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Pointwise approximation by Meyer–König and Zeller operators

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Abstract. We study the rate of pointwise convergence of Meyer–König and Zeller operators for bounded functions, and get an asymptotically optimal estimate.

1. Introduction. For a function f defined on $[0,1]$, the Meyer–König and Zeller operators M_n are given by

$$M_n(f,x)=\sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)m_{nk}(x),\qquad 0\le x<1, \tag{1}$$

$$M_n(f,1)=f(1),\qquad m_{nk}(x)=\binom{n+k}{k}x^k(1-x)^{n+1}.$$

The approximation-theoretical behaviour of the operators (1), such as direct approximation, best asymptotic constants, global approximation, L_p-approximation, moment estimates, etc., has been the subject of extensive investigation (cf. [1–3, 5, 9, 11, 12]).

The rates of convergence on functions of bounded variation were obtained for various operators (see [4, 6, 8, 13, 14]). In this paper we consider the rate of convergence of the operators (1) for a more general class of functions:

$$I_B=\{f\mid f \text{ is bounded on } [0,1]\}.$$

In order that our work includes the case of functions of bounded variation and gives a real improvement, we introduce the following three quantities:

$$\Omega_{x-}(f,\delta_1)=\sup_{t\in[x-\delta_1,x]}|f(t)-f(x)|,\qquad \Omega_{x+}(f,\delta_2)=\sup_{t\in[x,x+\delta_2]}|f(t)-f(x)|,$$

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$$\Omega(x,f,\lambda)=\sup_{t\in[x-x/\lambda,\,x+(1-x)/\lambda]}|f(t)-f(x)|,$$

where $f\in I_B$, $x\in[0,1]$ is fixed, $0\le\delta_1\le x$, $0\le\delta_2\le 1-x$ and $\lambda\ge 1$. It is clear that

(i) $\Omega_{x-}(f,\delta_1)$ and $\Omega_{x+}(f,\delta_2)$ are non-decreasing in δ_1 and in δ_2 respectively; $\Omega(x,f,\lambda)$ is non-increasing in λ.

(ii) If f is continuous at x, then we have $\lim_{\delta_1\to 0+}\Omega_{x-}(f,\delta_1)=0$, $\lim_{\delta_2\to 0+}\Omega_{x+}(f,\delta_2)=0$ and $\lim_{\lambda\to\infty}\Omega(x,f,\lambda)=0$.

(iii) $\Omega_{x-}(f,\delta_1)\le\Omega(x,f,x/\delta_1)$ and $\Omega_{x+}(f,\delta_2)\le\Omega(x,f,(1-x)/\delta_2)$.

If f is of bounded variation on $[a,b]$, and $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$, then

(iv) $\Omega_{x-}(f,\delta_1)\le\bigvee\limits_{x-\delta_1}^{x}(f)$, $\Omega_{x+}(f,\delta_2)\le\bigvee\limits_{x}^{x+\delta_2}(f)$,

$$\Omega(x,f,\lambda)\le\bigvee_{x-x/\lambda}^{x+(1-x)/\lambda}(f).$$

Now let us state our main result:

THEOREM. *If f is bounded on $[0,1]$, and $f(x+)$ and $f(x-)$ exist at a fixed point $x\in(0,1)$, then for all $n>1$ we have*

$$\left|M_n(f,x)-\frac{1}{2}(f(x+)+f(x-))\right|\le\frac{6}{nx+1}\sum_{k=1}^{n}\Omega(x,g_x,\sqrt{k})+\frac{4}{\sqrt{nx}+1}(|f(x+)-f(x-)|+\varepsilon_n(x)|f(x)-f(x-)|),\tag{2}$$

where

$$\varepsilon_n(x)=\begin{cases}1 & \text{if } x=k'/(n+k') \text{ for some } k'\in\mathbb{N},\\ 0 & \text{if } x\ne k/(n+k) \text{ for any } k\in\mathbb{N},\end{cases}$$

and $g_x(t)$ is defined as

$$g_x(t)=\begin{cases}f(t)-f(x+), & x<t\le 1,\\ 0, & t=x,\\ f(t)-f(x-), & 0\le t<x.\end{cases}\tag{3}$$

Inequality (2) *holds at $x=0$* (*resp.* $x=1$) *if we set $\frac12 f(x+)+\frac12 f(x-)=f(0)$* (*resp.* $f(1)$).

In the last part of the paper, we shall show that our estimate is asymptotically optimal.

2. Preliminary results. We first give several preliminary results, which mainly are estimates concerning the basis functions and moments of Meyer–

König and Zeller operators. Some results and techniques of probability theory play an important role in this section.

LEMMA 1. *For $n \geq 2$ and $x \in [0,1]$, we have*

$$\frac{x(1-x)^2}{2n} \leq M_n((t-x)^2, x) \leq \frac{2x(1-x)^2}{n}, \tag{4}$$

and for $x \in (0,1]$ and n sufficiently large,

$$M_n((t-x)^4, x) \leq \frac{4x^2(1-x)^4}{n^2}. \tag{5}$$

Proof. By [3, Lemma 2.1],

$$\left(1+\frac{2x}{n+2}\right)\frac{x(1-x)^2}{n+1} \leq M_n((t-x)^2, x) \leq \left(1+\frac{2x}{n-1}\right)\frac{x(1-x)^2}{n+1},$$

which yields (4) by a simple calculation.

In addition, for $x \in (0,1]$ and n sufficiently large, by [1, p. 359, Corollary] we get by direct calculation

$$\begin{aligned} M_n((t-x)^4, x) &= M_n(t^4,x) - 4xM_n(t^3,x) + 6x^2M_n(t^2,x) - 4x^3M_n(t,x) + x^4 \\ &= \frac{3x^2(1-x)^4}{n^2} + \frac{x(1-x)^2(25x^4 - 112x^3 + 82x^2 - 2x + 1)}{n^3} + O(n^{-4}), \end{aligned}$$

which yields the inequality (5).

LEMMA 2. *For all $k \in \mathbb{N}$ and $x \in [0,1]$, we have*

$$m_{nk}(x) < \frac{2}{1+\sqrt{nx}}. \tag{6}$$

Proof. From Theorem 2 of [12] it is known that

$$m_{nk}(x) < \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{(n+1)x}}.$$

Since $m_{nk}(x) \leq 1$, it follows that

$$m_{nk}(x)\sqrt{nx} + m_{nk}(x) < \frac{1}{\sqrt{2e}} + 1.$$

The inequality (6) is proved.

LEMMA 3. *Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of independent random variables with the same geometric distribution*

$$P(\xi_i = k) = x^k(1-x), \quad k \in \mathbb{N}, \quad x \in (0,1).$$

Then

$$E\xi_1 = x/(1-x), \quad E(\xi_1 - E\xi_1)^2 = x/(1-x)^2,$$
$$E(\xi_1 - E\xi_1)^3 = (x^2+x)/(1-x)^3, \quad E|\xi_1 - E\xi_1|^3 \leq 3x/(1-x)^3.$$

Proof. Direct calculation gives

$$\sum_{k=0}^{\infty} x^k(1-x) = 1, \qquad E\xi_1 = \sum_{k=0}^{\infty} kx^k(1-x) = \frac{x}{1-x},$$

$$E\xi_1^2 = \sum_{k=0}^{\infty} k^2x^k(1-x) = \frac{x^2+x}{(1-x)^2},$$

$$E\xi_1^3 = \sum_{k=0}^{\infty} k^3x^k(1-x) = \frac{x^3+4x^2+x}{(1-x)^3},$$

$$E\xi_1^4 = \sum_{k=0}^{\infty} k^4x^k(1-x) = \frac{x^4+11x^3+11x^2+x}{(1-x)^4}.$$

Hence it is easy to show that

$$E(\xi_1 - E\xi_1)^2 = x/(1-x)^2, \qquad E(\xi_1 - E\xi_1)^3 = (x^2+x)/(1-x)^3,$$

and

$$\begin{aligned} E(\xi_1 - E\xi_1)^4 &= E\xi_1^4 - 4E\xi_1 E\xi_1^3 + 6(E\xi_1)^2 E\xi_1^2 - 4(E\xi_1)^3 E\xi_1 + (E\xi_1)^4 \\ &= \frac{x^4+11x^3+11x^2+x}{(1-x)^4} - \frac{4x}{1-x}\frac{x^3+4x^2+x}{(1-x)^3} \\ &\quad + \frac{6x^2}{(1-x)^2}\frac{x(1+x)}{(1-x)^2} - \frac{3x^4}{(1-x)^4} \\ &= \frac{x^3+7x^2+x}{(1-x)^4}. \end{aligned}$$

By the Hölder inequality we get

$$\begin{aligned} E|\xi_1 - E\xi_1|^3 &\le \sqrt{E(\xi_1 - E\xi_1)^4 E(\xi_1 - E\xi_1)^2} \\ &= \sqrt{\frac{(x^3+7x^2+x)x}{(1-x)^4(1-x)^2}} \le \frac{3x}{(1-x)^3}. \end{aligned}$$

The proof of Lemma 3 is complete.

Lemmas 4 and 5 below are the well-known Berry–Esseen bound and the asymptotic expression for the central limit theorem of probability theory. They can be used to get upper and lower bounds for partial sums of Meyer–König and Zeller basis functions. Their proofs can be found in Feller [7, pp. 540–543] and Shiryayev [10, p. 432].

LEMMA 4. *Let* $\{\xi_k\}_{k=1}^{\infty}$ *be a sequence of independent and identically distributed random variables with* $E(\xi_1) = a_1$, $E(\xi_1 - a_1)^2 = \sigma^2 > 0$, $E|\xi_1 - a_1|^3 = \varrho < \infty$, *and let* F_n *stand for the distribution function of* $\sum_{k=1}^{n}(\xi_k - a_1)/(\sigma\sqrt{n})$. *Then there exists an absolute constant* C, $1/\sqrt{2\pi} \le$

$C < 0.8$, *such that for all* t *and* n,

$$\left|F_n(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\,du\right| < \frac{C\varrho}{\sigma^3\sqrt{n}}. \tag{7}$$

LEMMA 5. *Under the conditions of Lemma* 4 ($E|\xi_1 - a_1|^3 < \infty$ *can be reduced to* $E(\xi_1 - a_1)^3 < \infty$), *assume* F_n *to be a lattice distribution. Then at all points* t *of the lattice we have*

$$\frac{F_n(t) + F_n(t-)}{2} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2}\,du - \frac{E(\xi_1 - a_1)^3}{6\sigma^3\sqrt{n}} \cdot \frac{1-t^2}{\sqrt{2\pi}} e^{-t^2/2} = o(n^{-1/2}). \tag{8}$$

LEMMA 6. *For* $x \in [0,1)$, *we have*

$$\left|\sum_{k > nx/(1-x)} m_{nk}(x) - \frac{1}{2}\right| \le \frac{4}{\sqrt{nx+1}}. \tag{9}$$

Proof. Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with the same geometric distribution $P(\xi_i = k) = x^k(1-x)$, $k \in \mathbb{N}$, $x \in (0,1)$, and let $\eta_{n+1} = \sum_{i=1}^{n+1} \xi_i$. Then the probability distribution of the random variable η_{n+1} is

$$P(\eta_{n+1} = k) = \binom{n+k}{k} x^k (1-x)^{n+1} = m_{nk}(x).$$

So

$$\begin{aligned}\sum_{k > nx/(1-x)} m_{nk}(x) &= P\Big(\eta_{n+1} > \frac{nx}{1-x}\Big) = 1 - P\Big(\eta_{n+1} \le \frac{nx}{1-x}\Big)\\ &= 1 - F_{n+1}\Big(\frac{-\sqrt{x}}{\sqrt{n+1}}\Big).\end{aligned}$$

By Lemmas 3 and 4 we get

$$\begin{aligned}\left|\sum_{k > nx/(1-x)} m_{nk}(x) - \frac{1}{2}\right| &= \left|\frac{1}{2} - F_{n+1}\Big(\frac{-\sqrt{x}}{\sqrt{n+1}}\Big)\right|\\ &= \left|F_{n+1}\Big(\frac{-\sqrt{x}}{\sqrt{n+1}}\Big) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-u^2/2}\,du\right|\\ &\le \frac{C\varrho}{\sigma^3\sqrt{n+1}} + \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}/\sqrt{n+1}}^{0} e^{-u^2/2}\,du\\ &< 0.8 \frac{3x}{(1-x)^3} \cdot \frac{(1-x)^3}{x^{3/2}\sqrt{n+1}} + \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{x}}{\sqrt{n+1}} \le \frac{3}{\sqrt{nx}},\end{aligned}$$

and since $|\sum_{k > nx/(1-x)} m_{nk}(x) - 1/2| \le 1$, we obtain (9).

3. Proof of Theorem. For any $f \in I_B$, if $f(x+)$ and $f(x-)$ exist at x, we decompose f into

$$f(t) = \frac{f(x+)+f(x-)}{2} + g_x(t) + \frac{f(x+)-f(x-)}{2}\operatorname{sgn}(t-x) + \delta_x(t)\left[f(x) - \frac{f(x+)+f(x-)}{2}\right], \tag{10}$$

where $g_x(t)$ is defined in (3) and

$$\operatorname{sgn}(t) = \begin{cases} 1, & t>0, \\ 0, & t=0, \\ -1, & t<0, \end{cases} \qquad \delta_x(t) = \begin{cases} 1, & t=x, \\ 0, & t \neq x. \end{cases}$$

Hence

$$\left|M_n(f,x) - \frac{1}{2}f(x+) - \frac{1}{2}f(x-)\right| \le |M_n(g_x,x)| + \left|\frac{f(x+)-f(x-)}{2}M_n(\operatorname{sgn}(t-x),x) + \left[f(x) - \frac{f(x+)+f(x-)}{2}\right]M_n(\delta_x,x)\right|. \tag{11}$$

Direct calculation gives

$$M_n(\delta_x,x) = \varepsilon_n(x)m_{nk'}(x) \tag{12}$$

and

$$\begin{aligned} M_n(\operatorname{sgn}(t-x),x) &= \sum_{k=0}^{\infty} \operatorname{sgn}\left(\frac{k}{n+k} - x\right)m_{nk}(x) \\ &= -\sum_{k<nx/(1-x)} m_{nk}(x) + \sum_{k>nx/(1-x)} m_{nk}(x) \\ &= 2\sum_{k>nx/(1-x)} m_{nk}(x) - 1 + \varepsilon_n(x)m_{nk'}(x), \end{aligned} \tag{13}$$

where

$$\varepsilon_n(x) = \begin{cases} 1 & \text{if } x = k'/(n+k') \text{ for some } k' \in \mathbb{N}, \\ 0 & \text{if } x \neq k/(n+k) \text{ for any } k \in \mathbb{N}. \end{cases}$$

By (12), (13) and Lemmas 2, 6, we have

$$\left|\frac{f(x+)-f(x-)}{2}M_n(\operatorname{sgn}(t-x),x) + \left[f(x) - \frac{f(x+)+f(x-)}{2}\right]M_n(\delta_x,x)\right| \tag{14}$$

$$= \left| \frac{f(x+)-f(x-)}{2} \Big[2 \sum_{k>nx/(1-x)} m_{nk}(x) - 1 \Big] \right.$$
$$\left. + [f(x)-f(x-)]\varepsilon_n(x) m_{nk'}(x) \right|$$
$$\le \frac{4}{\sqrt{nx}+1} (|f(x+)-f(x-)| + \varepsilon_n(x)|f(x)-f(x-)|).$$

Now it is clear from (11) and (14) that the Theorem will be proved if we establish that

$$|M_n(g_x,x)| \le \frac{6}{nx+1} \sum_{k=1}^{n} \Omega(x, g_x, \sqrt{k}). \tag{15}$$

Recalling the Lebesgue–Stieltjes integral representations we have

$$M_n(g_x,x) = \int_0^1 g_x(t)\, d_t K_n(x,t), \tag{16}$$

where

$$K_n(x,t) = \begin{cases} \sum_{k\le nt/(1-t)} m_{nk}(x), & 0<t<1, \\ 1, & t=1, \\ 0, & t=0. \end{cases}$$

We decompose the integral of (16) into three parts:

$$\int_0^1 g_x(t)\, d_t K_n(x,t) = \triangle_{1,n}(g_x) + \triangle_{2,n}(g_x) + \triangle_{3,n}(g_x),$$

where

$$\triangle_{1,n}(g_x) = \int_0^{x-x/\sqrt{n}} g_x(t)\, d_t K_n(x,t),$$
$$\triangle_{2,n}(g_x) = \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} g_x(t)\, d_t K_n(x,t),$$
$$\triangle_{3,n}(g_x) = \int_{x+(1-x)/\sqrt{n}}^{1} g_x(t)\, d_t K_n(x,t).$$

We shall estimate $\triangle_{1,n}(g_x)$, $\triangle_{2,n}(g_x)$ and $\triangle_{3,n}(g_x)$ by the quantities $\Omega_{x-}(g_x,\delta_1)$, $\Omega_{x+}(g_x,\delta_2)$ and $\Omega(x,g_x,\lambda)$ (for convenience, below we write them as $\Omega_{x-}(\delta_1)$, $\Omega_{x+}(\delta_2)$ and $\Omega(x,\lambda)$ respectively). Firstly, for $\triangle_{2,n}(g_x)$

noting that $g_x(x)=0$ we have

$$(17)\qquad |\triangle_{2,n}(g_x)| \le \int_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} |g_x(t)-g_x(x)|\,d_tK_n(x,t) \le \Omega(x,\sqrt{n}).$$

Next we estimate $|\triangle_{1,n}(g_x)|$. Since $\Omega_{x-}(\delta_1)$ is non-decreasing in δ_1, it follows that

$$|\triangle_{1,n}(g_x)| = \left|\int_0^{x-x/\sqrt{n}} g_x(t)\,d_tK_n(x,t)\right| \le \int_0^{x-x/\sqrt{n}} \Omega_{x-}(x-t)\,d_tK_n(x,t).$$

Using partial integration with $y = x - x/\sqrt{n}$, we have

$$(18)\qquad \int_0^{x-x/\sqrt{n}} \Omega_{x-}(x-t)\,d_tK_n(x,t) \le \Omega_{x-}(x-y)K_n(x,y+) + \int_0^y \widehat{K}_n(x,t)\,d(-\Omega_{x-}(x-t)),$$

where $\widehat{K}_n(x,t)$ is the normalized form of $K_n(x,t)$. Since $\widehat{K}_n(x,t) \le K_n(x,t)$ and $K_n(x,y+) = K_n(x,y)$ on (0, 1), using the inequality (4), we deduce that

$$\widehat{K}_n(x,t) \le K_n(x,t) \le \sum_{k/(n+k)\le t} m_{nk}(x) \le \sum_{k/(n+k)\le t} \frac{|k/(n+k)-x|^2}{(t-x)^2} m_{nk}(x) \le \frac{2x(1-x)^2}{n(t-x)^2}.$$

From (18) it follows that

$$(19)\qquad |\triangle_{1,n}(g_x)| \le \Omega_{x-}(x-y)\frac{2x(1-x)^2}{n(x-y)^2} + \frac{2x(1-x)^2}{n}\int_0^y \frac{d(-\Omega_{x-}(x-t))}{(x-t)^2}.$$

Since

$$\int_0^y \frac{d(-\Omega_{x-}(x-t))}{(x-t)^2} = -\frac{1}{(x-t)^2}\Omega_{x-}(x-t)\Big|_0^{y+} + \int_0^y \Omega_{x-}(x-t)\frac{2}{(x-t)^3}\,dt$$
$$= -\frac{1}{(x-t)^2}\Omega_{x-}(x-y) + \frac{\Omega_{x-}(x)}{x^2} + \int_0^y \Omega_{x-}(x-t)\frac{2}{(x-t)^3}\,dt,$$

from (19) we have

$$|\triangle_{1,n}(g_x)| \le \frac{2x(1-x)^2}{nx^2}\Omega_{x-}(x) + \frac{2x(1-x)^2}{n}\int\limits_0^{x-x/\sqrt{n}} \Omega_{x-}(x-t)\frac{2}{(x-t)^3}\,dt.$$

Putting $t = x - x/\sqrt{u}$ in the last integral we get

$$\int\limits_0^{x-x/\sqrt{n}} \Omega_{x-}(x-t)\frac{2}{(x-t)^3}\,dt = \frac{1}{x^2}\int\limits_1^n \Omega_{x-}(x/\sqrt{u})\,du.$$

Consequently,

$$|\triangle_{1,n}(g_x)| \le \frac{2(1-x)^2}{nx}\Big(\Omega_{x-}(x) + \int\limits_1^n \Omega_{x-}(x/\sqrt{u})\,du\Big). \tag{20}$$

Using a similar method for estimating $|\triangle_{3,n}(g_x)|$ we get

$$|\triangle_{3,n}(g_x)| \le \frac{2x^2}{nx}\Big(\Omega_{x+}(1-x) + \int\limits_1^n \Omega_{x+}((1-x)/\sqrt{u})\,du\Big). \tag{21}$$

From (17), (20) and (21) it follows that

$$\begin{aligned}|M_n(g_x,x)| &\le \Omega(x,\sqrt{n})\\ &\quad + \bigg(\frac{2(1-x)^2}{nx} + \frac{2x^2}{nx}\bigg)\Big(\Omega(x,1) + \int\limits_1^n \Omega(x,\sqrt{u})\,du\Big).\end{aligned} \tag{22}$$

By the monotonicity of $\Omega(x,\lambda)$ and noting that $(1-x)^2 + x^2 \le 1$, from (22) we get

$$\begin{aligned}|M_n(g_x,x)| &\le \frac{1}{n}\sum_{k=1}^n \Omega(x,\sqrt{k}) + \frac{2}{nx}\Big(\sum_{k=1}^n \Omega(x,\sqrt{k}) + \sum_{k=1}^n \Omega(x,\sqrt{k})\Big)\\ &\le \frac{5}{nx}\sum_{k=1}^n \Omega(x,\sqrt{k}).\end{aligned} \tag{23}$$

On the other hand

$$\begin{aligned}|M_n(g_x,x)| &= \Big|\int\limits_0^1 (g_x(t) - g_x(x))\,d_tK_n(x,t)\Big|\\ &\le \Omega(x,1) \le \sum_{k=1}^n \Omega(x,\sqrt{k}).\end{aligned} \tag{24}$$

The inequality (15) now follows from (23) and (24). The proof of the Theorem is complete.

4. Asymptotic optimality of our estimate. We now show that our estimate (2) is asympototically optimal. For $f \in I_B$, if x is a continuity point of f, then (2) becomes

$$|M_n(f,x)-f(x)| \le \frac{6}{nx+1}\sum_{k=1}^{n}\Omega(x,f,\sqrt{k}). \tag{25}$$

Taking the function $f_x(t)=|t-x|$, from (25) we have

$$\begin{aligned} |M_n(f_x,x)-f_x(x)| &= M_n(|t-x|,x) \\ &\le \frac{6}{nx+1}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \le \frac{12}{x\sqrt{n}+1/\sqrt{n}}. \end{aligned} \tag{26}$$

On the other hand, for any small positive number δ, it is easy to show that

$$\begin{aligned} &\delta\sum_{k=0}^{\infty}\left|\frac{k}{n+k}-x\right|m_{nk}(x)+\frac{1}{\delta^2}\sum_{k=0}^{\infty}\left(\frac{k}{n+k}-x\right)^4 m_{nk}(x) \\ &\ge \sum_{|k/(n+k)-x|\le\delta}\left(\frac{k}{n+k}-x\right)^2 m_{nk}(x)+\sum_{|k/(n+k)-x|>\delta}\left(\frac{k}{n+k}-x\right)^2 m_{nk}(x) \\ &= \sum_{k=0}^{\infty}\left(\frac{k}{n+k}-x\right)^2 m_{nk}(x). \end{aligned}$$

That is,

$$\begin{aligned} \sum_{k=0}^{\infty}\left|\frac{k}{n+k}-x\right|m_{nk}(x) &\ge \frac{1}{\delta}\sum_{k=0}^{\infty}\left(\frac{k}{n+k}-x\right)^2 m_{nk}(x) \\ &\quad -\frac{1}{\delta^3}\sum_{k=0}^{\infty}\left(\frac{k}{n+k}-x\right)^4 m_{nk}(x). \end{aligned}$$

Hence, from Lemma 1 for n sufficiently large, it follows that

$$\sum_{k=0}^{\infty}\left|\frac{k}{n+k}-x\right|m_{nk}(x) \ge \frac{x(1-x)^2}{2n\delta}-\frac{4}{\delta^3}\frac{x^2(1-x)^4}{n^2}.$$

Choose $\delta=4\sqrt{x(1-x)^2/n}$ to get

$$M_n(|t-x|,x)=\sum_{k=0}^{\infty}\left|\frac{k}{n+k}-x\right|m_{nk}(x) \ge \frac{1}{16}\frac{\sqrt{x(1-x)^2}}{\sqrt{n}}. \tag{27}$$

Therefore from (26) and (27) we see that (25) cannot be asymptotically improved.

To prove that the second term on the right hand side of (2) is asymptotically optimal, one needs an accurate estimate. If $g_x\equiv 0$, then (2) becomes

$$(28)\qquad \left|M_n(f,x)-\frac{f(x+)+f(x-)}{2}\right| \le \frac{4(|f(x+)-f(x-)|+\varepsilon_n(x)|f(x)-f(x-)|)}{\sqrt{nx}+1}.$$

We consider the function

$$f(t)=\begin{cases}1, & 0\le t<1/2,\\ 0, & 1/2\le t\le 1,\end{cases}$$

at the point $t=1/2$. Then

$$\left|M_n(f,1/2)-\frac{1}{2}\left(f\left(\frac{1}{2}+\right)+f\left(\frac{1}{2}-\right)\right)\right| = \left|\sum_{k<n} m_{nk}(1/2)-\frac{1}{2}\right|.$$

From Lemma 5 and a simple calculation for geometric distributions it follows that

$$\frac{1}{2}[F_{n+1}(0)+F_{n+1}(0-)]-\frac{1}{2}-\frac{1+x}{6\sqrt{n+1}\sqrt{x}}\cdot\frac{1}{\sqrt{2\pi}}=o(1/\sqrt{n+1}).$$

That is,

$$(29)\qquad \frac{1}{2}\Bigg[\sum_{k\le\frac{(n+1)x}{1-x}} m_{nk}(x)+\sum_{k<\frac{(n+1)x}{1-x}} m_{nk}(x)\Bigg]-\frac{1}{2}-\frac{1+x}{6\sqrt{n+1}\sqrt{x}}\cdot\frac{1}{\sqrt{2\pi}} = o(1/\sqrt{n+1}).$$

Taking $x=1/2$ in (29) we get

$$\sum_{k<n} m_{nk}(1/2)-1/2 = \frac{1}{4\sqrt{\pi}\sqrt{n+1}}-m_{nn}(1/2)-\frac{1}{2}m_{n,n+1}(1/2)+o(1/\sqrt{n+1}).$$

Using Stirling's formula $n!=(2\pi n)^{1/2}(n/e)^n e^{\theta_n/(12n)}$ $(0<\theta_n<1)$, we find that

$$\frac{1}{4\sqrt{\pi}\sqrt{n+1}}-\frac{1}{2}m_{n,n+1}(1/2)=o(1/\sqrt{n+1}),$$

and

$$\frac{1}{5\sqrt{n}}<m_{nn}(1/2)=\frac{(2n)!}{(n!)^2}(1/2)^{2n+1}=\frac{e^{\theta_{2n}/(24n)}}{2\sqrt{\pi}e^{\theta_n/(6n)}}\cdot\frac{1}{\sqrt{n}}<\frac{1}{2\sqrt{n}}.$$

Consequently, for n sufficiently large,

$$\frac{1}{6\sqrt{n}}\le\left|M_n(f,1/2)-\frac{1}{2}\left(f\left(\frac{1}{2}+\right)+f\left(\frac{1}{2}-\right)\right)\right| = \left|\sum_{k<n} m_{nk}(1/2)-1/2\right|\le\frac{1}{\sqrt{n}}.$$

Therefore (28) cannot be asymptotically improved as $n\to+\infty$.

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