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Oscillatory and nonoscillatory solutions of neutral differential equations

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Abstract. Neutral differential equations are studied. Sufficient conditions are obtained to have oscillatory solutions or nonoscillatory solutions. For the existence of solutions, the Schauder–Tikhonov fixed point theorem is used.

1. Introduction. In this paper we consider the neutral differential equation

$$\frac{d^n}{dt^n}[x(t)+\lambda x(t-\tau)]+f(t,x(g(t)))=0. \tag{1.1}$$

Throughout, the following conditions (H1)–(H3) are assumed:

(H1) $n \in \mathbb{N}$, $\lambda > 0$ and $\tau > 0$;

(H2) $g \in C[t_0,\infty)$ and $\lim_{t\to\infty} g(t) = \infty$;

(H3) $f \in C([t_0,\infty)\times\mathbb{R})$ and there exists $F \in C([t_0,\infty)\times[0,\infty))$ such that $F(t,u)$ is nondecreasing in $u \in [0,\infty)$ for each fixed $t \ge t_0$ and satisfies

$$|f(t,u)| \le F(t,|u|), \qquad (t,u) \in [t_0,\infty)\times\mathbb{R}.$$

By a *solution* of (1.1) we mean a function $x(t)$ which is continuous and satisfies (1.1) on $[t_x,\infty)$ for some $t_x \ge t_0$. Therefore, if $x(t)$ is a solution of (1.1), then $x(t)+\lambda x(t-\tau)$ is n times continuously differentiable on $[t_x,\infty)$. Note that, in general, $x(t)$ itself is not continuously differentiable.

A solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *nonoscillatory*. This means that a solution $x(t)$ is oscillatory if and only if there is a sequence $\{t_i\}_{i=1}^\infty$ such that $t_i \to \infty$ as $i \to \infty$ and $x(t_i) = 0$ $(i = 1,2,\ldots)$, and a solution $x(t)$ is nonoscillatory if and only if $x(t)$ is either eventually positive or eventually negative.

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There has been much current interest in the existence of oscillatory solutions and nonoscillatory solutions of neutral differential equations, and many results have been obtained. For typical results, we refer to the papers [1, 5–15] and the monographs [2, 3].

Neutral differential equations find numerous applications in natural science and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. See, for example, Hale [4].

Now consider the equation

$$\frac{d^n}{dt^n}[x(t) - \lambda x(t-\tau)] + f(t, x(g(t))) = 0. \tag{1.2}$$

Let $\omega, \omega_- \in C(\mathbb{R})$ satisfy $\omega(t+\tau) = -\omega(t)$ and $\omega_-(t+\tau) = \omega_-(t)$, respectively, for $t \in \mathbb{R}$. For example, $\omega(t) = \sin(\pi t/\tau)$ and $\omega_-(t) = \cos(2\pi t/\tau)$ are such functions. We easily see that $\lambda^{t/\tau}\omega(t)$ and $\lambda^{t/\tau}\omega_-(t)$ are solutions of the unperturbed equations

$$\frac{d^n}{dt^n}[x(t) + \lambda x(t-\tau)] = 0 \quad \text{and} \quad \frac{d^n}{dt^n}[x(t) - \lambda x(t-\tau)] = 0,$$

respectively. Thus it is natural to expect that, if f is small enough in some sense, equation (1.1) [resp. (1.2)] has a solution $x(t)$ which behaves like the function $\lambda^{t/\tau}\omega(t)$ [resp. $\lambda^{t/\tau}\omega_-(t)$] as $t \to \infty$. In fact, the following results have been established by Jaroš and Kusano [7].

Theorem A. *Suppose that* $0 < \lambda \le 1$ *and that there exist constants* $\mu \in (0, \lambda)$ *and* $a > 0$ *such that*

$$\int_{t_0}^{\infty} t^{n-1} \mu^{-t/\tau} F(t, a\lambda^{g(t)/\tau})\, dt < \infty.$$

Then

(i) *for each* $\omega \in C(\mathbb{R})$ *such that* $\omega(t+\tau) = -\omega(t)$ *for* $t \in \mathbb{R}$ *and* $\max_{t\in\mathbb{R}} |\omega(t)| < a$, *equation* (1.1) *has a solution* $x(t)$ *satisfying*

$$x(t) = \lambda^{t/\tau}[\omega(t) + o(1)] \quad (t \to \infty), \tag{1.3}$$

(ii) *for each* $\omega_- \in C(\mathbb{R})$ *such that* $\omega_-(t+\tau) = \omega_-(t)$ *for* $t \in \mathbb{R}$ *and* $\max_{t\in\mathbb{R}} |\omega_-(t)| < a$, *equation* (1.2) *has a solution* $x(t)$ *satisfying*

$$x(t) = \lambda^{t/\tau}[\omega_-(t) + o(1)] \quad (t \to \infty). \tag{1.4}$$

Theorem B. *Suppose that* $\lambda > 1$ *and that there exist constants* $\mu \in (1, \lambda)$ *and* $a > 0$ *such that*

$$\int_{t_0}^{\infty} \mu^{-t/\tau} F(t, a\lambda^{g^*(t)/\tau})\, dt < \infty,$$

where $g^*(t) = \max\{g(t), t\}$. *Then* (i) *and* (ii) *of Theorem A follow.*

We note that a solution $x(t)$ satisfying (1.3) is oscillatory if $\omega(t) \not\equiv 0$, and that a solution $x(t)$ satisfying (1.4) is oscillatory or nonoscillatory according to whether the function $\omega_-(t)$ is oscillatory or nonoscillatory. In particular, Theorems A and B are first results concerning the existence of oscillatory solutions of nonlinear neutral differential equations.

For equation (1.2), Theorems A and B have been extended to the following results by Kitamura and Kusano [9]. (See also [5, 8, 10, 14].)

THEOREM C. *Let* $\lambda = 1$. *Suppose that*

$$\int_{t_0}^{\infty} t^n F(t, a)\, dt < \infty \quad \textit{for some } a > 0.$$

Then, for each $\omega_- \in C(\mathbb{R})$ *such that* $\omega_-(t+\tau) = \omega_-(t)$ *for* $t \in \mathbb{R}$ *and* $\max_{t\in\mathbb{R}} |\omega_-(t)| < a$, *equation* (1.2) *has a solution* $x(t)$ *satisfying*

$$x(t) = \omega_-(t) + o(1) \quad (t \to \infty).$$

THEOREM D. *Let* $\lambda \neq 1$. *Suppose that*

$$\int_{t_0}^{\infty} \lambda^{-t/\tau} F(t, a\lambda^{g(t)/\tau})\, dt < \infty \quad \textit{for some } a > 0. \tag{1.5}$$

Then (ii) *of Theorem A follows.*

However, very little is known about extensions of Theorems A and B for equation (1.1) such as Theorems C and D. In this paper we obtain the following results which improve Theorems A and B for equation (1.1).

THEOREM 1.1. *Let* $\lambda = 1$. *Suppose that*

$$\int_{t_0}^{\infty} t^{n-1} F(t, a)\, dt < \infty \quad \textit{for some } a > 0. \tag{1.6}$$

Then, for each $c \in \mathbb{R}$ *and* $\omega \in C(\mathbb{R})$ *such that* $\omega(t+\tau) = -\omega(t)$ *for* $t \in \mathbb{R}$ *and* $\max_{t\in\mathbb{R}} |\omega(t)| + |c| < a$, *equation* (1.1) *has a solution* $x(t)$ *satisfying*

$$x(t) = \omega(t) + c + o(1) \quad \textit{as } t \to \infty. \tag{1.7}$$

THEOREM 1.2. *Let* $\lambda \neq 1$. *Suppose that* (1.5) *holds. Then* (i) *of Theorem A follows.*

REMARK 1.1. The solution obtained in Theorem 1.1 is oscillatory or nonoscillatory according to whether the function $\omega(t) + c$ is oscillatory or nonoscillatory. Since condition (1.6) is independent of the choice of the function $\omega(t)+c$, equation (1.1) has both oscillatory solutions and nonoscillatory solutions if (1.6) holds. For the case $\omega(t) \not\equiv 0$, the solution of (1.1) obtained in Theorem 1.2 is oscillatory.

The proof of Theorem 1.1 is given in Section 2. The proof of Theorem 1.2 is divided into the cases $0 < \lambda < 1$ and $\lambda > 1$. These are considered in Sections 3 and 4, respectively. To prove the existence of solutions, we use the Schauder–Tikhonov fixed point theorem.

2. Proof of Theorem 1.1. Equation can be replaced by (1.1) with $\lambda = 1$.

Let T and T_* be constants with $T-\tau \geq T_* \geq t_0$. We denote by $U[T_*,\infty)$ the set of all functions $u \in C[T_*,\infty)$ such that $\sum_{i=1}^{\infty}(-1)^{i+1}u(t+i\tau)$ converges for each fixed $t \in [T-\tau,\infty)$. To each $u \in U[T_*,\infty)$ we assign the function Φu on $[T_*,\infty)$ by

$$(\Phi u)(t) = \begin{cases} \sum_{i=1}^{\infty}(-1)^{i+1}u(t+i\tau), & t \geq T-\tau, \\ (\Phi u)(T-\tau), & t \in [T_*, T-\tau]. \end{cases}$$

Then we see that

$$(\Phi u)(t) + (\Phi u)(t-\tau) = u(t), \qquad t \geq T,\ u \in U[T_*,\infty). \tag{2.1}$$

In fact,

$$\begin{aligned}(\Phi u)(t) + (\Phi u)(t-\tau) &= \sum_{i=1}^{\infty}(-1)^{i+1}u(t+i\tau) + \sum_{i=1}^{\infty}(-1)^{i+1}u(t+(i-1)\tau) \\ &= \sum_{i=1}^{\infty}(-1)^{i+1}u(t+i\tau) - \sum_{i=0}^{\infty}(-1)^{i+1}u(t+i\tau) \\ &= u(t), \qquad t \geq T,\ u \in U[T_*,\infty).\end{aligned}$$

Hereafter, $C[T_*,\infty)$ is regarded as the Fréchet space of all continuous functions on $[T_*,\infty)$ with the topology of uniform convergence on every compact subinterval of $[T_*,\infty)$ (the $C[T_*,\infty)$-*topology*).

We prepare the next proposition for the proof of Theorem 1.1.

Lemma 2.1. *Let T and T_* be constants with $T-\tau \geq T_* \geq t_0$. Suppose that $\eta \in C[T-\tau,\infty)$ is such that $\eta(t) \geq 0$ for $t \geq T-\tau$ and $\lim_{t\to\infty}\eta(t) = 0$ and define*

$$V = \{v \in U[T_*,\infty) : |(\Phi v)(t)| \leq \eta(t),\ t \geq T-\tau\}.$$

Then Φ maps V into $C[T_*,\infty)$ and is continuous on V in the $C[T_*,\infty)$-topology.

P r o o f. If $v \in V$, then

$$\sup_{t\in[T-\tau,\infty)}\Big|\sum_{i=p+1}^{\infty}(-1)^{i+1}v(t+i\tau)\Big| \tag{2.2}$$
$$= \sup_{t\in[T-\tau,\infty)}\Big|\sum_{i=1}^{\infty}(-1)^{i+1}v(t+p\tau+i\tau)\Big|$$
$$\le \sup_{t\in[T-\tau,\infty)} \eta(t+p\tau)$$
$$= \sup_{t\in[T+(p-1)\tau,\infty)} \eta(t), \qquad p = 0,1,2,\ldots,$$

which means that the series $\sum_{i=1}^{\infty}(-1)^{i+1}v(t+i\tau)$ converges uniformly on $[T-\tau,\infty)$. Consequently, Φv is continuous on $[T_*,\infty)$ for each $v\in V$ and Φ maps V into $C[T_*,\infty)$.

Now we prove that Φ is continuous on V. It suffices to show that if $\{v_j\}_{j=1}^{\infty}$ is a sequence in $C[T_*,\infty)$ converging to $v \in C[T_*,\infty)$ in the $C[T_*,\infty)$-topology, then also Φv_j converges to Φv in this topology.

For any $\varepsilon > 0$, there is an integer $p \ge 1$ such that

$$\sup_{t\in[T+(p-1)\tau,\infty)} \eta(t) < \frac{\varepsilon}{3}. \tag{2.3}$$

Take an arbitrary compact subinterval I of $[T-\tau,\infty)$. There exists an integer $j_0 \ge 1$ such that

$$\sum_{i=1}^{p}|v_j(t+i\tau)-v(t+i\tau)| < \frac{\varepsilon}{3}, \qquad t\in I,\ j\ge j_0.$$

It follows from (2.2) and (2.3) that

$$|(\Phi v_j)(t)-(\Phi v)(t)| \le \sum_{i=1}^{p}|v_j(t+i\tau)-v(t+i\tau)|$$
$$+\Big|\sum_{i=p+1}^{\infty}(-1)^{i+1}v_j(t+i\tau)\Big| + \Big|\sum_{i=p+1}^{\infty}(-1)^{i+1}v(t+i\tau)\Big|$$
$$< \varepsilon, \qquad t\in I,\ j\ge j_0,$$

which implies that Φv_j converges to Φv uniformly on I. In view of the fact that $(\Phi v)(t) = (\Phi v)(T-\tau)$ for $t\in[T_*,T-\tau]$ and $v\in V$, we conclude that Φ is continuous on V. The proof is complete.

Proof of Theorem 1.1. Put $\delta = a-|c|-\max_{t\in\mathbb{R}}|\omega(t)|>0$. Take a number $T\ge t_0$ so large that

$$T_* = \min\{T-\tau, \inf\{g(t): t\ge T\}\} \ge t_0$$

and

$$\int_T^\infty s^{n-1}F(s,a)\,ds < \delta. \tag{2.4}$$

Let

$$G(t) = \begin{cases} \int_t^\infty \dfrac{(s-t)^{n-2}}{(n-2)!}F(s,a)\,ds, & n \geq 2, \\ F(t,a), & n = 1, \end{cases}$$

for $t \geq T$. Notice that

$$\int_t^\infty G(s)\,ds = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}F(s,a)\,ds, \qquad t \geq T. \tag{2.5}$$

Denote by Y the set of all functions $y \in C[T_*,\infty)$ such that

$$y(t) = y(T) \quad \text{for } t \in [T_*,T], \qquad |y(t)| \leq \int_t^\infty G(s)\,ds \quad \text{for } t \geq T$$

and

$$|y(t) - y(t+\tau)| \leq \int_t^{t+\tau} G(s)\,ds \qquad \text{for } t \geq T.$$

Obviously, Y is a closed convex subset of $C[T_*,\infty)$.

Now we claim that if $y \in Y$, then

$$\Big|\sum_{i=1}^m (-1)^{i+1} y(t+i\tau)\Big| \leq \int_{t+\tau}^\infty G(s)\,ds, \qquad t \geq T-\tau \tag{2.6}$$

for $m = 1,2,\ldots$ If m is odd, then

$$\begin{aligned} \Big|\sum_{i=1}^m (-1)^{i+1} y(t+i\tau)\Big| & \\ &= \Big|\sum_{j=1}^{(m-1)/2} [y(t+(2j-1)\tau) - y(t+2j\tau)] + y(t+m\tau)\Big| \\ &\leq \sum_{j=1}^{(m-1)/2} \int_{t+(2j-1)\tau}^{t+2j\tau} G(s)\,ds + \int_{t+m\tau}^\infty G(s)\,ds \\ &\leq \int_{t+\tau}^\infty G(s)\,ds, \qquad t \geq T-\tau,\ y \in Y. \end{aligned}$$

For the case where m is even, using the equality

$$\sum_{i=1}^{m}(-1)^{i+1}y(t+i\tau)=\sum_{j=1}^{m/2}[y(t+(2j-1)\tau)-y(t+2j\tau)],\qquad t\geq T-\tau,$$

we get (2.6).

According to (2.6), if $m\geq p\geq 1$ and $t\in[T-\tau,\infty)$, then

$$\begin{aligned}\Big|\sum_{i=p}^{m}(-1)^{i+1}y(t+i\tau)\Big| &= \Big|\sum_{i=1}^{m-p+1}(-1)^{i+p}y(t+(i+p-1)\tau)\Big|\\ &= \Big|\sum_{i=1}^{m-p+1}(-1)^{i+1}y(t+(p-1)\tau+i\tau)\Big|\\ &\leq \int_{t+p\tau}^{\infty}G(s)\,ds\to 0\qquad\text{as } p\to\infty\end{aligned}$$

for each $y\in Y$. Hence, $Y\subset U[T_*,\infty)$. Letting $m\to\infty$ in (2.6), we obtain

$$|(\Phi y)(t)|\leq\int_{t+\tau}^{\infty}G(s)\,ds,\qquad t\geq T-\tau,\ y\in Y.$$

Lemma 2.1 implies that Φ maps Y into $C[T_*,\infty)$ and is continuous on Y. From (2.4), (2.5) and the last inequality, it follows that

$$\lim_{t\to\infty}(\Phi y)(t)=0\quad\text{and}\quad|(\Phi y)(t)|\leq\delta,\qquad t\geq T_*,\ y\in Y.$$

Set

$$(\Omega y)(t)=\omega(t)+c+(-1)^{n-1}(\Phi y)(t),\qquad t\geq T_*,\ y\in Y. \tag{2.7}$$

Then we find that

$$(\Omega y)(t)=\omega(t)+c+o(1)\qquad(t\to\infty) \tag{2.8}$$

and

$$|(\Omega y)(t)|\leq|\omega(t)|+|c|+\delta\leq a,\qquad t\geq T_* \tag{2.9}$$

for each $y\in Y$.

We define the mapping $\mathcal{F}:Y\to C[T_*,\infty)$ as follows:

$$(\mathcal{F}y)(t)=\begin{cases}\displaystyle\int_{t}^{\infty}\frac{(s-t)^{n-1}}{(n-1)!}f(s,(\Omega y)(g(s)))\,ds, & t\geq T,\\ (\mathcal{F}y)(T), & t\in[T_*,T].\end{cases}$$

By (H3) and (2.9), the mapping $\mathcal{F}$ is well defined. We have $\mathcal{F}(Y)\subset Y$. In fact, if $t \ge T$ and $y \in Y$, then

$$|(\mathcal{F}y)(t)| \le \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)} F(s,a)\,ds = \int_t^\infty G(s)\,ds,$$

by (2.5), and

$$\begin{aligned}|(\mathcal{F}y)(t) - (\mathcal{F}y)(t+\tau)| &= \Big|\int_t^{t+\tau} f(s,(\Omega y)(g(s)))\,ds\Big| \\ &\le \int_t^{t+\tau} F(s,a)\,ds = \int_t^{t+\tau} G(s)\,ds\end{aligned}$$

for $n=1$, and

$$\begin{aligned}|(\mathcal{F}y)(t) - (\mathcal{F}y)(t+\tau)| &= \Big|\int_t^{t+\tau}\int_s^\infty \frac{(r-s)^{n-2}}{(n-2)!} f(r,(\Omega y)(g(r)))\,dr\,ds\Big| \\ &\le \int_t^{t+\tau}\int_s^\infty \frac{(r-s)^{n-2}}{(n-2)!} F(r,a)\,dr\,ds = \int_t^{t+\tau} G(s)\,ds\end{aligned}$$

for $n \ne 1$.

Since Ω is continuous on Y, the Lebesgue dominated convergence theorem shows that $\mathcal{F}$ is continuous on Y.

Now we claim that $\mathcal{F}(Y)$ is relatively compact. We note that $\mathcal{F}(Y)$ is uniformly bounded on every compact subinterval of $[T_*,\infty)$, because of $\mathcal{F}(Y) \subset Y$. By the Ascoli–Arzelà theorem, it suffices to verify that the family $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[T_*,\infty)$. Observe that

$$|(\mathcal{F}y)'(t)| \le \begin{cases} F(t,a), & n=1, \\ \int_T^\infty s^{n-2}F(s,a)\,ds, & n \ne 1, \end{cases} \qquad t \ge T,\ y \in Y.$$

Let I be an arbitrary compact subinterval of $[T,\infty)$. Then we see that $\{(\mathcal{F}y)'(t) : y \in Y\}$ is uniformly bounded on I. The mean value theorem implies that $\mathcal{F}(Y)$ is equicontinuous on I. Since $|(\mathcal{F}y)(t_1) - (\mathcal{F}y)(t_2)| = 0$ for $t_1,\, t_2 \in [T_*,T]$, we conclude that $\mathcal{F}(Y)$ is equicontinuous on every compact subinterval of $[T_*,\infty)$. Thus $\mathcal{F}(Y)$ is relatively compact as claimed.

Consequently, we are able to apply the Schauder–Tikhonov fixed point theorem to the operator $\mathcal{F}$ and find that there exists a $\widetilde{y} \in Y$ such that $\widetilde{y} = \mathcal{F}\widetilde{y}$. Set $x(t) = (\Omega\widetilde{y})(t)$. From (2.8) it follows that $x(t)$ satisfies (1.7). By (2.7) and (2.1), we obtain

$$\begin{aligned}x(t) + x(t-\tau) &= \omega(t) + \omega(t-\tau) + 2c + (-1)^{n-1}[(\Phi\widetilde{y})(t) + (\Phi\widetilde{y})(t-\tau)] \\ &= 2c + (-1)^{n-1}\widetilde{y}(t), \\ &= 2c + (-1)^{n-1}(\mathcal{F}\widetilde{y})(t), \qquad t \ge T.\end{aligned}$$

Therefore we see that

$$\frac{d^n}{dt^n}[x(t)+x(t-\tau)] = (-1)^{n-1}(\mathcal{F}\widetilde{y})^{(n)}(t) = -f(t,x(g(t))), \qquad t \ge T,$$

so that $x(t)$ is a solution of (1.1). The proof is complete.

3. Proof of Theorem 1.2 $(0<\lambda<1)$**.** We need a few lemmas.

Let T and T_* be constants such that $T-\tau \ge T_* \ge t_0$. We denote by $S[T_*,\infty)$ the set of all functions $u \in C[T_*,\infty)$ such that the series

$$\sum_{i=1}^{\infty} |u(t+i\tau)| \tag{3.1}$$

converges uniformly on $[T-\tau,\infty)$. It is easy to see that $S[T_*,\infty) \subset U[T_*,\infty)$ and Φ maps $S[T_*,\infty)$ into $C[T_*,\infty)$.

LEMMA 3.1. *Let T and T_* be constants with $T-\tau \ge T_* \ge t_0$. Suppose that $\varphi \in S[T_*,\infty)$ satisfies $\varphi(t) \ge 0$ for $t \ge T$ and define*

$$W = \{w \in C[T_*,\infty) : |w(t)| \le \varphi(t),\ t \ge T\}.$$

Then $W \subset S[T_*,\infty)$ and Φ is continuous on W in the $C[T_*,\infty)$-topology.

Proof. It is clear that $W \subset S[T_*,\infty)$. Let $\varepsilon > 0$. There is an integer $p \ge 1$ such that

$$\sum_{i=p+1}^{\infty} \varphi(t+i\tau) < \frac{\varepsilon}{3}, \qquad t \ge T-\tau.$$

Take an arbitrary compact subinterval I of $[T-\tau,\infty)$. Let $\{w_j\}_{j=1}^{\infty}$ be a sequence in W converging to $w \in W$ in the $C[T_*,\infty)$-topology. There exists an integer $j_0 \ge 1$ such that

$$\sum_{i=1}^{p} |w_j(t+i\tau) - w(t+i\tau)| < \frac{\varepsilon}{3}, \quad t \in I,\ j \ge j_0.$$

We see that

$$\begin{aligned}
|(\Phi w_j)(t) - (\Phi w)(t)| &\le \sum_{i=1}^{p} |w_j(t+i\tau) - w(t+i\tau)| \\
&\quad + \sum_{i=p+1}^{\infty} |w_j(t+i\tau)| + \sum_{i=p+1}^{\infty} |w(t+i\tau)| \\
&< \frac{\varepsilon}{3} + 2\sum_{i=p+1}^{\infty} \varphi(t+i\tau) < \varepsilon, \qquad t \in I,\ j \ge j_0,
\end{aligned}$$

implying that Φw_j converges to Φw uniformly on I. For $t \in [T_*, T-\tau]$, we have $|(\Phi w_j)(t) - (\Phi w)(t)| = |(\Phi w_j)(T-\tau) - (\Phi w)(T-\tau)|$. Therefore, Φ is continuous on W.

LEMMA 3.2. *Let $u \in C[T_*, \infty)$. Then $u \in S[T_*, \infty)$ if and only if the series* (3.1) *converges for each fixed $t \in [T-\tau, \infty)$ and*

$$\lim_{t\to\infty} \sum_{i=1}^{\infty} |u(t+i\tau)| = 0. \tag{3.2}$$

Proof. We note that if the series (3.1) converges for each fixed $t \in [T-\tau, \infty)$, then

$$\begin{aligned}\sup_{t\in[T+m\tau,\infty)} \sum_{i=1}^{\infty} |u(t+i\tau)| &= \sup_{t\in[T-\tau,\infty)} \sum_{i=1}^{\infty} |u(t+(m+1)\tau+i\tau)| \\ &= \sup_{t\in[T-\tau,\infty)} \sum_{i=m+2}^{\infty} |u(t+i\tau)|\end{aligned} \tag{3.3}$$

for $m = 1, 2, \ldots$

First we prove the "only if" part. Assume that $u \in S[T_*, \infty)$. Clearly, the series (3.1) converges for each fixed $t \in [T-\tau, \infty)$. Letting $m \to \infty$ in (3.3), we have

$$\lim_{m\to\infty} \sup_{t\in[T+m\tau,\infty)} \sum_{i=1}^{\infty} |u(t+i\tau)| = 0,$$

which implies (3.2).

Conversely, suppose that the series (3.1) converges for each fixed $t \in [T-\tau, \infty)$ and (3.2) holds. By (3.3) again, we obtain

$$\lim_{m\to\infty} \sup_{t\in[T-\tau,\infty)} \sum_{i=m+2}^{\infty} |u(t+i\tau)| = 0.$$

This shows that the series (3.1) converges uniformly on $[T-\tau, \infty)$. Hence, the "if" part follows.

LEMMA 3.3. *Let $0 < \lambda < 1$ and $k \in \mathbb{N} \cup \{0\}$. Suppose that $G \in C[t_0, \infty)$ satisfies*

$$G(t) \geq 0 \quad \text{for } t \geq t_0 \quad \text{and} \quad \int_{t_0}^{\infty} \lambda^{-t/\tau} G(t)\,dt < \infty, \tag{3.4}$$

and define the function φ on $[t_0, \infty)$ by

$$\varphi(t) = \lambda^{-t/\tau} \int_{t}^{\infty} (s-t)^k G(s)\,ds, \qquad t \geq t_0.$$

Then $\sum_{i=1}^{\infty} \varphi(t+i\tau)$ converges for each fixed $t \in [t_0-\tau, \infty)$ and tends to 0 as $t \to \infty$.

P r o o f. Let $t \ge t_0 - \tau$ be fixed. Observe that

$$
(3.5)\quad \sum_{i=1}^{\infty} \varphi(t+i\tau) = \sum_{i=1}^{\infty} \lambda^{-(t+i\tau)/\tau} \sum_{j=i}^{\infty} \int\limits_{t+j\tau}^{t+(j+1)\tau} (s-t-i\tau)^k G(s)\,ds
$$

$$
= \sum_{j=1}^{\infty} \int\limits_{t+j\tau}^{t+(j+1)\tau} \sum_{i=1}^{j} \lambda^{(s-t-i\tau)/\tau} (s-t-i\tau)^k \lambda^{-s/\tau} G(s)\,ds.
$$

If $s \in [t+j\tau, t+(j+1)\tau]$, then $(j-i)\tau \le s-t-i\tau \le (j+1-i)\tau$. Hence

$$
(3.6)\quad \sum_{i=1}^{j} \lambda^{(s-t-i\tau)/\tau}(s-t-i\tau)^k \le \tau^k \sum_{i=1}^{j} \lambda^{j-i}(j+1-i)^k
$$

$$
= \tau^k \sum_{l=1}^{j} \lambda^{l-1} l^k \le \tau^k K
$$

for $s \in [t+j\tau, t+(j+1)\tau]$, $K = \sum_{i=1}^{\infty} \lambda^{i-1} i^k$. By (3.5) and (3.6), we obtain

$$
\sum_{i=1}^{\infty} \varphi(t+i\tau) \le \tau^k K \int\limits_{t+\tau}^{\infty} \lambda^{-s/\tau} G(s)\,ds.
$$

This completes the proof.

Proof of Theorem 1.2 ($0 < \lambda < 1$). Let $0 < \lambda < 1$. Put $\delta = a - \max_{t\in\mathbb{R}} |\omega(t)| > 0$, $G(t) = F(t, a\lambda^{g(t)/\tau})$, and

$$
\varphi(t) = \lambda^{-t/\tau} \int\limits_{t}^{\infty} (s-t)^{n-1} G(s)\,ds \ge 0, \qquad t \ge t_0.
$$

From Lemma 3.3 it follows that

$$
\eta(t) \equiv \sum_{i=1}^{\infty} \varphi(t+i\tau)
$$

converges for each fixed $t \in [t_0 - \tau, \infty)$ and $\lim_{t\to\infty} \eta(t) = 0$. Thus we can choose a number $T \ge t_0$ so large that $\eta(t) \le \delta$ for $t \ge T - \tau$ and

$$
T_* = \min\{T-\tau, \inf\{g(t) : t \ge T\}\} \ge t_0.
$$

Lemma 3.2 implies $\varphi|_{[T_*,\infty)} \in S[T_*,\infty)$. Define

$$
(3.7)\qquad Y = \{y \in C[T_*,\infty) : |y(t)| \le \varphi(t) \text{ for } t \ge T_*\}.
$$

Then Y is closed and convex. By Lemma 3.1, the mapping Φ is continuous on Y. Put

$$
(\Omega y)(t) = \omega(t) + (-1)^{n-1}(\Phi y)(t), \qquad t \ge T_*,\ y \in Y.
$$

Since

$$|(\Phi y)(t)| \le \sum_{i=1}^{\infty} \varphi(t+i\tau) = \eta(t) \le \delta, \qquad t \ge T-\tau,\ y \in Y,$$

we have

$$|(\Omega y)(t)| \le |\omega(t)| + \delta \le a, \qquad t \ge T_*,\ y \in Y, \tag{3.8}$$

and

$$(\Omega y)(t) = \omega(t) + o(1) \quad (t \to \infty), \qquad y \in Y. \tag{3.9}$$

To each $y \in Y$ we assign the function $\mathcal{F}y$ on $[T_*, \infty)$ by

$$(\mathcal{F}y)(t) = \begin{cases} \lambda^{-t/\tau} \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} f(s, (\Omega y)(g(s))\lambda^{g(s)/\tau})\, ds, & t \ge T, \\ (\mathcal{F}y)(T), & t \in [T_*, T]. \end{cases}$$

In view of (3.8), we easily see that $\mathcal{F}$ is well defined and maps Y into itself. Using the same arguments as in the proof of Theorem 1.1, we conclude that $\mathcal{F}$ is continuous and $\mathcal{F}(Y)$ is relatively compact. Application of the Schauder–Tikhonov fixed point theorem shows that there exists $\widetilde{y} \in Y$ such that $\widetilde{y} = \mathcal{F}\widetilde{y}$. Put $x(t) = (\Omega\widetilde{y})(t)\lambda^{t/\tau}$. Then we obtain

$$\begin{aligned} x(t) + \lambda x(t-\tau) &= (\Omega\widetilde{y})(t)\lambda^{t/\tau} + \lambda(\Omega\widetilde{y})(t-\tau)\lambda^{(t-\tau)/\tau} \\ &= \lambda^{t/\tau}[(\Omega\widetilde{y})(t-\tau) + (\Omega\widetilde{y})(t-\tau)] \\ &= \lambda^{t/\tau}[\omega(t) + \omega(t-\tau) + (-1)^{n-1}\{(\Phi\widetilde{y})(t) + (\Phi\widetilde{y})(t-\tau)\}] \\ &= (-1)^{n-1}\lambda^{t/\tau}\widetilde{y}(t) = (-1)^{n-1}\lambda^{t/\tau}(\mathcal{F}\widetilde{y})(t) \\ &= (-1)^{n-1} \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} f(s, x(g(s)))\, ds, \qquad t \ge T. \end{aligned}$$

By differentiation of the above equality, we conclude that $x(t)$ is a solution of (1.1). From (3.9) it follows that $x(t)$ satisfies (1.3). This completes the proof of Theorem 1.2 for the case $0 < \lambda < 1$.

4. Proof of Theorem 1.2 ($\lambda > 1$).

First we prove two lemmas.

Lemma 4.1. *Let $\lambda > 1$ and $k \in \mathbb{N} \cup \{0\}$. Suppose that $G \in C[t_0, \infty)$ satisfies* (3.4). *Then*

$$\lim_{t\to\infty} \lambda^{-t/\tau} \int_{t_0}^{t+\tau} (t+\tau-s)^k G(s)\, ds = 0. \tag{4.1}$$

Proof. It suffices to give the proof for the case $k = 0$. In fact, if

$$\lim_{t\to\infty} \lambda^{-t/\tau} \int_{t_0}^{t+\tau} G(s)\,ds = 0,$$

then for $k \neq 0$ we have

$$\lim_{t\to\infty} \lambda^{-t/\tau} \int_{t_0}^{t+\tau} (t+\tau-s)^k G(s)\,ds = \lim_{t\to\infty} \frac{d^k}{dt^k} \int_{t_0}^{t+\tau} (t+\tau-s)^k G(s)\,ds \Big/ \frac{d^k}{dt^k} \lambda^{t/\tau}$$
$$= \lim_{t\to\infty} k! \left[\frac{\tau}{\log\lambda}\right]^k \lambda^{-t/\tau} \int_{t_0}^{t+\tau} G(s)\,ds = 0.$$

Put $\psi(t) = \lambda^{-t/\tau} \int_{t_0}^t G(s)ds$. An easy computation shows that

$$\int_{t_0}^{t} \psi(s)\,ds = \frac{\tau}{\log\lambda} \Big[\int_{t_0}^{t} \lambda^{-s/\tau} G(s)\,ds - \psi(t) \Big], \qquad t \geq t_0. \tag{4.2}$$

Then we have

$$0 \leq \int_{t_0}^{t} \psi(s)\,ds \leq \frac{\tau}{\log\lambda} \int_{t_0}^{\infty} \lambda^{-s/\tau} G(s)\,ds, \qquad t \geq t_0,$$

which implies that ψ is integrable on $[t_0,\infty)$. It follows from (4.2) that $l = \lim_{t\to\infty}\psi(t)$ exists and is a nonnegative finite value. Since ψ is integrable on $[t_0,\infty)$, it is impossible that $l > 0$. Consequently, (4.1) holds for the case $k = 0$. This completes the proof.

LEMMA 4.2. *Let $\lambda > 1$ and $k \in \mathbb{N} \cup \{0\}$. Suppose that $G \in C[t_0,\infty)$ satisfies* (3.4), *and define the function φ on $[t_0,\infty)$ by*

$$\varphi(t) = \lambda^{-t/\tau} \int_{t_0}^{t} (t-s)^k G(s)\,ds, \qquad t \geq t_0.$$

Then $\sum_{i=1}^{\infty} \varphi(t+i\tau)$ converges for each fixed $t \in [t_0-\tau,\infty)$ and tends to 0 *as $t \to \infty$.*

Proof. Let $t \geq t_0 - \tau$ be fixed. We observe that

$$\sum_{i=1}^{\infty} \varphi(t+i\tau) = \sum_{i=1}^{\infty} \lambda^{-(t+i\tau)/\tau} \int_{t_0}^{t+\tau} (t+i\tau-s)^k G(s)\,ds$$
$$+ \sum_{i=2}^{\infty} \lambda^{-(t+i\tau)/\tau} \sum_{j=1}^{i-1} \int_{t+j\tau}^{t+(j+1)\tau} (t+i\tau-s)^k G(s)\,ds$$

$$= \lambda^{-t/\tau} \sum_{i=1}^{\infty} \lambda^{-i} \int_{t_0}^{t+\tau} (t + i\tau - s)^k G(s)\, ds$$

$$+ \sum_{j=1}^{\infty} \int_{t+j\tau}^{t+(j+1)\tau} \sum_{i=j+1}^{\infty} \lambda^{-(t+i\tau-s)/\tau} (t + i\tau - s)^k \lambda^{-s/\tau} G(s)\, ds$$

$$\equiv I_1(t) + I_2(t).$$

We have

$$(t + i\tau - s)^k = [(t + \tau - s) + (i - 1)\tau]^k \le 2^k [(t + \tau - s)^k + (i - 1)^k \tau^k]$$

for $s \in [t_0, t + \tau]$, because $(u + v)^k \le 2^k (u^k + v^k)$ for $u \ge 0$ and $v \ge 0$. Therefore

$$I_1(t) \le \lambda^{-t/\tau} 2^k \sum_{i=1}^{\infty} \lambda^{-i} \int_{t_0}^{t+\tau} (t + \tau - s)^k G(s)\, ds$$

$$+ \lambda^{-t/\tau} 2^k \tau^k \sum_{i=1}^{\infty} \lambda^{-i} (i - 1)^k \int_{t_0}^{t+\tau} G(s)\, ds$$

$$= \frac{2^k}{\lambda - 1} \lambda^{-t/\tau} \int_{t_0}^{t+\tau} (t + \tau - s)^k G(s)\, ds + L \lambda^{-t/\tau} \int_{t_0}^{t+\tau} G(s)\, ds,$$

where $L = 2^k \tau^k \sum_{i=1}^{\infty} \lambda^{-i} (i - 1)^k$. By Lemma 4.1 we obtain $\lim_{t\to\infty} I_1(t) = 0$.

If $s \in [t + j\tau, t + (j + 1)\tau]$, then $(i - j - 1)\tau \le t + i\tau - s \le (i - j)\tau$. Thus

$$\sum_{i=j+1}^{\infty} \lambda^{-(t+i\tau-s)/\tau} (t + i\tau - s)^k \le \tau^k \sum_{i=j+1}^{\infty} \lambda^{-(i-j-1)} (i - j)^k$$

$$= \tau^k \sum_{l=1}^{\infty} \lambda^{-l+1} l^k \equiv M$$

for $s \in [t + j\tau, t + (j + 1)\tau]$, and so

$$I_2(t) \le M \int_{t+\tau}^{\infty} \lambda^{-s/\tau} G(s)\, ds,$$

which implies that $\lim_{t\to\infty} I_2(t) = 0$. This completes the proof.

Proof of Theorem 1.2 ($\lambda > 1$). Define $\delta = a - \max_{t\in\mathbb{R}} |\omega(t)| > 0$, $G(t) = F(t, a\lambda^{g(t)/\tau})$, and let

$$\varphi(t) = \lambda^{-t/\tau} \int_{t_0}^{t} (t - s)^{n-1} G(s) ds \ge 0, \qquad t \ge t_0.$$

In view of Lemma 4.2, we find that

$$\eta(t) \equiv \sum_{i=1}^{\infty} \varphi(t + i\tau)$$

converges for each fixed $t \in [t_0 - \tau, \infty)$ and $\lim_{t\to\infty} \eta(t) = 0$. Take $T \geq t_0$ such that $\eta(t) \leq \delta$ for $t \geq T - \tau$ and

$$T_* \equiv \min\{T - \tau, \inf\{g(t) : t \geq T\}\} \geq t_0.$$

By virtue of Lemma 3.2, we have $\varphi|_{[T_*,\infty)} \in S[T_*, \infty)$. We define the set Y by (3.7). To each $y \in Y$ we assign the functions Ωy and $\mathcal{F}y$ on $[T_*, \infty)$ by

$$(\Omega y)(t) = \omega(t) - (\Phi y)(t), \qquad t \geq T_*,$$

and

$$(\mathcal{F}y)(t) = \begin{cases} \lambda^{-t/\tau} \int_T^t \dfrac{(t-s)^{n-1}}{(n-1)!} f(s, (\Omega y)(g(s))\lambda^{g(s)/\tau})\, ds, & t \geq T, \\ 0, & t \in [T_*, T], \end{cases}$$

respectively. By the same argument as in the proof of Theorem 1.2 for the case $0 < \lambda < 1$, we conclude that $\mathcal{F}\widetilde{y} = \widetilde{y}$ for some $\widetilde{y} \in Y$, and that $x(t) \equiv (\Omega\widetilde{y})(t)\lambda^{t/\tau}$ is a solution of (1.1) satisfying (1.3). This completes the proof of Theorem 1.2.

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