNALES
POLONICI MATHEMATICI
LXXIII.3 (2000)

On the Kuramoto–Sivashinsky equation in a disk

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Abstract. We consider the first initial-boundary value problem for the 2-D Kuramoto–Sivashinsky equation in a unit disk with homogeneous boundary conditions, periodicity conditions in the angle, and small initial data. Apart from proving the existence and uniqueness of a global in time solution, we construct it in the form of a series in a small parameter present in the initial conditions. In the stable case we also obtain the uniform in space long-time asymptotic expansion of the constructed solution and its asymptotics with respect to the nonlinearity constant. The method can work for other dissipative parabolic equations with dispersion.

1. Introduction. In this paper we shall consider the Kuramoto–Sivashinsky equation in two space dimensions. It can be written as

$$\partial_t u + \nu \Delta^2 u + \Delta u = \beta |\nabla u|^2, \tag{1.1.}$$

where $u = u(x_1, x_2, t)$, Δ is the Laplace operator in $x_1, x_2, \nabla u = \operatorname{grad} u$, $\nu = \text{const} > 0$, and $\beta = \text{const} \in \mathbb{R}$.

The equation (1.1) arises in the theory of long waves in thin films [4], [33], of long waves at an interface between two viscous liquids [13], in systems of the reaction-diffusion type [15], [16], and in the description of the nonlinear evolution of a linearly unstable flame front [29], [30]. The linear terms in (1) describe the interaction of long-wavelength pumping and short-wavelength dissipation, and the nonlinear term characterizes energy redistribution between various modes.

The Kuramoto–Sivashinsky equation and related model equations have been studied extensively in the eighties (mostly in the spatially one-dimensional case), both in the context of inertial manifolds and in numerical simulations of dynamical behavior (see [2], [7], [8], [23], [24], and the references there). Michelson [18] investigated special solutions $u(x,t) = -c^2 t + v(x)$ of the spatially one-dimensional equation (1.1). Setting $y = v'(x)$ he reduced

2000 *Mathematics Subject Classification*: 35K55, 35K65.

Key words and phrases: Kuramoto–Sivashinsky equation, disk, first initial-boundary value problem, long-time asymptotics.

it to the ordinary differential equation

$$y''' + y' = c^2 - y^2 \tag{1.2}$$

which he studied numerically. The equation (1.2) was examined analytically in [26], [34], and in the latter paper from the point of view of singular perturbations. In [19], [20] Michelson showed that a slight modification of (1.1),

$$\partial_t u + \nu \Delta^2 u + \Delta u + |\nabla u|^2 = c^2, \tag{1.3}$$

possesses stationary solutions. In the context of combustion theory these solutions represent Bunsen flames on infinite linear or circular burners. In [20] Michelson examined the linear stability of the radially symmetric solutions of (1.3) in a disk with the boundary conditions $u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0$. We shall also use these conditions below in our analysis of the long-time behavior of solutions of the spatially two-dimensional equation (1.1) in a circular domain.

As regards spatially periodic solutions of the Kuramoto–Sivashinsky equation and their stability, we must point out that Nicolaenko, Scheurer, and Temam [24] showed that the existence of a global absorbing ball implied the existence of a global attractor and gave an upper estimate of its Hausdorff dimension. Under the assumption that the initial data is odd, they proved the existence of a bounded global absorbing set in $L_2(0, l)$ for the derivative Kuramoto–Sivashinsky equation. Collet, Eckmann, Epstein, and Stubbe [6] and independently Goodman [12] got rid of this antisymmetry requirement. Berloff and Howard [3] considered the generalized derivative Kuramoto–Sivashinsky equation

$$\partial_t u + \partial_x^4 u + \sigma \partial_x^3 u + \partial_x^2 u + 2u\partial_x u = 0$$

and constructed a periodic wave train solution by means of the singular manifold method and partial fraction decomposition.

In the two-dimensional case an important problem was to show the existence of a bounded absorbing set in $L_2(\Omega)$. Sell and Taboada [28] gave the first answer to this question by means of proving the existence of a bounded local absorbing set in $\mathbf{H}^1_{\mathrm{per}}([0, 2\pi] \times [0, 2\pi\varepsilon])$ for ε small enough. They adapted the method used by Raugel and Sell [27] for the Navier–Stokes equation in a three-dimensional thin domain. In his interesting study [21] Molinet improved the results of [28] and examined the local stability of the solutions of the reduced Kuramoto–Sivashinsky equation with spatially periodic boundary conditions in a thin rectangular domain. He gave a sufficient condition on the width l_2 of the domain depending on the length l_1, so that there exists a bounded local absorbing set in $L_{2,\mathrm{per}}$, and estimated this set. As well as Sell and Taboada, Molinet used the derivative form of (1.1), that is, he set $\nabla u = (v_1, v_2)$ and reduced (1.1) to the system

$$\partial_t v_1 + \nu \Delta^2 v_1 + \Delta v_1 + v_1 \partial_x v_1 + v_2 \partial_x v_2 = 0,$$
$$\partial_t v_2 + \nu \Delta^2 v_2 + \Delta v_2 + v_1 \partial_y v_1 + v_2 \partial_y v_2 = 0,$$
$$\partial_y v_1 = \partial_x v_2,$$

which is convenient for obtaining some estimates.

In our present investigation of the Kuramoto–Sivashinsky equation we shall not use this reduction and will study (1.1) in its original form. We shall consider the first initial-boundary value problem for (1.1) in a unit disk with "small" initial conditions, homogeneous boundary conditions, and periodicity conditions in the angle. We shall prove the existence of a global in time strong solution by means of constructing it in the form of a series in a small parameter present in the initial conditions. The uniqueness will be proved via showing that the difference of two solutions from the required function space equals zero. We shall also obtain a uniform in space long-time asymptotic representation of the solution in question. The method applied includes the use of eigenfunction expansions and the theory of perturbations. In order to explain its origins we have to give a bit of history.

One of the powerful methods of studying Cauchy problems for nonlinear evolution equations is the inverse scattering transform (IST) (see [1]). Nevertheless, solving initial-boundary value problems by this method remained an open question until the breakthrough made by Fokas [9] and Fokas and Its [10], [11]. However, IST does not work for a wide class of dissipative equations which are not completely integrable. Another approach was used by Naumkin and Shishmarëv [22] who considered nonlocal dissipative equations of the first order in time. Having applied the Fourier transform and the theory of perturbations, they solved a number of Cauchy problems with small initial data and calculated the major terms of the long-time asymptotic expansions of their solutions. In [35]–[39] this method was further developed and adapted for solving Cauchy problems, spatially periodic problems, and spatially 1-D initial-boundary value problems for nonlinear dissipative equations of the second and third order in time. A radially symmetric mixed problem in a circle was considered in [40].

In the present paper we shall show how this approach can be applied for solving a spatially two-dimensional initial-boundary value problem in a disk via the use of eigenfunction expansions. As a result of examining a general spatially 2-D case, we shall not observe the effect of the "loss of smoothness", as in [40]. The increase of the regularity of the initial data via imposing more periodicity conditions can still influence the smoothness of the solution in question. After constructing the solution, we shall obtain its uniform in space long-time asymptotic expansion in the stable case $\nu > 1/\lambda_{01}^2$, where λ_{01} is the first positive zero of the Bessel function $J_0(z)$, and examine its growth in time for $0 < \nu \le 1/\lambda_{01}^2$.

2. Statement of the problem, notations, and technical lemmas. We consider the first initial-boundary value problem for the Kuramoto–Sivashinsky equation in the unit disk with small initial data and homogeneous boundary conditions. Using polar coordinates (r,θ) we can write it as follows:

$$\begin{aligned}
&\partial_t u + \nu\Delta^2 u + \Delta u = \beta|\nabla u|^2, \quad (r,\theta)\in\Omega,\ t>0,\\
&u(r,\theta,0) = \varepsilon^2\varphi(r,\theta), \quad (r,\theta)\in\Omega,\\
&u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0,\\
&|u(0,\theta,t)| < +\infty,\\
&\text{periodicity conditions in } \theta \text{ with period } 2\pi \text{ for } u\\
&\text{and its derivatives included in the equation,}
\end{aligned} \tag{2.1}$$

where $\Delta = \frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\theta^2$, $\Omega = \{(r,\theta) : |r| < 1,\ \theta\in[-\pi,\pi]\}$; ε, $\nu = \text{const} > 0$, $\beta = \text{const} \in \mathbb{R}$, and $\varphi(r,\theta)$ is a given real-valued function.

Our main tool in examining (2.1) will be the expansions in the series of the eigenfunctions of the Laplace operator in the disk. For a function $f(r,\theta) \in L_{2,r}(\Omega)$ ($L_2(\Omega)$ with a weight r) the corresponding expansion is (see [31])

$$f(r,\theta) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \widehat{f}_{mn}\chi_{mn}(r,\theta) = \sum_{m,n}\widehat{f}_{mn}\chi_{mn}(r,\theta), \tag{2.2}$$

where $\chi_{mn}(r,\theta)$ are the eigenfunctions of the Laplace operator in the disk, i.e., nontrivial solutions of the problem

$$\begin{aligned}
&\Delta\chi = -\Lambda\chi, \quad (r,\theta)\in\Omega,\\
&\chi|_{\partial\Omega} = 0,\ \chi(r,\theta) = \chi(r,\theta+2\pi), \quad |\chi(0,\theta)| < \infty.
\end{aligned}$$

These eigenfunctions and the corresponding eigenvalues are given by the formulas

$$\chi_{mn}(r,\theta) = J_m(\lambda_{mn}r)e^{im\theta}, \quad \Lambda_{mn} = \lambda_{mn}^2, \quad m\in\mathbb{Z},\ n\in\mathbb{N},$$

where $J_m(z)$ are the Bessel functions of index m, λ_{mn} are its positive zeros numbered in increasing order, and $n = 1, 2, \ldots$ is the number of the zero.

The system of functions $\{\chi_{mn}(r,\theta)\}_{m\in\mathbb{Z},\, n\in\mathbb{N}}$ is orthogonal and complete in the space $L_{2,r}(\Omega)$ (see [31]). Denoting the scalar product in $L_{2,r}(\Omega)$ by $(\cdot,\cdot)_{r,0}$ and the corresponding norm by $\|\cdot\|_{r,0}$ we can write

$$(\chi_{mn}, \chi_{kl})_{r,0} = \delta_{mk}\delta_{nl}\|\chi_{mn}\|_{r,0}^2,$$

where δ_{ij} is the Kronecker symbol.

We also have the Parseval identity in $L_{2,r}(\Omega)$,

$$\|f\|_{r,0}^2 = \sum_{m,n} |\widehat{f}_{mn}|^2 \|\chi_{mn}\|_{r,0}^2.$$

The coefficients of the expansion (2.2) are expressed by the formulas

$$\widehat{f}_{mn} = \frac{(f,\chi_{mn})_{r,0}}{\|\chi_{mn}\|_{r,0}^2} = \frac{1}{\|\chi_{mn}\|_{r,0}^2} \int_0^1 r J_m(\lambda_{mn} r)\, dr \int_{-\pi}^{\pi} f(r,\theta) e^{-im\theta}\, d\theta.$$

It will be convenient to use iterated integrals in what follows and to expand first in θ and then in r (the absolute convergence of the integrals will permit us to do that).

We shall need the weighted space $L_{2,r}(0,1)$ ($L_2(0,1)$ with a weight r) and shall denote the corresponding scalar product by $(\cdot,\cdot)_r$ and the norm by $\|\cdot\|_r$. Then

$$\|\chi_{mn}\|_{r,0}^2 = 2\pi \|J_m(\lambda_{mn} r)\|_r^2.$$

For a fixed integer m the system of functions $\{J_m(\lambda_{mn} r)\}_{n=1}^{\infty}$ is orthogonal and complete in $L_{2,r}(0,1)$ (see [31], [41]). Expansions of the type

$$f(r) = \sum_{n=1}^{\infty} \widehat{f}_{m,n} J_m(\lambda_{mn} r), \qquad \widehat{f}_{m,n} = \frac{(f, J_m(\lambda_{mn} r))_r}{\|J_m(\lambda_{mn} r)\|_r^2},$$

called Fourier–Bessel series are often used for solving radially symmetric problems in a disk (see [32]). However, if m is not fixed, the system $\{J_m(\lambda_{mn} r)\}_{m\in\mathbb{Z},\, n\in\mathbb{N}}$ is not orthogonal in $L_{2,r}(0,1)$.

Note that [32, p. 219]

$$\|J_m(\lambda_{mn} r)\|_r^2 = \int_0^1 r J_m^2(\lambda_{mn} r)\, dr = J_{m+1}^2(\lambda_{mn})/2$$

and for sufficiently large positive λ,

$$C_1/\lambda \le \|J_m(\lambda r)\|_r^2 \le C_2/\lambda. \tag{2.3}$$

We shall also need some properties of the zeros of the Bessel functions $J_m(z)$, $m \ge 0$. For bounded m large positive zeros of $J_m(z)$ have the following uniform asymptotics (called McMahon's expansion; see [14, p. 153], [25, p. 247]):

$$\lambda_{mn} = \mu + O\left(\frac{1}{\mu_{mn}}\right), \qquad \mu_{mn} = \left(m + 2n - \frac{1}{2}\right)\frac{\pi}{2}, \qquad n \to \infty.$$

For large m and n the major term of this formula still holds [41, p. 514]:

$$\lambda_{mn} \sim (m+2n)\frac{\pi}{2}. \tag{2.4}$$

In what follows we shall need the weighted Sobolev spaces $H_r^s(\Omega)$, $s \in \mathbb{R}$, which differ from the usual Sobolev spaces $H^s(\Omega)$ in that instead of $L_2(\Omega)$

we use the weighted space $L_{2,r}(\Omega)$. We introduce the norm in $H^s_r(\Omega)$ by the formula (see [17])

$$\|f\|^2_{r,s} = \sum_{m,n} \lambda^{2s}_{mn} |\widehat{f}_{mn}|^2 \, \|\chi_{mn}\|^2_{r,0}.$$

Here $\lambda_{mn} > 0$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Evidently, $H^0_r(\Omega) \equiv L_{2,r}(\Omega)$.

Our elementary space operator in (2.1) is Δ, and thanks to the orthogonality of the functions $\chi_{mn}(r,\theta)$ in the space $L_{2,r}(\Omega)$ we have

$$\begin{aligned}\|\Delta f\|^2_{r,0} &= (\Delta f, \Delta f)_{r,0} = \Big(\sum_{m,n}(-\lambda^2_{mn})\widehat{f}_{mn}\chi_{mn}, \sum_{k,l}(-\lambda^2_{kl})\widehat{f}_{kl}\chi_{kl}\Big)_{r,0} \\ &= \sum_{m,n,k,l} \lambda^2_{mn}\lambda^2_{kl}|\widehat{f}_{mn}| \cdot |\widehat{f}_{kl}|(\chi_{mn},\chi_{kl})_{r,0} = \sum_{m,n}\lambda^4_{mn}|\widehat{f}_{mn}|^2\|\chi_{mn}\|^2_{r,0}.\end{aligned}$$

We need to introduce the Banach space $C^k([0,\infty), H^s_r(\Omega))$ equipped with the norm

$$\|u\|_{C^k} = \sum_{j=0}^{k} \sup_{t\in[0,\infty)} \|\partial^j_t u(t)\|_{s,r}.$$

Now we prove two lemmas that will enable us to estimate the magnitude of the coefficients $\widehat{f}_{mn}$ in the eigenfunction expansion (2.2). The first lemma is the extension of the proposition given in [41, p. 595] to the case when the integrand depends on a parameter, that is, $f = f(x,\alpha) = f_\alpha(x)$, $x \in [0,1]$, $\alpha \in [a,b]$, $-\infty < a, b < \infty$. We denote by $V^1_0(f_\alpha(x))$ the total variation of the function $f_\alpha(x)$ in $x \in [0,1]$.

Consider the integral

$$I_m(\lambda,\alpha) = \int_0^1 x f(x,\alpha) J_m(\lambda x)\, dx, \qquad m \ge 0,\ \lambda > 0,\ \alpha \in [a,b].$$

LEMMA 1. *If for each fixed $\alpha \in [a,b]$ the function $\sqrt{x} f(x,\alpha)$ has a bounded total variation in x on $[0,1]$, $V^1_0(\sqrt{x} f_\alpha(x)) = V_\alpha$; $\lim_{x\to 0} \sqrt{x} f(x,\alpha) = F_\alpha$, and $V_\alpha, F_\alpha \in L_1(a,b)$, then for $m \ge 0$, $\lambda > 0$, $\alpha \in [a,b]$,*

$$|I_m(\lambda,\alpha)| \le C_\alpha/\lambda^{3/2},$$

where C_α is independent of m and λ and $C_\alpha \in L_1(a,b)$.

P r o o f. From the asymptotic formula as $x \to \infty$ for the Bessel functions we see that for any $z \in (0,\infty)$,

$$\Big|\int_0^z \sqrt{x} J_m(x)\, dx\Big| \le c < \infty,$$

where c is independent of m and z.

Set $\sqrt{x} f_\alpha(x) = \psi_\alpha(x)$. We can represent $\psi_\alpha(x)$ as

$$\psi_\alpha(x) = \psi_{\alpha,1}(x) - \psi_{\alpha,2}(x),$$

where $\psi_{\alpha,1}(x) = V_0^x(\psi_\alpha(x))$ is the variation of $\psi_\alpha(x)$ in $[0,x]$, $x \in [0,1]$, and $\psi_{\alpha,2}(x) = V_0^x(\psi_\alpha(x)) - \psi_\alpha(x)$. The functions $\psi_{\alpha,1}(x)$ and $\psi_{\alpha,2}(x)$ are non-decreasing in x for each fixed $\alpha \in [a,b]$. Then

$$\begin{aligned}
&\psi_{\alpha,1}(0) = 0, \qquad \psi_{\alpha,1}(1) = V_0^1(\psi_\alpha(x)) = V_\alpha \in L_1(a,b),\\
&\psi_{\alpha,2}(0) = -\psi_\alpha(0) = -F_\alpha \in L_1(a,b),\\
&\psi_{\alpha,2}(1) = V_0^1(\psi_\alpha(x)) - \psi_\alpha(1) = V_\alpha - \psi_\alpha(1).
\end{aligned}$$

We also have $|\psi_\alpha(1) - \psi_\alpha(0)| \le V_0^1(\psi_\alpha(x))$, which implies that

$$|\psi_\alpha(1)| \le |\psi_\alpha(0)| + V_0^1(\psi_\alpha(x)) = |F_\alpha| + V_\alpha.$$

Applying the second mean value theorem for integrals we obtain

$$\begin{aligned}
&\Big|\int_0^1 \psi_{\alpha,1}(x)\sqrt{x} J_m(\lambda x)\,dx\Big|\\
&\qquad \le \Big|\psi_{\alpha,1}(0)\int_0^\xi \sqrt{x} J_m(\lambda x)\,dx + \psi_{\alpha,1}(1)\int_\xi^1 \sqrt{x} J_m(\lambda x)\,dx\Big| \le cV_\alpha \lambda^{-3/2},\\
&\Big|\int_0^1 \psi_{\alpha,2}(x)\sqrt{x} J_m(\lambda x)\,dx\Big|\\
&\qquad\qquad \le \Big|\psi_{\alpha,2}(0)\int_0^\eta \sqrt{x} J_m(\lambda x)\,dx + \psi_{\alpha,2}(1)\int_\eta^1 \sqrt{x} J_m(\lambda x)\,dx\Big|\\
&\qquad\qquad \le c[2|F_\alpha| + V_\alpha]\lambda^{-3/2}.
\end{aligned}$$

Hence follows the necessary estimate which completes the proof. ∎

The next statement gives a tool for increasing the decay of $I_m(\lambda,\alpha)$ in λ.

Lemma 2. *If $f(x,\alpha)$ has partial derivatives in x in $[0,1]$ through the third order and $f(0,\alpha) = \partial_x f(0,\alpha) = \partial_x^2 f(0,\alpha) = 0$ (in case $m = 0$ only $\partial_x f(0,\alpha) = 0$), $f(1,\alpha) = \partial_x f(1,\alpha) = \partial_x^2 f(1,\alpha) = 0$, $\lim_{x\to 0+} \sqrt{x}\partial_x^3 f(x,\alpha) = \widetilde{F}_\alpha \in L_1(a,b)$, and for each fixed α the function $\sqrt{x}\partial_x^3 f(x,\alpha)$ has a bounded total variation in x in $[0,1]$ which is absolutely integrable in $\alpha \in (a,b)$, i.e. $V_0^1(\sqrt{x}\partial_x^3 f_\alpha(x)) = \widetilde{V}_\alpha \in L_1(a,b)$, then for $m \ge 0$, $\lambda > 0$,*

$$|I_m(\lambda,\alpha)| \le C_\alpha (m+1)^3/\lambda^{9/2},$$

where C_α is independent of m and λ and $C_\alpha \in L_1(a,b)$.

Proof. By the well known formula (see [32, p. 205]), we have

$$\frac{d}{dx}(x^{m+1}J_{m+1}(x)) = x^{m+1}J_m(x), \qquad m \geq 0.$$

Below we shall use the notation $f_\alpha(x) = f(x,\alpha)$, and a prime will denote the derivative in x. Changing the variable of integration $\xi = \lambda x$ and integrating by parts we deduce that

$$\begin{aligned} I_m(\lambda,\alpha) = \frac{1}{\lambda^2}\int_0^1 \xi f_\alpha(\xi/\lambda)J_m(\xi)\,d\xi &= \frac{\xi}{\lambda^2} f_\alpha(\xi/\lambda)J_{m+1}(\xi)\Big|_0^\lambda \\ &- \frac{1}{\lambda^2}\int_0^\lambda \left[\frac{1}{\lambda}f'_\alpha(\xi/\lambda) - \frac{m}{\xi}f_\alpha(\xi/\lambda)\right]\xi J_{m+1}(\xi)\,d\xi. \end{aligned}$$

Here the first term on the right is zero because of the condition $f_\alpha(1) = 0$. Integrating two more times by parts and using the boundary conditions $f'_\alpha(1) = f''_\alpha(1) = 0$ we obtain

$$\begin{aligned} I_m(\lambda,\alpha) = &-\frac{1}{\lambda^2}\int_0^\lambda \bigg\{\bigg\{\xi\left[\frac{1}{\lambda^3}f'''_\alpha(\xi/\lambda) - \frac{2m}{\xi^3}f_\alpha(\xi/\lambda) + \frac{2m}{\lambda\xi^2}f'_\alpha(\xi/\lambda)\right. \\ &\left.- \frac{m}{\lambda^2\xi}f''_\alpha(\xi/\lambda)\right] + \left[\frac{1}{\lambda^2}f''_\alpha(\xi/\lambda) + \frac{m}{\xi^2}f(\xi/\lambda) - \frac{m}{\lambda\xi}f'_\alpha(\xi/\lambda)\right] \\ &- (m+1)\left[\frac{1}{\lambda^2}f''_\alpha(\xi/\lambda) + \frac{m}{\xi^2}f_\alpha(\xi/\lambda) - \frac{m}{\lambda\xi}f'_\alpha(\xi/\lambda)\right]\bigg\} \\ &- \frac{m+3}{\xi}\bigg\{\xi\left[\frac{1}{\lambda^2}f''_\alpha(\xi/\lambda) + \frac{m}{\xi^2}f_\alpha(\xi/\lambda) - \frac{m}{\lambda\xi}f'_\alpha(\xi/\lambda)\right] \\ &- (m+1)\left[\frac{1}{\lambda}f'_\alpha(\xi/\lambda) - \frac{m}{\xi}f_\alpha(\xi/\lambda)\right]\bigg\}\bigg\}J_{m+3}(\xi)\,d\xi \\ = &-\frac{1}{\lambda^3}\int_0^1 \bigg\{xf'''_\alpha(x) - 3(m+1)f''_\alpha(x) - m\left[f'_\alpha(x) - \frac{f_\alpha(x)}{x}\right] \\ &+ \frac{(m+1)(m+3)}{x}\left[f'_\alpha(x) - \frac{m}{x}f_\alpha(x)\right]\bigg\}J_{m+3}(\lambda x)\,dx. \end{aligned}$$

Now we have to justify the formal calculations. Since $V_0^1(\sqrt{x}f'''_\alpha(x)) = \widetilde{V}_\alpha \in L_1(a,b), \lim_{x\to 0+}\sqrt{x}f'''_\alpha(x) = \widetilde{F}_\alpha \in L_1(a,b)$, there exists a constant $M_\alpha \in L_1(a,b)$ such that $|\sqrt{x}f'''_\alpha(x)| \leq M_\alpha$ for $x \in [0,1]$. Therefore $f'''_\alpha(x)$ is absolutely integrable in x in $(0,1)$. Expanding $f_\alpha(x)$, $f'_\alpha(x)$, and $f''_\alpha(x)$ around $x_0 = 0$ in the integrand and using the boundary conditions $f_\alpha(0) = f'_\alpha(0) = f''_\alpha(0) = 0$ we can write that for $x \in (0,1]$,

$$f_\alpha(x) = \frac{f_\alpha'''(\vartheta_1 x)}{3!}x^3, \quad 0 < \vartheta_1 < 1,$$
$$f_\alpha'(x) = \frac{f_\alpha'''(\vartheta_2 x)}{2!}x^2, \quad 0 < \vartheta_2 < 1,$$
$$f_\alpha''(x) = f_\alpha'''(\vartheta_3 x)x, \quad 0 < \vartheta_3 < 1.$$

Substituting these expansions into the integrand we get

$$\begin{aligned} I_m(\lambda,\alpha) = &-\frac{1}{\lambda^3}\int_0^1 x\bigg\{f_\alpha'''(x) - 3(m+1)f_\alpha'''(\vartheta_3 x) \\ &- m\bigg[\frac{f_\alpha'''(\vartheta_2 x)}{2!} - \frac{f_\alpha'''(\vartheta_1 x)}{3!}\bigg] \\ &+ (m+1)(m+3)\bigg[\frac{f_\alpha'''(\vartheta_2 x)}{2!} - m\frac{f_\alpha'''(\vartheta_1 x)}{3!}\bigg]\bigg\}J_{m+3}(\lambda x)\,dx. \end{aligned}$$

All the transformations performed above are valid for $m \geq 0$, but in the special case $m = 0$ it is possible to reduce the number of the boundary conditions at $x_0 = 0$. Indeed, we have

$$I_0(\lambda,\alpha) = -\frac{1}{\lambda^3}\int_0^1 \bigg\{x f_\alpha'''(x) + 3\bigg[\frac{f_\alpha'(x)}{x} - f_\alpha''(x)\bigg]\bigg\}J_3(\lambda x)\,dx.$$

By Taylor's theorem and the condition $f_\alpha'(0) = 0$ we have for $x \in (0,1]$,

$$\frac{f_\alpha'(x)}{x} - f_\alpha''(x) = \bigg[\frac{f_\alpha'''(\vartheta_2 x)}{2!} - \frac{f_\alpha'''(\vartheta_3 x)}{3!}\bigg]x, \quad 0 < \vartheta_{2,3} < 1.$$

Therefore, applying Lemma 1 we obtain the required estimate of $I_0(\lambda,\alpha)$. ∎

We shall need eigenfunction expansions of type (2.2) for the nonlinearity of the equation in (2.1). The coefficients $F_{mn}(|\nabla u|^2)$ of these equations will be represented by the quadruple series

$$\sum_{p,q,l,s} a_{mnpqls}\widehat{u}_{pq}(t)\widehat{u}_{ls}(t) \quad \text{and} \quad \sum_{p,q,l,s} b_{mnpqls}\widehat{u}_{pq}(t)\widehat{u}_{ls}(t).$$

The following lemma permits us to estimate the coefficients of these series.

Let m, n, p, q, l, s be nonnegative integers, $n,q,s \geq 1$, and

$$\begin{aligned} a_{mnpqls} &= \frac{\lambda_{pq}\lambda_{ls}}{\|J_m(\lambda_{mn}r)\|_r^2}\int_0^1 rJ_m(\lambda_{mn}r)J_p'(\lambda_{pq}r)J_l'(\lambda_{ls}r)\,dr, \quad m,p,l\geq 0, \\ b_{mnpqls} &= \frac{pl}{\|J_m(\lambda_{mn}r)\|_r^2}\int_0^1 \frac{1}{r}J_m(\lambda_{mn}r)J_p(\lambda_{pq}r)J_l(\lambda_{ls}r)\,dr, \quad m,p,l\geq 1, \end{aligned} \tag{2.5}$$

where λ_{kj}, $k = 0,1,\ldots$ and $j = 1,2,\ldots$, are positive zeros of the Bessel function $J_k(z)$ arranged in increasing order, and the prime denotes differentiation with respect to the argument.

Lemma 3. *The following estimates are valid*:

$$|a_{mnpqls}| \le c(\lambda_{mn}\lambda_{pq}\lambda_{ls})^{1/2}, \qquad |b_{mnpqls}| \le c(\lambda_{mn}\lambda_{pq}\lambda_{ls})^{1/2}. \tag{2.6}$$

Proof. Representing the derivatives of the Bessel functions in the first integral by the formulas (see [32, p. 207])

$$J_k'(z) = \tfrac{1}{2}[J_{k-1}(z) - J_{k+1}(z)], \quad k \ge 1, \qquad J_0'(z) = -J_1(z),$$

and using (2.3) and the inequality

$$|J_k(z)| \le c/\sqrt{z}, \qquad z > 0, \tag{2.7}$$

we deduce the first estimate in (2.6).

Making use of the formula [32, p. 207]

$$\frac{J_k(z)}{z} = \frac{1}{2k}[J_{k-1}(z) + J_{k+1}(z)], \qquad k \ge 1,$$

we have for $k \ge 1,\ j \ge 1$,

$$\frac{J_k(\lambda_{kj}r)}{r} = \frac{\lambda_{kj}}{2k}[J_{k-1}(\lambda_{kj}r) + J_{k+1}(\lambda_{kj}r)].$$

Then applying (2.7) we obtain the second estimate in (2.6). ∎

3. The main results. In this section we present several theorems concerning the existence, uniqueness, construction of the global in time solution of the problem (2.1), and its uniform in space long-time asymptotic expansion.

Let $\Omega_\delta = \{(r,\theta) : r \in [\delta, 1],\ \theta \in [-\pi,\pi]\}$, where $\delta > 0$ is small.

Theorem 1. *Suppose that* $\nu > 1/\lambda_{01}^2$, *where* λ_{01} *is the first positive zero of* $J_0(z)$,

$$\varphi(r,-\pi) = \varphi(r,\pi), \quad \partial_\theta\varphi(r,-\pi) = \partial_\theta\varphi(r,\pi), \quad \partial_\theta^2\varphi(r,-\pi) = \partial_\theta^2\varphi(r,\pi),$$

$\varphi(r,\theta)$ *satisfies the hypotheses of Lemma* 2 *with* $m = 0$, *i.e.*,

$$\partial_r\varphi(0,\theta) = \varphi(1,\theta) = \partial_r\varphi(1,\theta) = \partial_r^2\varphi(1,\theta) = 0,$$
$$\lim_{r\to 0+} \sqrt{r}\,\partial_r^3\varphi(r,\theta) = \Phi_\theta \in L_1(-\pi,\pi),$$
$$V_0^1(\sqrt{r}\,\partial_r^3\varphi(r,\theta)) = V_\theta \in L_1(-\pi,\pi),$$

and $\partial_\theta^3\varphi(r,\theta)$ *satisfies the hypotheses of Lemma* 2 *in the general case*, *i.e.*

$$\partial_\theta^3\varphi(0,\theta) = \partial_r\partial_\theta^3\varphi(0,\theta) = \partial_r^2\partial_\theta^3\varphi(0,\theta) = \partial_\theta^3\varphi(1,\theta)$$
$$= \partial_r\partial_\theta^3\varphi(1,\theta) = \partial_r^2\partial_\theta^3\varphi(1,\theta) = 0,$$

$$\lim_{r\to 0+} \sqrt{r}\partial_r^3\partial_\theta^3\varphi(r,\theta) = \widetilde{\Phi}_\theta \in L_1(-\pi,\pi),$$

$$V_0^1(\sqrt{r}\partial_r^3\partial_\theta^3\varphi(r,\theta)) = \widetilde{V}_\theta \in L_1(-\pi,\pi).$$

Then there is ε_0 *such that for* $0<\varepsilon\le\varepsilon_0$ *there exists a unique solution of the problem* (2.1) *from the class* $C^1([0,\infty),H_r^{-1-\gamma}(\Omega))\cap C^0([0,\infty),H_r^{3-\gamma}(\Omega))$ *with* $\Delta u\in C^0([0,\infty),H_r^{1-\gamma}(\Omega))$ *and* $\Delta^2 u\in C^0([0,\infty),H_r^{-1-\gamma}(\Omega))$ *for any* $\gamma>0$.

Moreover, u *and* ∇u *are continuous and bounded in* $\overline{\Omega}\times[0,\infty)$ *and* Δu *is continuous and bounded in* $\Omega_\delta\times[0,\infty)$.

This solution can be represented as

$$u(r,\theta,t) = \sum_{N=0}^{\infty}\varepsilon^{N+1}u^{(N)}(r,\theta,t), \tag{3.1}$$

where the functions $u^{(N)}$ *will be defined in the proof* (*see* (4.8) *and* (4.12)). *The series* (3.1) *converges absolutely and uniformly with respect to* $(r,\theta)\in\overline{\Omega}$, $t\in[0,\infty)$, $\varepsilon\in[0,\varepsilon_0]$ *together with* ∇u *which can be calculated termwise.*

In the next statement (and only there) we denote the solution of the nonlinear problem (2.1) by $u_\beta(r,\theta,t)$ and the solution of the corresponding linear problem (with $\beta=0$) by $u_0(r,\theta,t)$. The existence and uniqueness of the latter is evident.

Corollary. *Under the assumptions of Theorem* 1, *the following estimate holds*:

$$\sup_{\overline{\Omega}\times[0,\infty)} |u_\beta(r,\theta,t)-u_0(r,\theta,t)| \le C|\beta|, \qquad \beta\in\mathbb{R},$$

where the constant C *is independent of* r, θ, t, *and* ε.

Remark 1. The parameter $\varepsilon\in(0,\varepsilon_0]$ which controls the initial data guarantees the absolute and uniform convergence of the series (3.1). The latter is a series of regular perturbations and can be used as an asymptotic series with respect to ε. The estimate of ε_0 will be made clear in the proof.

Remark 2. The solution presented above is a strong solution. The equation in (2.1) is satisfied in the distributional sense, i.e., in $H_r^{-1-\gamma}(\Omega)$, $\gamma>0$, for each fixed $t>0$. In the same sense the periodicity conditions are satisfied for $\partial_t u$ and $\Delta^2 u$. The boundary conditions $u|_{\partial\Omega}=\Delta u|_{\partial\Omega}=0$, the initial condition, and the periodicity conditions for u and ∇u are satisfied in the classical sense. For Δu the periodicity conditions are satisfied in $H_r^{1-\gamma}(\Omega)$ (and in Ω_δ they are satisfied in the classical sense).

Remark 3. It is not difficult to construct a function $\varphi(r,\theta)$ satisfying the hypothesis of Theorem 1 by separation of variables, i.e., $\varphi(r,\theta)=R(r)\Theta(\theta)$, where $R^{(k)}(0)=R^{(k)}(1)$, $k=0,1,2$; $\lim_{r\to 0+}\sqrt{r}R'''(r)=c_1<$

∞, $V_0^1(\sqrt{r}R'''(r)) = c_2 < \infty$, $\Theta^{(k)}(-\pi) = \Theta^{(k)}(\pi)$, $k = 0,1,2$; $\Theta''' \in L_1(-\pi,\pi)$. The fact that $\Theta \in L_1(-\pi,\pi)$ follows from the absolute integrability of Θ''' in $(-\pi,\pi)$ and the existence of $\Theta^{(k)}(-\pi)$, $k = 0,1,2$.

Theorem 2. *Under the hypotheses of Theorem* 1, *the solution of* (2.1) *has the following asymptotic representation as* $t \to \infty$:

$$u(r,\theta,t) = \exp(-\kappa_{01}t)[A_\varepsilon J_0(\lambda_{01}r) + O(\exp(-\kappa_{01}t))], \tag{3.2}$$

where $\kappa_{01} = \lambda_{01}^2(\nu\lambda_{01}^2 - 1) > 0$, *and the coefficient* A_ε *will be defined in the proof* (*see* (7.1) *and* (7.3)). *The estimate of the residual term is uniform with respect to* $(r,\theta) \in \overline{\Omega}$ *and* $\varepsilon \in [0,\varepsilon_0]$.

Consider the problem (2.1) on a bounded time interval $[0,T]$, $T < \infty$, and denote it by (2.1*).

Theorem 3. *If* $0 < \nu \le 1/\lambda_{01}^2$ *and the remaining assumptions of Theorem* 1 *hold*, *then for any* $T > 0$ *there is* $\varepsilon_0(T) > 0$ *such that for* $0 < \varepsilon \le \varepsilon_0(T)$ *there exists a unique solution of the problem* (2.1*) *from the class stated in Theorem* 1 *with* $[0,\infty)$ *replaced by* $[0,T]$. *This solution is represented in the form* (3.1), *where* $\varepsilon_0(T) \to 0$ *as* $T \to \infty$. *For any fixed* ε *there exists* $\overline{T} < \infty$ *such that the solution of* (2.1*) *cannot be extended beyond this point.*

The rest of the paper is organized as follows. In Section 4 we construct a solution of the problem (2.1) and verify that it belongs to the required function space. Section 5 is devoted to the proof of uniqueness. This completes the proof of Theorem 1. The Corollary is proved in Section 6. The long-time asymptotic expansion of the solution for $\nu > 1/\lambda_{01}^2$ which forms the content of Theorem 2 is obtained in Section 7. Theorem 3 deals with the case $0 < \nu \le 1/\lambda_{01}^2$ and is proved in Section 8. Some final remarks are given in Section 9.

4. Existence and construction of solutions: proof of Theorem 1. In order to satisfy the boundary and periodicity conditions in (2.1) we seek a solution of this problem in the form (2.2), namely:

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \widehat{u}_{mn}(t)\chi_{mn}(r,\theta), \tag{4.1}$$

where

$$\widehat{u}_{mn}(t) = \frac{(u,\chi_{mn})_{r,0}}{\|\chi_{mn}\|_{r,0}^2}, \quad m \in \mathbb{Z},\ n \in \mathbb{N}.$$

Since $J_{-m}(z) = (-1)^m J_m(z)$ for integer $m \ge 0$, the zeros of $J_m(z)$ and $J_{-m}(z)$ coincide ($\lambda_{-m,n} = \lambda_{mn}$, $m \ge 0$, $n \ge 1$), and (4.1) can be rewritten in the form of a double series with $m \ge 0$. Using the fact that $u(r,\theta,t)$ is a real-valued function we can write

$$\widehat{u}_{mn}(t) = \frac{1}{\|\chi_{mn}\|_{r,0}^2} \int_0^1 r J_m(\lambda_{mn} r)\, dr \int_{-\pi}^{\pi} e^{-im\theta} u(r,\theta,t)\, d\theta,$$

$$\begin{aligned}\widehat{u}_{-m,n}(t) &= \frac{1}{\|\chi_{mn}\|_{r,0}^2} \int_0^1 r J_{-m}(\lambda_{-m,n} r)\, dr \int_{-\pi}^{\pi} e^{im\theta} u(r,\theta,t)\, d\theta \\ &= \frac{(-1)^m}{\|\chi_{mn}\|_{r,0}^2} \int_0^1 r J_m(\lambda_{mn} r)\, dr \int_{-\pi}^{\pi} e^{im\theta} u(r,\theta,t)\, d\theta, \qquad m \ge 0,\ n \ge 1.\end{aligned}$$

Therefore,

$$\widehat{u}_{mn}(t) = (-1)^m \overline{\widehat{u}_{-m,n}(t)}, \qquad m \ge 0,\ n \ge 1,$$

where the bar denotes complex conjugation (this notation should not be confused with $\overline{\Omega}$, where the bar denotes closure).

We can rewrite the expression (4.1) as

$$\begin{aligned}u(r,\theta,t) &= \sum_{n=1}^{\infty} \widehat{u}_{0n}(t) J_0(\lambda_{0n} r) \\ &\quad + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} J_m(\lambda_{mn} r)[\widehat{u}_{mn}(t) e^{im\theta} + \overline{\widehat{u}_{mn}(t)} e^{-im\theta}] \\ &= \sum_{m,n}{}^{*}\, \widehat{u}_{mn}(t) J_{mn}(\lambda_{mn} r) e^{im\theta}.\end{aligned} \tag{4.2}$$

Here and below the double sum with an asterisk includes the "usual" double sum $\sum_{m,n=1}^{\infty}$ and $\sum_{n=1}^{\infty}$.

Expanding $|\nabla u|^2$ in the series of type (2.2) and denoting its coefficients by $F_{mn}(|\nabla u|^2)(t)$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$, we substitute this expansion and (4.1) into (2.1) and obtain the following Cauchy problem for $\widehat{u}_{mn}(t)$:

$$\begin{aligned}&\widehat{u}'_{mn}(t) + \kappa_{mn}\widehat{u}_{mn}(t) = \beta F_{mn}(|\nabla u|^2)(t), && t > 0, \\ &\widehat{u}_{mn}(0) = \varepsilon^2 \widehat{\varphi}_{mn}, && m \in \mathbb{Z},\ n \in \mathbb{N},\end{aligned} \tag{4.3}$$

where

$$\kappa_{mn} = \lambda_{mn}^2(\nu\lambda_{mn}^2 - 1) > 0, \qquad F_{mn}(|\nabla u|^2)(t) = \frac{(|\nabla u|^2(t), \chi_{mn})_{r,0}}{\|\chi_{mn}\|_{r,0}^2},$$

and $\widehat{\varphi}_{mn}$ are the coefficients of the type (2.2) expansion of $\varphi(r,\theta)$, that is,

$$\varphi(r,\theta) = \sum_{m,n} \widehat{\varphi}_{mn}(t)\chi_{mn}(r,\theta), \qquad \widehat{\varphi}_{mn} = \frac{(\varphi, \chi_{mn})_{r,0}}{\|\chi_{mn}\|_{r,0}^2}.$$

We now prove the following estimates:

$$|\widehat{\varphi}_{mn}| \le c/\lambda_{mn}^{7/2}, \qquad m \ge 0,\ n \ge 1. \tag{4.4}$$

For $m = 0$ we can write

$$\begin{aligned}\widehat{\varphi}_{0n} &= \frac{1}{2\pi\|J_0(\lambda_{0n}r)\|_r^2}\int_0^1 rJ_0(\lambda_{0n}r)\,dr\int_{-\pi}^{\pi}\varphi(r,\theta)\,d\theta \\ &= \frac{1}{2\pi\|J_0(\lambda_{0n}r)\|_r^2}\int_{-\pi}^{\pi}d\theta\int_0^1 rJ_0(\lambda_{0n}r)\varphi(r,\theta)\,dr.\end{aligned}$$

Since $\varphi(r,\theta)$ satisfies the hypothesis of Lemma 2 with $m = 0$ we have

$$\Big|\int_0^1 r\varphi(r,\theta)J_0(\lambda_{0n}r)\,dr\Big| \le C_\theta/\lambda_{0n}^{9/2},$$

where $C_\theta \in L_1(-\pi,\pi)$. Taking into account (2.3) we obtain the required estimate.

It remains to justify the change of the order of integration performed above. We observe that the conditions

$$V_0^1(\sqrt{r}\partial_r^3\varphi(r,\theta)) = V_\theta \in L_1(-\pi,\pi), \qquad \lim_{r\to 0+}\sqrt{r}\partial_r^3\varphi(r,\theta) = \varPhi_\theta \in L_1(-\pi,\pi)$$

imply that there exists $N_\theta \in L_1(-\pi,\pi)$ such that

$$|\partial_r^3\varphi(r,\theta)| \le N_\theta/\sqrt{r}, \qquad r \in (0,1).$$

Therefore, using the boundary conditions $\varphi(1,\theta) = \partial_r\varphi(1,\theta) = \partial_r^2\varphi(1,\theta) = 0$ we deduce that

$$|\varphi(r,\theta)| \le \Big|\int_1^r dr_1\int_1^{r_1}dr_2\int_1^{r_3}|\partial_r^3\varphi(r_3,\theta)|\,dr_3\Big| \le cN_\theta$$

uniformly with respect to $r \in [0,1]$. The change of the order of integration is justified.

Assume now that $m \ge 1$. Defining

$$\phi_m(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-im\theta}\varphi(r,\theta)\,d\theta$$

we can write

$$\widehat{\varphi}_{mn} = \frac{1}{\|J_m(\lambda_{mn}r)\|_r^2}\int_{-\pi}^{\pi}rJ_m(\lambda_{mn}r)\phi_m(r)\,dr.$$

Integrating three times by parts in the integral defining $\phi_m(r)$ and using the periodicity conditions in θ for $\varphi(r,\theta)$, $\partial_\theta\varphi(r,\theta)$, and $\partial_\theta^2\varphi(r,\theta)$ we get

$$\phi_m(r) = \frac{i}{2\pi m^3}\int_{-\pi}^{\pi} e^{-im\theta}\partial_\theta^3\varphi(r,\theta)\,d\theta, \qquad m \ge 1.$$

Changing the order of integration in the integral representation of $\widehat{\varphi}_{mn}$ we obtain

$$\widehat{\varphi}_{mn} = \frac{i}{2\pi m^3\|J_m(\lambda_{mn}r)\|_r^2}\int_{-\pi}^{\pi} e^{-im\theta}\,d\theta\int_0^1 rJ_m(\lambda_{mn}r)\partial_\theta^3\varphi(r,\theta)\,dr.$$

Since $\partial_\theta^3\varphi(r,\theta)$ satisfies the hypothesis of Lemma 2 in r, for $m \ge 1$ we have

$$\Big|\int_0^1 rJ_m(\lambda_{mn}r)\partial_\theta^3\varphi(r,\theta)\,dr\Big| \le \frac{C_\theta(m+1)^3}{\lambda_{mn}^{9/2}},$$

where $C_\theta \in L_1(-\pi,\pi)$. The last inequality and the estimate (2.3) imply (4.4) with $m \ge 1$.

In order to show that the change of the order of integration is valid we observe that there exists $P_\theta \in L_1(-\pi,\pi)$ such that

$$|\partial_r^3\partial_\theta^3\varphi(r,\theta)| \le P_\theta/\sqrt{r}, \qquad r \in (0,1).$$

Therefore, using the boundary conditions $\partial_\theta^3\varphi(0,\theta) = \partial_r\partial_\theta^3\varphi(0,\theta) = \partial_r^2\partial_r^3\varphi(0,\theta) = 0$ we deduce that

$$|\partial_\theta^3\varphi(r,\theta)| \le \int_0^r dr_1\int_0^{r_1} dr_2\int_0^{r_2} dr_3\,|\partial_r^3\partial_r^3\varphi(r_3,\theta)| \le cP_\theta$$

uniformly with respect to $r \in [0,1]$. The estimates (4.4) are established for all integer $m \ge 0$, $n \ge 1$.

Integrating the Cauchy problem (4.3) in time we obtain

$$\widehat{u}_{mn}(t) = \varepsilon\widehat{\Phi}_{mn}\exp(-\kappa_{mn}t) + \beta\int_0^t \exp[-\kappa_{mn}(t-\tau)]F_{mn}(|\nabla u|^2)(\tau)\,d\tau, \tag{4.5}$$

where $m \ge 0$, $n \ge 1$, and $\widehat{\Phi}_{mn} = \varepsilon\widehat{\varphi}_{mn}$. It is convenient to keep one small parameter in the coefficients $\widehat{\Phi}_{mn}$ in order to simplify some estimates.

In order to calculate $F_{mn}(|\nabla u|^2)$ we set

$$F_{mn}(|\nabla u|^2) = F_{mn}[(\partial_r u)^2] + F_{mn}[(\partial_\theta u)^2/r^2], \tag{4.6}$$

where

$$F_{mn}[(\partial_r u)^2] = \frac{1}{\|\chi_{mn}\|^2_{r,0}} \int\limits_0^1 r J_m(\lambda_{mn} r)\, dr \int\limits_{-\pi}^{\pi} d\theta\, e^{-im\theta}$$

$$\times\, \partial_r \Big[\sum_{q=1}^{\infty} \Big\{ \widehat{u}_{0q}(t) J_0(\lambda_{0q} r) + \sum_{p=1}^{\infty} J_p(\lambda_{pq} r) [\widehat{u}_{pq}(t) e^{ip\theta} + \overline{\widehat{u}_{pq}(t)} e^{-ip\theta}] \Big\} \Big]$$

$$\times\, \partial_r \Big[\sum_{s=1}^{\infty} \Big\{ \widehat{u}_{0s}(t) J_{0s}(\lambda_{0s} r) + \sum_{l=1}^{\infty} J_l(\lambda_{ls} r) [\widehat{u}_{ls}(t) e^{il\theta} + \overline{\widehat{u}_{ls}(t)} e^{-il\theta}] \Big\} \Big]$$

$$= \sum_{p,q,l,s}{}' a_{mnpqls} \widehat{u}_{pq}(t) \widehat{u}_{ls}(t),$$

where

$$\sum_{p,q,l,s}{}' a_{mnpqls} \widehat{u}_{pq}(t) \widehat{u}_{ls}(t) = \sum_{\substack{p,l \geq 0;\ q,s \geq 1 \\ p+l=m}} a_{mnpqls}\, \widehat{u}_{pq}(t) \widehat{u}_{ls}(t)$$

$$+ \sum_{\substack{p,q,l,s \geq 1 \\ l-p=m}} a_{mnpqls} \overline{\widehat{u}_{pq}(t)} \widehat{u}_{ls}(t)$$

$$+ \sum_{\substack{p,q,l,s \geq 1 \\ p-l=m}} a_{mnpqls} \widehat{u}_{pq}(t) \overline{\widehat{u}_{ls}(t)},$$

and the coefficients a_{mnpqls} are defined by (2.5). Here we have used the relations

$$\int\limits_{-\pi}^{\pi} e^{i(-m+p+l)\theta}\, d\theta = \begin{cases} 2\pi, & p+l=m, \\ 0, & p+l \neq m, \end{cases}$$

$$\int\limits_{-\pi}^{\pi} e^{i(-m-p+l)\theta}\, d\theta = \begin{cases} 2\pi, & l-p=m, \\ 0, & l-p \neq m, \end{cases}$$

$$\int\limits_{-\pi}^{\pi} e^{i(-m+p-l)}\, d\theta = \begin{cases} 2\pi, & p-l=m, \\ 0, & p-l \neq m, \end{cases}$$

$$\int\limits_{-\pi}^{\pi} e^{-i(m+p+l)\theta}\, d\theta = \begin{cases} 2\pi, & m=p=l=0, \\ 0, & m \geq 0,\ p \geq 1,\ l \geq 1. \end{cases}$$

Next, we have

$$F_{mn}[(\partial_\theta u)^2/r^2] = \frac{1}{\|\chi_{mn}\|^2} \int\limits_0^1 \frac{1}{r} J_m(\lambda_{mn} r) dr \int\limits_{-\pi}^{\pi} e^{-im\theta}$$

$$\times \sum_{q,p=1}^{\infty} J_p(\lambda_{pq} r) ip [\widehat{u}_{pq}(t) e^{ip\theta} - \overline{\widehat{u}_{pq}(t)} e^{-ip\theta}]$$

$$\times \sum_{s,l=1}^{\infty} J_l(\lambda_{ls} r) il [\widehat{u}_{ls}(t) e^{il\theta} - \overline{\widehat{u}_{ls}(t)} e^{-il\theta}]\, d\theta$$
$$= \sum_{p,q,l,s}'' b_{mnpqls} \widehat{u}_{pq}(t) \widehat{u}_{ls}(t),$$

where

$$\begin{aligned}\sum_{p,q,l,s}'' b_{mnpqls} \widehat{u}_{pq}(t) \widehat{u}_{ls}(t) = & - \sum_{\substack{p,l \geq 0;\, q,s \geq 1 \\ p+l=m \geq 1}} b_{mnpqls} \widehat{u}_{pq}(t) \widehat{u}_{ls}(t) \\ & + \sum_{\substack{p,q,l,s \geq 1 \\ l-p=m}} b_{mnpqls} \overline{\widehat{u}_{pq}(t)} \widehat{u}_{ls}(t) \\ & + \sum_{\substack{p,q,l,s \geq 1 \\ p-l=m}} b_{mnpqls} \widehat{u}_{pq}(t) \overline{\widehat{u}_{ls}(t)},\end{aligned}$$

and the coefficients b_{mnpqls} are defined by (2.5).

The three sums with the additional conditions $p+l=m$, $l-p=m$, and $p-l=m$ are of the convolution type. Note that in the sum $\sum_{p,q,l,s}''$ the analog of the term $\sum_{q,s=1}^{\infty} a_{0n0q0s} \widehat{u}_{0q}(t) \widehat{u}_{0s}(t)$ corresponding to $p+l=m=0$ and representing the "purely radial part" is absent as a result of differentiating with respect to θ.

To solve the nonlinear integral equation (4.5) we use perturbation theory. Representing $\widehat{u}_m(t)$ as a formal series in ε,

$$\widehat{u}_{mn}(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} \widehat{v}_{mn}^{(N)}(t), \tag{4.7}$$

we substitute it into (4.5) and compare the coefficients of equal powers of ε. As a result, we obtain the following representations for $N \geq 0$, $m \geq 0$, $n \geq 1$, $t > 0$:

$$\begin{aligned}&\widehat{v}_{mn}^{(0)}(t) = \widehat{\Phi}_{mn} \exp(-\kappa_{mn} t), \\ &\widehat{v}_{mn}^{(N)}(t) = \beta \int_0^t \exp[-\kappa_{mn}(t-\tau)] Q_{mn}^{(N)}(\widehat{v}(\tau))\, d\tau, \qquad N \geq 1,\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}Q_{mn}^{(N)}(\widehat{v}(t)) = & \sum_{p,q,l,s}' a_{mnpqls} \sum_{j=1}^{N} \widehat{v}_{pq}^{(j-1)}(t) \widehat{v}_{ls}^{(N-j)}(t) \\ & + \sum_{p,q,l,s}'' b_{mnpqls} \sum_{j=1}^{N} \widehat{v}_{pq}^{(j-1)}(t) \widehat{v}_{ls}^{(N-j)}(t).\end{aligned}$$

Now we have to prove that the formally constructed function (4.2), (4.7), (4.8) is really a solution of the problem (2.1) in the required function space. To this end we study the convergence of the series

$$u(r,\theta,t)=\sum_{m,n}^{*}\Big[\sum_{N=0}^{\infty}\varepsilon^{N+1}\widehat{v}_{mn}^{(N)}(t)\Big]J_m(\lambda_{mn}r)e^{im\theta}. \tag{4.9}$$

First, we establish the following estimates for $N\ge 0$, $m\ge 0$, $n\ge 1$, $t>0$:

$$|\widehat{v}_{mn}^{(N)}(t)|\le c_\beta^N(N+1)^{-2}\lambda_{mn}^{-7/2}\exp(-\kappa_{01}t), \tag{4.10}$$

where $c_\beta=c|\beta|$, $\beta\in\mathbb{R}$. We use induction on N. For $N=0$ and sufficiently small ε we have, from (4.4) and (4.8),

$$|\widehat{v}_{mn}^{(0)}(t)|=|\varepsilon\widehat{\varphi}_{mn}|\exp(-\kappa_{mn}t)\le\lambda_{mn}^{-7/2}\exp(-\kappa_{01}t).$$

Assuming that (4.10) is valid for $|\widehat{v}_{mn}^{(s)}(t)|$ with $0\le s\le N-1$ we prove it for $s=N$. For $1\le j\le N$ we have

$$j^{-2}(N+1-j)^{-2}\le 2^2(N+1)^{-2}[j^{-2}+(N+1-j)^{-2}].$$

By means of Lemma 3 we can estimate a typical term on the right-hand side of (4.8). Indeed,

$$|\Im|\le c|\beta|\int_0^t\exp[-\kappa_{mn}(t-\tau)]\sum_{\substack{p,l\ge 0;\,q,s\ge 1\\ p+l=m}}|a_{mnpqls}|\sum_{j=1}^{N}|\widehat{v}_{pq}^{(j-1)}(\tau)|\cdot|\widehat{v}_{ls}^{(N-j)}(\tau)|\,d\tau$$

$$\le\frac{c_\beta\sqrt{\lambda_{mn}}}{(N+1)^2}S_{mn}(t)\sum_{p,q}^{*}\frac{1}{\lambda_{pq}^3}\sum_{l,s}^{*}\frac{1}{\lambda_{ls}^3}\sum_{j=1}^{N}c_\beta^{j-1}c_\beta^{N-j}[j^{-2}+(N+1-j)^{-2}],$$

where

$$S_{mn}(t)=\exp(-\kappa_{mn}t)\int_0^t\exp(L_{mn}\tau)\,d\tau,$$

$$L_{mn}=\kappa_{mn}-2\kappa_{01}=\lambda_{mn}^2(\nu\lambda_{mn}^2-1)-2\lambda_{01}^2(\nu\lambda_{01}^2-1).$$

The convergence of the double series in p,q and in l,s in the last inequality follows from the asymptotics (2.4) and the fact that the series

$$\sum_{\substack{m,n=-\infty\\ m,n\ne 0}}^{\infty}\frac{1}{(m+2n)^\sigma}$$

converges for $\sigma>2$ and diverges for $\sigma\le 2$ (see [5]). Now we have to consider three cases.

(i) If $m=0$, $n=1$, then $L_{01}=-\kappa_{01}$, and

$$S_{01}(t)=\exp(-\kappa_{01}t)\frac{1-\exp(-\kappa_{01}t)}{\kappa_{01}}\le\frac{\exp(-\kappa_{01}t)}{\lambda_{01}^4(\nu-1/\lambda_{01}^4)}.$$

(ii) If $m = 0$, $n \geq 2$, then for the positive zeros of $J_0(z)$ we have (see [14]) $\lambda_{0n}^2 \geq \lambda_{02}^2 > 2\lambda_{01}^2$, and therefore,

$$L_{0n} = \lambda_{0n}^2 \nu(\lambda_{0n}^2 - \lambda_{01}^2) + (\nu\lambda_{01}^2 - 1)(\lambda_{0n}^2 - 2\lambda_{01}^2) > 0.$$

Then

$$\begin{aligned} S_{0n}(t) &= \exp(-\kappa_{0n}t)\frac{\exp(L_{0n}t) - 1}{L_{0n}} \\ &\leq \frac{\exp(-2\kappa_{01}t)}{\nu\lambda_{0n}^2(\lambda_{0n}^2 - \lambda_{01}^2) + (\nu\lambda_{01}^2 - 1)(\lambda_{0n}^2 - 2\lambda_{01}^2)} \\ &\leq \frac{\exp(-2\kappa_{01}t)}{\nu\lambda_{0n}^4(1 - \lambda_{01}^2/\lambda_{0n}^2)} \leq c(\nu)\frac{\exp(-\kappa_{01}t)}{\lambda_{0n}^4}. \end{aligned}$$

(iii) If $m \geq 1$, $n \geq 1$, then $\lambda_{mn}^2 - 2\lambda_{01}^2 \geq \lambda_{11}^2 - 2\lambda_{01}^2 > 0$ (see [14]), and consequently,

$$\begin{aligned} L_{mn} &= \nu\lambda_{mn}^2(\lambda_{mn}^2 - \lambda_{01}^2) + (\nu\lambda_{01}^2 - 1)(\lambda_{mn}^2 - 2\lambda_{01}^2) \\ &\geq \nu\lambda_{11}^2(\lambda_{11}^2 - \lambda_{01}^2) + (\nu\lambda_{01}^2 - 1)(\lambda_{11}^2 - 2\lambda_{01}^2) > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} S_{mn}(t) &= \exp(-\kappa_{mn}t)\frac{\exp(L_{mn}t) - 1}{L_{mn}} \\ &\leq \frac{\exp(-2\kappa_{01}t)}{\nu\lambda_{mn}^4(1 - \lambda_{01}^2/\lambda_{mn}^2)} \leq c(\nu)\frac{\exp(-\kappa_{01}t)}{\lambda_{mn}^4}. \end{aligned}$$

The estimates (4.10) are proved. Moreover, in items (ii) and (iii) we have established that for $t > 0$, $N \geq 0$, $m = 0$, $n \geq 2$ and $m \geq 1$, $n \geq 1$,

$$|\widehat{v}_{0n}^{(N)}(t)| \leq c_\beta^N (N+1)^{-2}\lambda_{0n}^{-7/2}\exp(-2\kappa_{01}t), \tag{4.11}$$

where $c_\beta = c|\beta|$, $\beta \in \mathbb{R}$ (in these two cases the estimate for $\widehat{v}_{mn}^{(0)}(t)$ can also be rewritten with $\exp(-2\kappa_{01}t)$ since $\kappa_{mn} \geq 2\kappa_{01}$). The inequalities (4.11) will be used later for calculating the long-time asymptotics of the solution.

In order to obtain the representation (3.1) we interchange the order of summation in the series (4.9) to get

$$u(r,\theta,t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(r,\theta,t), \tag{4.12}$$

where

$$u^{(N)}(r,\theta,t) = \sum_{l,s}^{*} \widehat{v}_{mn}^{(N)}(t) J_m(\lambda_{mn}r)e^{im\theta}$$

and $\widehat{v}_{mn}^{(N)}(t)$ are defined by (4.8). This interchange is possible due to the absolute and uniform (in $(r,\theta) \in \overline{\Omega}$, $t \geq 0$, $\varepsilon \in [0, \varepsilon_0]$) convergence of the series in question, which in turn follows from (4.8) with $0 < \varepsilon \leq \varepsilon_0 < c_\beta^{-1}$.

Differentiating (4.8) with respect to t we find that

$$\partial_t \widehat{v}_{mn}^{(0)}(t) = -\kappa_{mn}\widehat{\Phi}_{mn}\exp(-\kappa_{mn}t),$$

$$\partial_t \widehat{v}_{mn}^{(N)}(t) = \beta\Big[\kappa_{mn}\int_0^t \exp[-\kappa_{mn}(t-\tau)]Q_{mn}^{(N)}(\widehat{v}(\tau))\,d\tau + Q_{mn}^{(N)}(\widehat{v}(t))\Big], \quad N\ge 1,$$

where $Q_{mn}^{(N)}(\widehat{v}(t))$ is defined by (4.8).

Taking into account the expression for κ_{mn} (see (4.3)) and (4.7) we deduce that for $m\ge 0$, $n\ge 1$, $t\ge 0$, $k=0,1$,

$$|\partial_t \widehat{u}_{mn}(t)| \le c\lambda_{mn}^{4k-7/2}\exp(-\kappa_{01}t). \tag{4.13}$$

Recalling the asymptotics (2.4) and using (4.13) with $k=0$ we conclude that the series

$$\sum_{m,n}\lambda_{mn}^{2s}|\widehat{u}_{mn}(t)|^2\|J_m(\lambda_{mn}r)\|_r^2$$

with $s=3-\gamma$, $\gamma>0$, converges uniformly with respect to $t\ge 0$. Therefore, $u\in C^0([0,\infty),H_r^{3-\gamma}(\Omega))$. Moreover, thanks to (4.13) the series (4.2) converges absolutely and uniformly with respect to $(r,\theta)\in\overline{\Omega}$, $t\ge 0$, and $\varepsilon\in[0,\varepsilon_0]$. Therefore, $u(r,\theta,t)$ is continuous and bounded in this domain.

Calculating ∇u by means of (4.2) we can see that for $m\ge 0$, $n\ge 1$, $t\ge 0$,

$$|F_{mn}(\nabla u)(t)| \le \frac{c}{\lambda_{mn}^{5/2}}\exp(-\kappa_{01}t)$$

and, therefore, the series

$$\sum_{m,n}^{\ *} F_{mn}(\nabla u)J_m(\lambda_{mn}r)e^{im\theta}$$

converges absolutely and uniformly in $(r,\theta)\in\overline{\Omega}$, $t\ge 0$.

As regards Δu, for $s=1-\gamma$, $\gamma>0$, the series

$$\|\Delta u\|_{r,s}^2 = \sum_{m,n}\lambda_{mn}^{2s}|F_{mn}(\Delta u)(t)|^2\|J_m(\lambda_{mn}r)\|_r^2$$

converges uniformly in $t\ge 0$, and this implies that $\Delta u\in C^0([0,\infty),H_r^{1-\gamma}(\Omega))$.

However, the convergence of the series of type (4.2) representing Δu in the pointed region Ω_δ is better than in the domain Ω. Indeed, for $(r,\theta)\in\Omega_\delta$, $t\ge 0$ we have

$$\Delta u = \sum_{m,n}^{\ *} F_{mn}(\Delta u)(t)J_m(\lambda_{mn}r)e^{im\theta}, \tag{4.14}$$

where

$$|F_{mn}(\Delta u)(t)| \le \frac{c}{\lambda_{mn}^{3/2}}\exp(-\kappa_{01}t), \qquad |J_m(\lambda_{mn}r)| \le \frac{c}{\sqrt{\lambda_{mn}}}.$$

To analyze the convergence of (4.14) we take the series

$$\sum_{n=A}^{\infty}\sum_{m=B}^{\infty} F_{mn}(\Delta u)J_m(\lambda_{mn}r)e^{im\theta}$$

with sufficiently large positive A and B and compare it with the integral

$$\widetilde{I}=\int_A^\infty dn\int_B^\infty dm\,\frac{e^{im\theta}}{\lambda_{mn}^2}.$$

Using the asymptotics (2.4) we integrate by parts in m to obtain

$$\widetilde{I}=\frac{1}{i}\int_A^\infty\left[\frac{ie^{iB\theta}}{B(B+2n)^2}+\int_B^\infty\frac{3m+2n}{m^2(m+2n)^3}e^{im\theta}\,dm\right]dn,$$

which implies that

$$|\widetilde{I}|\le\frac{c}{B}\int_A^\infty\frac{dn}{(B+2n)^2}\le\frac{c}{B(B+2A)}.$$

Therefore, the series (4.14) converges uniformly in Ω_δ and the boundary condition $\Delta u|_{\partial\Omega}=0$ is satisfied in the classical sense.

For u_t and Δ^2u the corresponding norms in $H_r^s(\Omega)$ are

$$\|u_t(t)\|_{r,s}^2=\sum_{m,n}\lambda_{mn}^{2s}|F_{mn}(u_t)|^2\|J_m(\lambda_{mn}r)\|_r^2,$$
$$\|\Delta^2u(t)\|_{r,s}^2=\sum_{m,n}\lambda_{mn}|F_{mn}(\Delta^2u)(t)|^2\|J_m(\lambda_{mn}r)\|_r^2,$$

where

$$|F_{mn}(u_t)(t)|\le c\lambda_{mn}^{1/2}\exp(-\kappa_{01}t),\qquad |F_{mn}(\Delta^2u)(t)|\le c\lambda_{mn}^{1/2}\exp(-\kappa_{01}t).$$

Hence, these series converge uniformly with respect to $t\ge0$ for $s=-1-\gamma$, $\gamma>0$, and this implies that u_t and Δ^2u belong to $C^0([0,\infty),H_r^{-1-\gamma}(\Omega))$. This completes the proof of the existence of a global in time solution of (2.1).

5. Uniqueness of solutions: proof of Theorem 1 (continuation). Assume that there exist two solutions $u^{(1)}$ and $u^{(2)}$ in the class stated in the theorem. Setting $w=u^{(1)}-u^{(2)}$ and expanding w into the series (4.2) we have

$$w(r,\theta,t)=\sum_{m,n}^{*}\widehat{w}_{mn}(t)J_m(\lambda_{mn}r)e^{im\theta},$$

where the estimates (4.13) with $k=0$ are valid for $\widehat{w}_{mn}(t)$. Since the linear part in the expression (4.5) equals zero the coefficients $\widehat{w}_{mn}(t)$ satisfy the integral equation

$$\widehat{w}_{mn}(t) = \beta\int\limits_0^t \exp[-\kappa_{mn}(t-\tau)][F_{mn}(|\nabla u^{(1)}|^2)(\tau) - F_{mn}(|\nabla u^{(2)}|^2)(\tau)]\,d\tau,$$

where $F_{mn}(|\nabla u|^2)$ are defined by (4.6). We can represent a typical term in the integrand in the last formula as follows:

$$\begin{aligned}
&\sum_{p,q,l,s}{}' a_{mnpqls}\widehat{u}_{pq}^{(1)}(t)\widehat{u}_{ls}^{(1)}(t) - \sum_{p,q,l,s}{}' a_{mnpqls}\widehat{u}_{pq}^{(2)}(t)\widehat{u}_{ls}^{(2)}(t)\\
&\qquad = \sum_{p,q,l,s}{}' a_{mnpqls}\{\widehat{u}_{pq}^{(1)}(t)[\widehat{u}_{ls}^{(1)}(t) - \widehat{u}_{ls}^{(2)}(t)] + \widehat{u}_{ls}^{(2)}(t)[\widehat{u}_{pq}^{(1)}(t) - \widehat{u}_{pq}^{(2)}(t)]\}\\
&\qquad = \sum_{p,q,l,s}{}' a_{mnpqls}[\widehat{u}_{pq}^{(1)}(t)\widehat{w}_{ls}(t) + \widehat{u}_{ls}^{(2)}(t)\widehat{w}_{pq}(t)].
\end{aligned}$$

Note that $w(r,\theta,t) \in H_r^{3-\gamma}(\Omega)$ for all $t \ge 0$ and any $\gamma > 0$ since $u^{(1)}$ and $u^{(2)}$ belong to the same space.

In order to estimate the integrand we shall use the Cauchy–Schwarz inequality and the relations

$$\begin{aligned}
\|w(t)\|_{r,1}^2 &= \sum_{m,n}\lambda_{mn}^2|\widehat{w}_{mn}(t)|^2\|J_m(\lambda_{mn}r)\|_r^2 < \infty,\\
\|u^{(k)}(t)\|_{r,1}^2 &< \infty, \qquad k = 1,2.
\end{aligned}$$

Since the inequalities (2.6) are valid for the coefficients a_{mnpqls} we can write

$$\begin{aligned}
&\Big|\sum_{p,q,l,s}{}' a_{mnpqls}\widehat{u}_{pq}^{(1)}(t)\widehat{w}_{ls}(t)\Big|\\
&\qquad \le c\sqrt{\lambda_{mn}}\sum_{p,q,l,s=1}^{\infty}\sqrt{\lambda_{pq}}|\widehat{u}_{pq}^{(1)}(t)|\sqrt{\lambda_{ls}}|\widehat{w}_{ls}(t)|\\
&\qquad \le c\sqrt{\lambda_{mn}}\Big(\sum_{p,q=1}^{\infty}\lambda_{pq}|\widehat{u}_{pq}^{(1)}(t)|^2\Big)^{1/2}\Big(\sum_{l,s=1}^{\infty}\lambda_{ls}|\widehat{w}_{ls}(t)|^2\Big)^{1/2}\\
&\qquad \le c\sqrt{\lambda_{mn}}\Big(\sum_{p,q=1}^{\infty}\lambda_{pq}^2|\widehat{u}_{pq}^{(1)}(t)|^2\|J_p(\lambda_{pq}r)\|_r^2\Big)^{1/2}\\
&\qquad\quad \times\Big(\sum_{l,s=1}^{\infty}\lambda_{ls}^2|\widehat{w}_{ls}(t)|^2\|J_l(\lambda_{ls}r)\|_r^2\Big)^{1/2}\\
&\qquad \le c\sqrt{\lambda_{mn}}\|u(t)\|_{r,1}\|w(t)\|_{r,1}.
\end{aligned}$$

Analogous estimates hold for $\sum''_{p,q,l,s} b_{mnpqls}\widehat{u}^{(k)}_{pq}(t)\widehat{w}_{ls}(t),\ k=1,2$. Therefore,

$$|\widehat{w}_{mn}(t)| \le c\sqrt{\lambda_{mn}}\int_0^t \exp[-\kappa_{mn}(t-\tau)]\|w(\tau)\|_{r,1}\,d\tau.$$

Squaring both sides, multiplying the result by $\lambda^2_{mn}\|J_m(\lambda_{mn}r)\|^2_r$, and summing in m and n we deduce that for some $T_1>0$ and $t\in[0,T_1]$,

$$\begin{aligned}\|w(t)\|^2_{r,1} &\le c\sum_{m,n}\lambda^3_{mn}\|J_m(\lambda_{mn}r)\|^2_r\Big(\int_0^t \exp[-\kappa_{mn}(t-\tau)]\|w(\tau)\|_{r,1}\,d\tau\Big)^2\\ &\le c(\sup_{t\in[0,T_1]}\|w(t)\|_{r,1})^2\sum_{m,n}\lambda^2_{mn}\Big(\frac{1-\exp(-\kappa_{mn}t)}{\kappa_{mn}}\Big)^2.\end{aligned}$$

This implies that

$$(\sup_{t\in[0,T_1]}\|w(t)\|_{r,1})^2 \le c(T_1)(\sup_{[0,T_1]}\|w(t)\|_{r,1})^2,$$

where the constant $c(T_1)$ can be made less than one by the appropriate choice of T_1. This contradiction allows to us complete the proof of uniqueness for $t\in[0,T_1]$. Continuing this process for the integrals $[T_1,T_2]$, $[T_2,T_3],\ldots$ $\ldots,[T_n,T_{n+1}],\ldots$ with $T_n\to\infty$ we obtain the same result for all $t>0$. This completes the proof of Theorem 1. ∎

6. Asymptotics with respect to β: proof of the Corollary. First, we construct the solution of the linear problem corresponding to (2.1) with $\beta=0$. Putting $\beta=0$ in (4.3) we obtain the following expression for the eigenfunction expansion coefficients:

$$\widetilde{u}_{mn}(t)=\varepsilon^2\widehat{\varphi}_{mn}\exp(-\kappa_{mn}t),$$

where $\kappa_{mn}=\lambda^2_{mn}(\nu\lambda^2_{mn}-1)$ and the estimates (4.4) are valid for $\widehat{\varphi}_{mn}$. The solution of the linear problem is

$$u_0(r,\theta,t)=\sum^*_{m,n}\widetilde{u}_{mn}(t)J_m(\lambda_{mn}r)e^{im\theta},$$

where $\sum^*_{m,n}$ is defined by (4.2). The uniqueness of the solution in the corresponding function space is evident.

Note that, according to (4.7) and (4.8),

$$\widehat{u}_{mn}(\beta,t)=\widetilde{u}_{mn}(t)+\sum_{N=1}^{\infty}\varepsilon^{N+1}\widehat{v}^{(N)}_{mn}(\beta,t).$$

Here we have shown the dependence on β in the notation of the coefficients. The estimates (4.10) are valid for $\widehat{v}^{(N)}_{mn}(\beta,t)$, $N\ge1$, that is,

$$|\widehat{v}^{(N)}_{mn}(\beta,t)|\le c^N|\beta|^N(N+1)^{-2}\lambda^{-7/2}_{mn}\exp(-\kappa_{01}t).$$

Therefore, we have for $\varepsilon_0 < (c|\beta|)^{-1}$ (this condition guarantees the absolute and uniform convergence of the series)

$$|\widehat{u}_{mn}(\beta,t) - \widetilde{u}_{mn}(t)| \le c|\beta|\lambda_{mn}^{-7/2}\exp(-\kappa_{01}t), \qquad \beta \in \mathbb{R},\ t > 0,$$

where the constant c is independent of β. This inequality yields the required estimate. ∎

7. Long-time asymptotics: proof of Theorem 2. Our idea of calculating the long-time asymptotics of the constructed solution consists in obtaining a subtle asymptotic estimate of $\widehat{u}_{01}(t)$ which will contribute to the major term of the asymptotics, and then estimating the remaining series (see (4.2))

$$R_0(r,t) = \sum_{n=2}^{\infty} \widehat{u}_{0n}(t) J_0(\lambda_{0n} r)$$

and

$$R_1(r,\theta,t) = \sum_{m,n=1}^{\infty} J_m(\lambda_{mn} r)[\widehat{u}_{mn}(t)e^{im\theta} + \overline{\widehat{u}_{mn}(t)}e^{-im\theta}].$$

In accordance with (4.7) and (4.8),

$$\widehat{u}_{01}(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1}\widehat{v}_{01}^{(N)}(t),$$

where $\widehat{v}_{01}^{(N)}(t)$ can be represented as

$$
\begin{aligned}
&\widehat{v}_{01}^{(0)}(t) = A_\varepsilon^{(0)} \exp(-\kappa_{01}t),\\
&\widehat{v}_{01}^{(N)}(t) = \exp(-\kappa_{01}t)[A_\varepsilon^{(N)} + R^{(N)}(t)], \qquad N \ge 1,\\
&A_\varepsilon^{(0)} = \widehat{\varPhi}_{01} = \varepsilon\widehat{\varphi}_{01}, \qquad A_\varepsilon^{(N)} = \beta \int_0^\infty \exp(\kappa_{01}\tau)\varXi(\tau)\,d\tau,\\
&R^{(N)}(t) = -\beta\int_t^\infty \exp(\kappa_{01}\tau)\varXi(\tau)\,d\tau,
\end{aligned}
\tag{7.1}
$$

where

$$
\begin{aligned}
\varXi(t) = {} & \sum_{p,q,l,s}{}' a_{01pqls} \sum_{j=1}^{N} \widehat{v}_{pq}^{(j-1)}(t)\widehat{v}_{ls}^{(N-j)}(t)\\
& + \sum_{p,q,l,s}{}'' b_{01pqls} \sum_{j=1}^{N} \widehat{v}_{pq}^{(j-1)}(t)\widehat{v}_{ls}^{(N-j)}(t),
\end{aligned}
$$

and the functions $\widehat{v}_{mn}^{(s)}(t)$, $s = 0,1,\ldots,N-1$, are defined by (4.8). Here we have added and subtracted the integrals from t to ∞ in the integral representations (4.8) for $\widehat{v}_{mn}^{(N)}(t), N \ge 1$.

Next, we have to prove that for $N \geq 1$, $t > 0$,

$$|R^{(N)}(t)| \leq c_\beta^N \exp(-\kappa_{01} t). \tag{7.2}$$

Using the estimates (2.6), (4.10), and (4.11) we get for $t \geq 0$,

$$\begin{aligned}|R^{(N)}(t)| &\leq c_\beta \Gamma_N \sqrt{\lambda_{01}} \int_t^\infty \exp(\kappa_{01}\tau)\Big[\exp(-2\kappa_{01}\tau) \sum_{q,s=1}^\infty \frac{1}{(\lambda_{0q}\lambda_{0s})^3} \\ &\quad + \exp(-4\kappa_{01}\tau) \sum_{p,q,l,s=1}^\infty \frac{1}{(\lambda_{pq}\lambda_{ls})^3}\Big]\, d\tau \\ &\leq c_\beta^N \exp(-k_{01} t),\end{aligned}$$

where

$$\begin{aligned}\Gamma_N &= \sum_{j=1}^\infty c_\beta^{j-1} c_\beta^{N-j} j^{-2}(N+1-j)^{-2} \\ &\leq 4(N+1)^{-2} c_\beta^{N-1} \sum_{j=1}^N [j^{-2} + (N+1-j)^{-2}] < \infty.\end{aligned}$$

Therefore, we have the following asymptotic expansion as $t \to \infty$:

$$\widehat{u}_{01}(t) = \exp(-\kappa_{01} t)[A_\varepsilon + O(\exp(-\kappa_{01} t)], \tag{7.3}$$

where

$$A_\varepsilon = \sum_{N=0}^\infty \varepsilon^{N+1} A_\varepsilon^{(N)}$$

and $A_\varepsilon^{(N)}$ are defined by (7.1). This series converges absolutely and uniformly in $\varepsilon \in [0, \varepsilon_0]$. The estimate of the remainder is uniform with respect to $(r,\theta) \in \overline{\Omega}$, $\varepsilon \in [0, \varepsilon_0]$, $\varepsilon_0 < c_\beta^{-1}$.

Now we can represent the solution as

$$\begin{aligned}u(r,\theta,t) &= \widehat{u}_{01}(t) J_0(\lambda_{01} r) + R_0(r,t) + R_1(r,\theta,t), \\ R_0(r,t) &= \sum_{n=2}^\infty J_0(\lambda_{0n} r) \sum_{N=0}^\infty \varepsilon^{N+1} \widehat{v}_{0n}^{(N)}(t), \\ R_1(r,\theta,t) &= \sum_{n,m=1}^\infty J_m(\lambda_{mn} r) \\ &\quad \times \Big[e^{im\theta} \sum_{N=0}^\infty \varepsilon^{N+1} \widehat{v}_{mn}^{(N)}(t) + e^{-im\theta} \sum_{N=0}^\infty \varepsilon^{N+1} \overline{\widehat{v}_{mn}^{(N)}(t)}\Big].\end{aligned} \tag{7.4}$$

By means of the estimates (4.11) we deduce that for $(r,\theta) \in \overline{\Omega}$, $t \geq 0$,

$$|R_0(r,t)| \leq c \exp(-2\kappa_{01} t), \qquad |R_1(r,\theta,t)| \leq c \exp(-2\kappa_{01} t). \tag{7.5}$$

Combining (7.3)–(7.5) we obtain (3.2). ∎

8. Growth of solutions in time in the case $0 < \nu \le 1/\lambda_{01}^2$: proof of Theorem 3. (i) Let $\nu = 1/\lambda_{01}^2$. Then the problem (4.3) for the coefficient $\widehat{u}_{01}(t)$ takes the form

$$\widehat{u}_{01}'(t) = \beta F_{01}(|\nabla u|^2)(t), \quad t \in (0,T], \quad \widehat{u}_{01}(0) = \varepsilon^2 \widehat{\varphi}_{01}, \tag{8.1}$$

where $F_{01}(|\nabla u|^2)(t)$ is defined by (4.6). For all $\widehat{u}_{mn}(t)$, $m \ge 1$, $n \ge 1$, we have (4.3) with $\kappa_{mn} > 0$. Therefore, integrating (8.1) in t and expanding $\widehat{u}_{01}(t)$ into the series (4.7) we deduce that

$$\widehat{v}_{01}^{(0)}(t) = \widehat{\varPhi}_{01} = \text{const}, \quad \widehat{v}_{01}^{(1)}(t) = \beta \int_0^t [a_{010101}\widehat{v}_{01}^{(0)}(\tau)\widehat{v}_{01}^{(0)}(\tau) + \ldots]\, d\tau,$$

$$\widehat{v}_{01}^{(2)}(t) = \beta \int_0^t [2a_{010101}\widehat{v}_{01}^{(0)}(\tau)\widehat{v}_{01}^{(1)}(\tau) + \ldots]\, d\tau, \ \ldots$$

Here we have shown only the terms that give the main contribution for large t. The dots in the integrand denote other terms that decrease exponentially with time. We can see that $\widehat{v}_{mn}^{(N)}(t)$ will exhibit polynomial growth with respect to t. Consequently, instead of (4.10) we shall obtain for $m \ge 0$, $n \ge 1$, $N \ge 0$, $t \ge 0$,

$$|\widehat{v}_{mn}^{(N)}(t)| \le [c_\beta(T)]^N (N+1)^{-2} \lambda_{mn}^{-7/2} \tag{8.2}$$

with $c(T) \to \infty$ as $T \to \infty$. In order to guarantee the absolute and uniform convergence of the series (3.1) we have to satisfy the condition $\varepsilon c_\beta(T) < 1$, or $0 < \varepsilon \le \varepsilon_0(T)$ with $\varepsilon_0(T) < [c_\beta(T)]^{-1}$. Naturally, $\varepsilon_0(T) \to 0$ as $T \to \infty$. The estimates (8.2) allow us to prove that the solution belongs to the required function space.

If ε is fixed, then there exists $\overline{T} > 0$ such that for $t = \overline{T}$ the necessary condition for the convergence of the series (3.1) is violated, and $u(x,t) \to \infty$ as $t \to \overline{T}$. The derivatives of (3.1) do not exist either, and this function ceases to be a solution.

By analogous arguments we obtain the same result for all values $\nu = 1/\lambda_{mn}^2$, $m \ge 1$, $n \ge 1$. The only difference is that in this case we have exponentially growing terms in the integral representations of $\widehat{v}_{mn}^{(N)}(t)$ since in some problems of type (4.3) we have $\kappa_{mn} < 0$.

(ii) Let $1/\lambda_{01}^2 < \nu < 1/\lambda_{11}^2$. Then for the coefficient $\widehat{u}_{01}(t)$ we have the following problem with $\kappa_{mn} = -|\kappa_{mn}| < 0$:

$$\begin{aligned} &\widehat{u}_{01}'(t) - |\kappa_{mn}|\widehat{u}_{01}(t) = \beta F_{01}(|\nabla u|^2)(t), \quad t \in (0,T], \\ &\widehat{u}_{01}(0) = \varepsilon^2 \widehat{\varphi}_{01}. \end{aligned} \tag{8.3}$$

All the coefficients $\widehat{u}_{mn}(t)$, $m \ge 1$, $n \ge 1$, satisfy (4.3) with $\kappa_{mn} > 0$.

Therefore, after integrating (8.3) with respect to t and using (4.7), we get

$$\widehat{v}_{01}^{(0)}(t) = \widehat{\Phi}_{01},$$

$$\widehat{v}_{01}^{(1)}(t) = \beta \int_0^t \exp(|\kappa_{01}|(t-\tau))[a_{010101}\widehat{v}_{01}^{(0)}(\tau)\widehat{v}_{01}^{(0)}(\tau) + \ldots]\, d\tau, \ \ldots,$$

$$\widehat{v}_{01}^{(2)}(t) = \beta \int_0^t \exp(|\kappa_{01}|(t-\tau))[2a_{010101}\widehat{v}_{01}^{(0)}(\tau)\widehat{v}_{01}^{(1)}(\tau) + \ldots]\, d\tau, \ \ldots$$

These relations again lead to the estimates (8.2) with $c(T) \to \infty$ as $T \to \infty$. Other cases, like $1/\lambda_{11}^2 < \nu < 1/\lambda_{21}^2$, can be considered in an analogous way.

The uniqueness of the solution on the interval $[0, T]$ can be proved by means of the same arguments as in Section 6. ∎

9. Discussion. Having used the eigenfunction expansions and perturbation theory, we succeeded in constructing the strong solution of the problem (2.1) in the form of a series of regular perturbations with respect to the initial conditions (series in $\varepsilon \in (0, \varepsilon_0]$). The solution in question is "small", and, according to (4.9), it can be represented as

$$u(r,\theta,t) = \varepsilon^2 U(r,\theta,t), \qquad |U(r,\theta,t)| \le c\exp(-\kappa_{01}t)\sum_{m,n}\frac{1}{\lambda_{mn}^{7/2}} \le c\exp(-\kappa_{01}t)$$

for $\varepsilon_0 c|\beta| < 1$. Therefore, $\varepsilon_0 < c|\beta|^{-1}$, and the smaller is the nonlinearity constant β, the bigger the interval $(0, \varepsilon_0]$ can be, for which (3.1) is valid.

In our studies of the radially symmetric problem for the damped Boussinesq equation (see [40]) we have encountered the effect of "the loss of smoothness", i.e., the increase of smoothness of the initial data does not lead to improving the regularity of the solution in question. In the general spatially 2-D case examined above this is no longer true. The "purely radial part"

$$\Re_n(t) = \sum_{q,s=1}^{\infty} a_{0n0q0s}\widehat{u}_{0q}(t)\widehat{u}_{ls}(t)$$

which forms the Fourier–Bessel coefficient of $|\nabla u|^2$ in the corresponding radially symmetric case and is responsible for the "loss of smoothness" is also present in the series expansion coefficient $F_{0n}(|\nabla u|^2)(t)$ (see (4.6)), namely:

$$F_{0n}(|\nabla u|^2)(t) = \Re_n(t) + \sum_{\substack{p,q,l,s\\ p+l\neq 0}}^{\infty}{}' a_{0nlqls}\widehat{u}_{lq}(t)\widehat{u}_{ls}(t) + \sum_{p,q,l,s}{}'' b_{0nlqls}\widehat{u}_{lq}(t)\widehat{u}_{ls}(t).$$

However, the convergence of the series expansion (4.2) is mainly determined by the decay properties of $F_{mn}(|\nabla u|^2)(t)$ for large m and n. Since in the representation of these coefficients we have convolutions with respect to the "angular indices" p and l, the decay in m can be improved by imposing more periodicity conditions on the initial data. The decay in n cannot be improved in an analogous way because we do not have convolutions with respect to the "radial indices" q and s. Therefore, we can say that a "partial loss of smoothness" still takes place in the problem in question.

In conclusion, we emphasize that the method employed can work for other parabolic dissipative equations with dispersion.

Acknowledgements. I would like to thank Professor Jerry Bona for having drawn my attention to the case $0 < \nu \leq 1/\lambda_{01}^2$, Professor Jairo Charris for many useful discussions, and Professor Luc Molinet for having sent me his article [21].

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Reçu par la Rédaction le 7.9.1999