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Two-dimensional real symmetric spaces with maximal projection constant

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Abstract. Let V be a two-dimensional real symmetric space with unit ball having $8n$ extreme points. Let $\lambda(V)$ denote the absolute projection constant of V. We show that $\lambda(V) \le \lambda(V_n)$ where V_n is the space whose ball is a regular $8n$-polygon. Also we reprove a result of [1] and [5] which states that $4/\pi = \lambda(l_2^{(2)}) \ge \lambda(V)$ for any two-dimensional real symmetric space V.

Introduction. Let X be a normed space and let V be a linear subspace of X. Denote by $\mathcal{P}(X,V)$ the set of all projections from X onto V, i.e., the set of all continuous extensions of $\mathrm{id} : \mathrm{V} \to \mathrm{V}$ onto X. Let

$$\lambda(V,X) = \inf\{\|P\| : P \in \mathcal{P}(X,V)\}, \tag{1.1}$$

$$\lambda(V) = \sup\{\lambda(V,X) : V \subset X\}. \tag{1.2}$$

We call $\lambda(V,X)$ the *relative projection constant* of V in X and $\lambda(V)$ the *absolute projection constant* of V. A projection $P \in \mathcal{P}(X,V)$ is called *minimal* if $\|P\| = \lambda(V,X)$. Observe that the problem of finding minimal projections is related to the Hahn–Banach theorem, since we are looking for a minimal norm extension of the identity operator on V.

In this note we show that, for any two-dimensional real symmetric space V with a polygonal unit ball having $8n$ extreme points,

$$\lambda(V) \le \lambda(V_n), \tag{1.3}$$

where V_n is the space whose unit ball is regular $8n$-polygon. As an application of (1.3) we reprove a result of [1] and [5] which states that

$$4/\pi = \lambda(l_2^{(2)}) \ge \lambda(V)$$

for any two-dimensional real symmetric space V.

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Now we introduce some notation which will be of use later. By S_V we will denote the unit sphere in a normed space V. The symbol $\mathrm{ext}(S_V)$ will stand for the set of all extreme points of S_V. Note that if V is a k-dimensional subspace of $l_1^{(n)}$ then each $P \in \mathcal{P}(l_1^{(n)}, V)$ has the form

$$Px = \sum_{i=1}^{k} u^i(x)v^i, \tag{1.4}$$

where $v^1, \ldots, v^k$ is a fixed basis of V and $u^1, \ldots, u^k \in l_\infty^{(n)}$ satisfy

$$u^j(v^i) = \sum_{l=1}^{n} u_l^j v_l^i = \delta_{ij}. \tag{1.5}$$

A point $x \in X$ is called a *norming point* for $f \in X^*$ if

$$x \in S_X \quad \text{and} \quad f(x) = \|f\|. \tag{1.6}$$

Definition 1.1. Let V be a finite-dimensional Banach space. It is called *symmetric* if there exists a basis $v^1, \ldots, v^k$ in V such that

$$\Big\| \sum_{i=1}^{k} |\alpha_i| v^i \Big\| = \Big\| \sum_{i=1}^{k} \alpha_{\pi(i)} v^i \Big\| \tag{1.7}$$

for any $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and any permutation π of the indices.

Now let $P = \sum_{i=1}^{k} u^i(\cdot)v^i \in \mathcal{P}(l_1^n, V)$. Define

$$\mathrm{crit}(P) = \{j \in \{1, \ldots, n\} : \|Pe_j\| = \|P\|\} \tag{1.8}$$

and for $j = 1, \ldots, n$,

$$V_j = (v_j^1, \ldots, v_j^k), \qquad U_j = (u_j^1, \ldots, u_j^k). \tag{1.9}$$

Theorem 1.2 [2, Th. 3, p. 294]. *Let* $P = \sum_{i=1}^{k} u^i(\cdot)v^i \in \mathcal{P}(l_1^{(n)}, V)$. *Then P is minimal if and only if there exists a nonzero $k \times k$ matrix M such that for every $j \in \mathrm{crit}(P)$,*

$$U_j = (u_j^1, \ldots, u_j^k) = \|P\| a^j, \tag{1.10}$$

where a^j is a norming point for the functional on V associated with MV_j, i.e.,

$$(MV_j)(x) = \sum_{i=1}^{k} (MV_j)_i x_i. \tag{1.11}$$

Here $x = \sum_{i=1}^{k} x_i v^i$.

Remark 1.3 (see e.g. [7]). If V is a symmetric space then M is the identity matrix.

Remark 1.4. By [3, Th. 1] it is easy to see that if M is invertible and $V_j \neq 0$ for $j = 1, \ldots, n$, then $\mathrm{crit}(P) = \{1, \ldots, n\}$ for any minimal projection P.

THEOREM 1.5 [8]. *Every two-dimensional real Banach space is linearly isometric to a subspace of* $L_1[-\pi/2, \pi/2]$.

THEOREM 1.6 [4, 6]. L_1 *is a maximal overspace for any two-dimensional real symmetric Banach space* V, *which means that*

$$\lambda(V) = \lambda(V, L_1). \tag{1.12}$$

II. Technical lemmas. In the above $\|x\| = \sum_{j=1}^{n} |V_j(x)| = 1$ determines S_V and the following lemma confirms that the "corners" of S_V ($\mathrm{ext}(S_V)$) are given by those x such that $V_j(x) = 0$ $k-1$ times.

LEMMA 2.1. *Let* $V = \mathrm{span}[v^1, \ldots, v^k]$ *be a* k-*dimensional subspace of* $l_1^{(n)}$. *Then* $x = \sum_{i=1}^{k} x_i v^i \in \mathrm{ext}(S_V)$ *if and only if the matrix* W *consisting of all vectors* V_j (see (1.9)) *orthogonal to* x *has rank* $k-1$ *and* $\|x\| = 1$. *We understand that* V_j *is orthogonal to* x *if*

$$V_j(x) = \sum_{i=1}^{k} (V_j)_i x_i = \sum_{i=1}^{k} v_j^i x_i = 0. \tag{2.1}$$

Proof. If $k = 1$, the result is obvious. So suppose that $k \geq 2$. Let $x \in \mathrm{ext}(S_V)$. Note that there is $j \in \{1, \ldots, n\}$ such that x is orthogonal to V_j, i.e., the jth coordinate of x with respect to the canonical basis of $\mathbb{R}^n$ is 0: if not, modifying slightly $x_1, \ldots, x_k$, we can construct $y, z \in S_V$ different from x such that $x = (y + z)/2$.

Now suppose that $\mathrm{rank}(W) < k - 1$ and $k > 2$. Put

$$S = \{j \in \{1, \ldots, n\} : x \text{ is orthogonal to } V_j\}$$

and let $l = \mathrm{card}(S)$. Set $Z = V \cap \bigcap_{j \in S} \ker(V_j)$ (we can consider Z as a subspace of $l_1^{(n-l)}$). Since $\mathrm{rank}(W) < k - 1$, $\dim(Z) \geq 2$. Since $x \in \mathrm{ext}(S_V)$ and $x \in Z$, $x \in \mathrm{ext}(S_Z)$. But, by the previous part of the proof, $V_j(x) = 0$ for some $j \notin S$, contrary to the definition of W.

Now take $x \in S_V$ and suppose that $\mathrm{rank}(W) = k - 1$. If $x \notin \mathrm{ext}(S_V)$, then

$$x = (x^1 + x^2)/2 \tag{2.2}$$

for some $x^1, x^2 \in S_V$ different from x. Fix $0 < c < 1$ and define a norm $\|\cdot\|_c$ on V by

$$\|y\|_c = c \sum_{j \in S} |V_j(y)| + \sum_{j \notin S} |V_j(y)| \tag{2.3}$$

(see (2.1)). Since $\mathrm{rank}(W) = k-1$, x^1 and x^2 are not perpendicular to all V_j for $j \in S$. Hence $\|x\|_c = 1$ and $\|x^i\|_c < 1$ for $i = 1, 2$, contrary to (2.2).

COROLLARY 2.2. *Let* $V = \mathrm{span}[v^1, v^2]$ *be a 2-dimensional subspace of* $l_1^{(n)}$. *Then* $x = x_1 v^1 + x_2 v^2 \in \mathrm{ext}(S_V)$ *if and only if there exists* $V_j \neq 0$ *which is orthogonal to* x *and* $\|x\|_1 = 1$.

DEFINITION 2.3. Given two nonnegative numbers a, b, $a > b$, we denote by $V_{a,b}$ the two-dimensional subspace of $l_1^{(4)}$ spanned by

$$v^1 = (a, b, -a, -b), \qquad v^2 = (b, a, b, a).$$

Let $W_{a,b}$ be the 2×4 matrix with rows v^1, v^2.

Analogously, let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two vectors with nonnegative coordinates and with $a_i > b_i$ for any $i \in \{1, \ldots, n\}$. Let $W_{[a,b]}$ be the $2 \times 4n$ matrix consisting of $W_{a_1,b_1}, \ldots, W_{a_n,b_n}$. Then we denote by $V_{[a,b]}$ the subspace of $l_1^{(4n)}$ spanned by the rows of $W_{[a,b]}$. We write $V_{[a,b]}$ for the space generated by a, b. Observe that $V_{[a,b]}$ is a symmetric space with respect to the basis v^1, v^2, where v^1, v^2 denote the rows of $W_{[a,b]}$.

REMARK 2.4. It is a simple consequence of [1, Lemma 1] that each two-dimensional real symmetric space with a polygonal unit ball having $8n$ extreme points is linearly isometric to $V_{[a,b]}$ for some $a, b \in \mathbb{R}^n$ with nonnegative coordinates.

LEMMA 2.5. *Let* $V_{[a,b]} \subset l_1^{(4n)}$ *be the space generated by* $a, b \in \mathbb{R}^n_+$. *Put*

$$\|(x, y)\| = \|xv^1 + yv^2\|_1. \tag{2.4}$$

For $j = 1, \ldots, 4n$ *define*

$$U_j = (V_j / \|V_j\|)\lambda, \tag{2.5}$$

where

$$\lambda = \frac{1}{\sum_{i=1}^n 2(a_i^2 + b_i^2)/\|(a_i, b_i)\|} \tag{2.6}$$

and V_j *are given by* (1.9). *For* $i = 1, 2$ *let* $u^i \in \mathbb{R}^{4n}$ *be the vector associated with* $U_1, \ldots, U_{4n}$ *by* (1.9). *Then the operator* $P_{[a,b]}$ *defined by*

$$P_{[a,b]}x = u^1(x)v^1 + u^2(x)v^2 \tag{2.7}$$

belongs to $\mathcal{P}(l_1^{(4n)}, V_{[a,b]})$ *and* $\|P_{[a,b]}\| = \lambda$.

Proof. Note that by the definition of v^1 and v^2, $u^1(v^2) = u^2(v^1) = 0$. Observe that by symmetry,

$$u^1(v^1) = u^2(v^2) = \lambda \sum_{i=1}^n 2(a_i^2 + b_i^2)/\|(a_i, b_i)\| = 1.$$

Hence the orthonormality conditions (1.5) are satisfied and consequently, $P_{[a,b]} \in \mathcal{P}(l_1^{(4n)}, V_{[a,b]})$.

To show that $\|P_{[a,b]}\| = \lambda$, observe that for any $j \in \{1,\ldots,4n\}$,

$$\|P_{[a,b]}e_j\| = \Big\|\sum_{j=1}^{4n} u_j^1 v^1 + u_j^2 v^2\Big\|_1$$
$$= \lambda\|(a_i,b_i)/\|(a_i,b_i)\|\| = \lambda \qquad \text{(by symmetry)},$$

which completes the proof. (Here i is so chosen that $j = 4i - k$, where $k \in \{0,1,2,3\}$.)

LEMMA 2.6. *Let $V_{[a,b]}$ be the space generated by $a, b \in \mathbb{R}^n$. Suppose that*

$$0 \le b_i < a_i, \qquad b_i/a_i < b_{i+1}/a_{i+1}. \tag{2.8}$$

Then

$$\|(a_i,b_i)\| = 2\Big(\sum_{j\le i} a_j(a_i+b_i) + \sum_{j>i} a_i(a_j+b_j)\Big), \tag{2.9}$$

where $\sum_{j>n} = 0$ by definition.

Proof. Note that, by (2.8), if $j \le i$, then

$$|a_ia_j + b_ib_j| + |-a_ia_j + b_ib_j| + |b_ia_j + a_ib_j| + |-b_ia_j + a_ib_j| = 2a_j(a_i+b_i).$$

Hence the result follows from the definition of v^1, v^2 and (2.4).

Now we recall the well known fact that a (weighted) harmonic mean of n nonnegative numbers is no greater than the (weighted) arithmetic mean of these numbers:

LEMMA 2.7 *Let $a_1,\ldots,a_n \in \mathbb{R}_+ \setminus \{0\}$. Then*

$$\frac{1}{\sum_{i=1}^n \lambda_i a_i^{-1}} \le \sum_{i=1}^n \lambda_i a_i, \tag{2.10}$$

where $0 \le \lambda_i \le 1$ and $\sum_{i=1}^n \lambda_i = 1$.

Proof. By cross-multiplying, (2.10) follows easily from the facts that

$$a_i/a_j + a_j/a_i \ge 2 \quad \text{and} \quad 1 = \sum_{i=1}^n \lambda_i^2 + 2\sum_{j<i}\lambda_i\lambda_j.$$

LEMMA 2.8. *The following trigonometric identities are satisfied*:

$$\frac{(\cos 2a - \sin 2a)\sin((a+b)/2)}{\cos a\cos((a-b)/2)} - \frac{2\sin a\,(\cos a + \sin a - \cos b - \sin b)}{\sin(a-b)} = \frac{\sin((b-3a)/2)}{\cos a\cos((a-b)/2)}; \tag{2.11}$$

$$\begin{aligned}(2.12)\quad &\frac{2(\cos b-\sin b)\sin((a+b)/2)}{\cos((a-b)/2)}\\ &\quad+\frac{(\cos 2b-\sin 2b)(\cos a+\sin a-\cos b-\sin b)}{\sin(a-b)(\cos b+\sin b)}\\ &\qquad=\frac{\sin((a-3b)/2)+\cos((a-3b)/2)}{\cos((a-b)/2)(\sin b+\cos b)};\end{aligned}$$

$$\begin{aligned}(2.13)\quad &2\cos((b-c)/2)(\cos b-\sin b)(\sin((a+b)/2)\\ &-(\cos 2b-\sin 2b)\sin((a-c)/2)\\ &-2\sin b\cos((a-b)/2)(\cos((b+c)/2)-\sin((b+c)/2))\\ &\qquad=\sin((a+c-2b)/2);\end{aligned}$$

$$\begin{aligned}(2.14)\quad &\frac{2\sin a\,(\cos b+\sin b)(\sin(b-c)+\sin(c-a)+\sin(a-b))}{\sin(a-b)\sin(b-c)}\\ &\quad-\frac{2\sin a(\cos b+\sin b-\cos c-\sin c)}{\sin(b-c)}\\ &\qquad=\frac{2\sin a\,(\cos b+\sin b-\cos a-\sin a)}{\sin(a-b)};\end{aligned}$$

$$\begin{aligned}(2.15)\quad &\cos((b-c)/2)\sin((a+b)/2)+\cos b\sin((c-a)/2)\\ &\qquad=\cos((a-b)/2)\sin((b+c)/2);\end{aligned}$$

$$(2.16)\qquad 1=2\sin\frac{\pi}{8l}\sum_{j=1}^{l}\left(\cos\frac{\pi(2j-1)}{8l}+\sin\frac{\pi(2j-1)}{8l}\right);$$

and

$$\begin{aligned}(2.17)\quad &\sin(b-a)+\sin(a-c)+\sin(c-b)\\ &\qquad=-4\sin((b-a)/2)\sin((c-a)/2)\sin((b-c)/2).\end{aligned}$$

Proof. Since to prove (2.11)–(2.17) we only use the basic trigonometric formulas and routine calculations, we restrict ourselves to indicating only the main steps.

To show (2.11) observe that

$$\begin{aligned}&\frac{(\cos 2a-\sin 2a)\sin((a+b)/2)}{\cos a\cos((a-b)/2)}-\frac{2\sin a\,(\cos a+\sin a-\cos b-\sin b)}{\sin(a-b)}\\ &=\frac{(\cos 2a-\sin 2a)\sin((a+b)/2)}{\cos a\cos((a-b)/2)}-\frac{\sin 2a\,(\cos((a+b)/2)-\sin((a+b)/2))}{\cos a\cos((a-b)/2)}\\ &=\frac{\sin((b-3a)/2)}{\cos a\cos((a-b)/2)}.\end{aligned}$$

To show (2.12) observe that

$$\begin{aligned}
&\frac{2(\cos b - \sin b)\sin((a+b)/2)}{\cos((a-b)/2)} \\
&\qquad + \frac{(\cos 2b - \sin 2b)(\cos a + \sin a - \cos b - \sin b)}{\sin(a-b)(\cos b + \sin b)} \\
&\quad = \frac{2\cos 2b \sin((a+b)/2)}{\cos((a-b)/2)(\cos b + \sin b)} \\
&\qquad + \frac{(\cos 2b - \sin 2b)(\cos((a+b)/2) - \sin((a+b)/2))}{2\cos((a-b)/2)(\cos b + \sin b)} \\
&\quad = \frac{\sin((a-3b)/2) + \cos((a-3b)/2)}{\cos((a-b)/2)(\sin b + \cos b)}.
\end{aligned}$$

To prove (2.13) note that

$$\begin{aligned}
&2\cos((b-c)/2)(\cos b - \sin b)\sin((a+b)/2) - (\cos 2b - \sin 2b)\sin((a-c)/2) \\
&\quad - 2\sin b\cos((a-b)/2)[\cos((b+c)/2) - \sin((b+c)/2)] \\
&= 2\cos((b-c)/2)(\cos b - \sin b)[\sin((a-b)/2)\cos b + \cos((a-b)/2)\sin b] \\
&\quad - (\cos 2b - \sin 2b)\sin((a-c)/2) \\
&\quad - 2\sin b\cos((a-b)/2)[\cos((c-b)/2)\cos b - \sin((c-b)/2)\sin b \\
&\quad - \sin((c-b)/2)\cos b - \cos((c-b)/2)\sin b] \\
&= 2\cos^2 b\,\cos((b-c)/2)\sin((a-b)/2) \\
&\quad - 2\sin^2 b\,\cos((b-c)/2)\sin((a-b)/2) \\
&\quad + \sin 2b\,[\cos((b-c)/2)\cos((a-b)/2) - \cos((b-c)/2)\sin((a-b)/2)] \\
&\quad - (\cos 2b - \sin 2b)\sin((a-c)/2) \\
&\quad - 2\sin^2 b\cos((a-b)/2)[-\sin((c-b)/2) - \cos((c-b)/2)] \\
&\quad - \sin 2b\cos((a-b)/2)[\cos((c-b)/2) - \sin((c-b)/2)] \\
&= 2\cos^2 b\cos((b-c)/2)\sin((a-b)/2) - \cos 2b\sin((a-c)/2) \\
&\quad - 2\sin^2 b\cos((a-b)/2)\sin((b-c)/2) \\
&= 2(\cos^2 b - \sin^2 b)[\cos((b-c)/2)\sin((a-b)/2) - \cos 2b\sin((a-c)/2)] \\
&\quad + (\sin^2 b - \cos^2 b)\sin[(a-b)/2 + (b-c)/2)] \\
&= \cos^2 b\cos((b-c)/2)\sin((a-b)/2) - \sin^2 b\cos((a-b)/2)\sin((b-c)/2) \\
&\quad + \sin^2 b\cos((b-c)/2)\sin((a-b)/2) \\
&\quad - \cos^2 b\cos((a-b)/2)\sin((b-c)/2) \\
&= \cos^2 b\,[\cos((b-c)/2)\sin((a-b)/2) - \cos((a-b)/2)\sin((b-c)/2)] \\
&\quad + \sin^2 b\,[-\cos((a-b)/2)\sin((b-c)/2) + \cos((b-c)/2)\sin((a-b)/2)] \\
&= \sin((a+c-2b)/2).
\end{aligned}$$

To show (2.14), note that

$$\frac{2\sin a\,(\cos b+\sin b)(\sin(b-c)+\sin(c-a)+\sin(a-b))}{\sin(a-b)\sin(b-c)} - \frac{2\sin a\,(\cos b+\sin b-\cos c-\sin c)}{\sin(b-c)}$$
$$= \frac{2\sin a\,(\cos b+\sin b-\sin a-\cos a)}{\sin(a-b)} + \frac{2\sin a\sin(b-c)(\cos a+\sin a)}{\sin(a-b)\sin(b-c)} + \frac{2\sin a\,(\cos b+\sin b)\sin(c-a)}{\sin(a-b)\sin(b-c)} + \frac{2\sin a\,(\cos c+\sin c)}{\sin(b-c)}.$$

To finish the proof we have to show that the sum of the second and third terms from the above formula is

$$-\frac{2\sin a(\cos c+\sin c)}{\sin(b-c)}.$$

But this follows immediately from the fact that

$$\sin(b-c)(\cos a+\sin a)+\sin(c-a)(\cos b+\sin b)=\sin(b-a)(\cos c+\sin c).$$

To show (2.15) note that

$$\begin{aligned}\cos((b-c)/2)\sin((a+b)/2)+\cos b\sin((c-a)/2)\\ &= \cos((b+c)/2-c)\sin((a+b)/2)+\cos b\sin((b+c)/2-(a+b)/2)\\ &= \sin((b+c)/2)(\cos b\,\cos((a+b)/2)+\sin b\,\sin((a+b)/2))\\ &= \cos((a-b)/2)\sin((b+c)/2).\end{aligned}$$

To prove (2.16), observe that for odd l,

$$\begin{aligned}2\sin\frac{\pi}{8l}\sum_{j=1}^{l}\left(\cos\frac{\pi(2j-1)}{8l}+\sin\frac{\pi(2j-1)}{8l}\right)\\ &= 4\sin\frac{\pi}{8l}\left(\cos\frac{\pi}{8}+\sin\frac{\pi}{8}\right)\left(\frac{1}{2}+\sum_{j=1}^{(l-1)/2}\cos\frac{\pi j}{4l}\right)\\ &= 2\sin\frac{\pi}{8l}\left(\cos\frac{\pi}{8}+\sin\frac{\pi}{8}\right)\frac{\sin\frac{((l-1)/2+1/2)\pi}{4l}}{\sin\frac{\pi}{8l}}\\ &= 2\sin\frac{\pi}{8}\left(\cos\frac{\pi}{8}+\sin\frac{\pi}{8}\right)=1.\end{aligned}$$

If l is even then

$$2\sin\frac{\pi}{8l}\sum_{j=1}^{l}\left(\cos\frac{\pi(2j-1)}{8l}+\sin\frac{\pi(2j-1)}{8l}\right)$$

$$=2\sin\frac{\pi}{4l}\sum_{j=1}^{l/2}\left(\cos\frac{\pi(2j-1)}{4l}+\sin\frac{\pi(2j-1)}{4l}\right),$$

which reduces the proof to the previous case.

The proof of (2.17) is an easy calculation, so we omit it.

LEMMA 2.9. *Let $A_n=(a_{ij})_{i,j=1}^n$ be an $n\times n$ symmetric matrix given by $a_{ij}=\cos d_i\,(\cos d_j+\sin d_j)$ for $1\le i\le j$. Let A_n^i be the matrix obtained from A_n by replacing the ith column by $(1,\dots,1)$. Then for any $d_1,\dots,d_n\in(0,\pi/2)$,*

$$\det(A_n)=\cos d_1\,(\cos d_n+\sin d_n)\prod_{j=1}^{n-1}\sin(d_j-d_{j+1}); \tag{2.18}$$

$$\det(A_n^1)=(\cos d_n+\sin d_n)\prod_{j=2}^{n-1}\sin(d_j-d_{j+1})(\cos d_2-\cos d_1); \tag{2.19}$$

$$\det(A_n^n)=\cos d_1\,(\cos d_{n-1}+\sin d_{n-1}-\cos d_n-\sin d_n)\times\prod_{j=1}^{n-2}\sin(d_j-d_{j+1}); \tag{2.20}$$

$$\det(A_n^i)=-4\cos d_1\,(\cos d_n+\sin d_n)\prod_{j=1}^{i-2}\sin(d_j-d_{j+1})\times\prod_{j=i+1}^{n-1}\sin(d_j-d_{j+1})\sin((d_{i-1}-d_i)/2)\times\sin((d_{i-1}-d_{i+1})/2)\sin((d_i-d_{i+1})/2), \tag{2.21}$$

for $i=2,\dots,n-1$. Here $\prod_{i=l}^{j}=1$ if $l>j$, by definition.

P r o o f. We prove the lemma by induction on n. If $n=2$, then easy calculations show that (2.18)–(2.20) hold true. To prove (2.18) for any $n\in\mathbb{N}$, add the second column of A_n multiplied by $-\cos d_1/\cos d_2$ to the first one, apply the induction hypothesis to the $(n-1)\times(n-1)$ matrix given by $d_2,\dots,d_n$ and the standard formula for calculating determinants.

To show (2.19) for any $n\in\mathbb{N}$, add the second column of A_n^1 multiplied by $-\cos d_1/\cos d_2$ to the first one and proceed as in the previous case. The same reasoning applies to (2.20). To show (2.21), we first consider the case

$n = 3$. Note that

$$\begin{aligned}\det(A_3^2) = &\ \cos d_1 (\cos d_3 + \sin d_3) \\ &\times (\sin(d_2 - d_1) + \sin(d_1 - d_3) + \sin(d_3 - d_2)) \\ = &\ -4\cos d_1 (\cos d_3 + \sin d_3) \sin((d_1 - d_2)/2) \\ &\times \sin((d_1 - d_3)/2) \sin(d_2 - d_3)/2) \qquad \text{(by (2.17))},\end{aligned}$$

as required.

To prove (2.21) for $n \ge 4$, for $i = 3, \ldots, n-1$ add the second column of A_n^i multiplied by $-\cos d_1/\cos d_2$ to the first one, and apply the induction hypothesis for $d_2, \ldots, d_n$ and $i-1$. For $i = 2$, add the $(n-1)$th column of A_n^i multiplied by $-(\cos d_n + \sin d_n)/(\cos d_{n-1} + \sin d_{n-1})$ and apply the induction hypothesis for $d_1, \ldots, d_{n-1}$ and i.

We also need the following obvious fact:

REMARK 2.10. Let f, g be real-valued functions defined on a set A, with $f \le g$. If $g(x) = \max_{y\in A} g(y)$ and $g(x) = f(x)$, then $f(x) = \max_{y\in A} f(y)$.

III. The main results

THEOREM 3.1. *Let V be a two-dimensional real symmetric space with unit ball having exactly $8n$ extreme points. Then*

$$\lambda(V) \le \lambda(V_n), \tag{3.1}$$

where V_n is the space whose unit ball is a regular polyhedron having exactly $8n$ vertices. Moreover, if $\lambda(V) = \lambda(V_n)$, then V is linearly isometric to V_n.

Proof. First we consider the case $V = V_{[a,b]}$, where $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, $0 < b_i < a_i$, $b_i/a_i < b_{i+1}/a_{i+1}$, for $i = 1, \ldots, n-1$. For $i = 1, \ldots, n$, put $r_i = \sqrt{a_i^2 + b_i^2}$, and let $d_i \in (0, \pi/4)$ be so chosen that $a_i = r_i \cos d_i$ and $b_i = r_i \sin d_i$. Without loss, dividing a_i and b_i by a constant, we can assume that

$$\sum_{i=1}^n r_i = 1. \tag{3.2}$$

Put $r = (r_1, \ldots, r_n)$, $d = (d_1, \ldots, d_n)$ and let

$$g_n(d, r) = \sum_{i=1}^n r_i \sum_{j=1}^n r_j a_{ij}(d), \tag{3.3}$$

where $a_{ij}(d) = \cos d_i (\cos d_j + \sin d_j)$ for $i \le j$ and $a_{ij} = a_{ji}$ for $i > j$. Define

$$D_n = \Big\{(d, r) \in \mathbb{R}^{2n} : 0 \le r_i,\ \sum_{i=1}^n r_i = 1,\ 0 \le d_1 \le \ldots \le d_n \le \pi/4\Big\}. \tag{3.4}$$

First we will show that for any $n \geq 2$, $\max_{(d,r)\in D_n} g_n(d,r)$ is achieved for $r = (1/n, \ldots, 1/n)$ and $d = \left(\frac{\pi}{8n}, \frac{3\pi}{8n}, \ldots, \frac{(2n-1)\pi}{8n}\right)$. To do this, we apply the Lagrange multiplier method. Consider in $\operatorname{int}(D_n)$ the function

$$w_n(d,r) = g_n(d,r) + \lambda\Big(1 - \sum_{i=1}^{n} r_i\Big).$$

Note that

$$\frac{\partial w_n}{\partial r_i}(d,r) = 2\sum_{i=1}^{n} a_{ij}(d) r_j - \lambda \tag{3.5}$$

for $i = 1, \ldots, n$. By Cramer's rule and Lemma 2.9, the system of equations

$$\frac{\partial w_n}{\partial r_i}(d,r) = 2\sum_{i=1}^{n} a_{ij}(d) r_j - \lambda = 0 \tag{3.6}$$

for $i = 1, \ldots, n$ has, in $\operatorname{int}(D_n)$, for fixed λ and d, the solution $r(d,\lambda) = (r_1(d,\lambda), \ldots, r_n(d,\lambda))$ given by

$$r_1(d,\lambda) = \frac{\lambda(\cos d_2 - \cos d_1)}{2\cos d_1 \sin(d_1 - d_2)}, \tag{3.7}$$

$$r_i(d,\lambda) = \frac{\lambda \sin((d_{i+1} - d_{i-1})/2)}{2\cos((d_{i-1} - d_i)/2)\cos((d_i - d_{i+1})/2)}, \quad i = 2, \ldots, n-1, \tag{3.8}$$

$$r_n(d,\lambda) = \frac{-\lambda}{2\sin(d_{n-1} - d_n)} + \frac{\lambda(\cos d_{n-1} + \sin d_{n-1})}{2\sin(d_{n-1} - d_n)(\cos d_n + \sin d_n)}. \tag{3.9}$$

Now we prove by induction on n that the system of equations

$$\frac{\partial w_n}{\partial d_i}(d, r(d,\lambda)) = 0, \qquad i = 1, \ldots, n, \tag{3.10}$$

reduces to the system

$$\sin((3d_1 - d_2)/2) = 0, \tag{3.11}$$

$$\sin((d_{i-1} + d_{i+1} - 2d_i)/2) = 0, \qquad i = 2, \ldots, n-1, \tag{3.12}$$

$$\cos((d_{n-1} - 3d_n)/2) + \sin((d_{n-1} - 3d_n)/2) = 0. \tag{3.13}$$

Note that for $n = 2$,

$$\begin{aligned}\frac{\partial w_2}{\partial d_1}(d, r(d,\lambda)) &= \lambda^2 r_1(d,\lambda)\bigg(\frac{(\cos 2d_1 - \sin 2d_1)\sin((d_1 + d_2)/2)}{\cos d_1 \cos((d_1 - d_2)/2)} \\ &\qquad - \frac{2\sin d_1\,(\cos d_1 + \sin d_1 - \cos d_2 - \sin d_2)}{\sin(d_1 - d_2)}\bigg) \\ &= \lambda^2 r_1(d,\lambda)\frac{\sin((d_2 - 3d_1)/2)}{\cos d_1 \cos((d_1 - d_2)/2)} \qquad \text{(by (2.11))}.\end{aligned}$$

Analogously,

$$\begin{aligned}
&\frac{\partial w_2}{\partial d_2}(d, r(d,\lambda)) \\
&\qquad = \lambda^2 r_2(d,\lambda)\biggl(\frac{2(\cos d_2 - \sin d_2)\sin((d_1+d_2)/2)}{\cos((d_1-d_2)/2)} \\
&\qquad\qquad + \frac{(\cos 2d_2 - \sin 2d_2)(\cos d_1 + \sin d_1 - \cos d_2 - \sin d_2)}{\sin(d_1-d_2)(\cos d_2 + \sin d_2)}\biggr) \\
&\qquad = \lambda^2 r_2(d,\lambda)\frac{\sin((d_1-3d_2)/2) + \cos((d_1-3d_2)/2)}{\cos((d_1-d_2)/2)(\sin d_2 + \cos d_2)} \qquad \text{(by (2.12))},
\end{aligned}$$

which proves our claim for $n = 2$.

Now for illustration consider first the case $n = 3$. Note that

$$\begin{aligned}
\frac{\partial w_3}{\partial d_1}(d, r(d,\lambda)) = (2\lambda)^2 r_1(d,\lambda)[&(\cos 2d_1 - \sin 2d_1) r_1(d,\lambda) \\
&- 2r_2(d,\lambda)\sin d_1\,(\cos d_2 + \sin d_2) \\
&- 2r_3(d,\lambda)\sin d_1\,(\cos d_3 + \sin d_3)].
\end{aligned}$$

By (3.7)–(3.9), (2.17) and (2.14) applied to d_1, d_2, d_3 and the second and third terms of the above equality, we get

$$\frac{\partial w_3}{\partial d_1}(d, r(d,\lambda)) = \frac{\partial w_2}{\partial d_1}(d_1, d_2, r(d_1, d_2, \lambda)),$$

which proves the result for $\frac{\partial w_3}{\partial d_1}(d, r(d,\lambda))$.

Observe that by (3.7)–(3.9),

$$\begin{aligned}
&\frac{\partial w_3}{\partial d_2}(d, r(d,\lambda)) \\
&\qquad = \lambda^2 r_2(d,\lambda)\biggl(\frac{2\cos((d_2-d_3)/2)(\cos d_2 - \sin d_2)\sin((d_1+d_2)/2)}{\cos((d_1-d_2)/2)\cos((d_2-d_3)/2)} \\
&\qquad\quad - \frac{(\cos 2d_2 - \sin 2d_2)\sin((d_1-d_3)/2)}{\cos((d_1-d_2)/2)\cos((d_2-d_3)/2)} \\
&\qquad\quad - \frac{2\sin d_2\,[\cos((d_2+d_3)/2) - \sin((d_2+d_3)/2)]\cos((d_1-d_2)/2)}{\cos((d_1-d_2)/2)\cos((d_2-d_3)/2)}\biggr) \\
&\qquad = \lambda^2 r_2(d,\lambda)\frac{\sin((d_1+d_3-2d_2)/2)}{\cos((d_1-d_2)/2)\cos((d_2-d_3)/2)} \qquad \text{(by (2.13))},
\end{aligned}$$

which proves our claim.

Note that

$$\frac{\partial w_3}{\partial d_3}(d, r(d,\lambda)) = (2\lambda)^2 r_3(d,\lambda)[2\cos d_1\,(\cos d_3 - \sin d_3) r_1(d,\lambda) + 2r_2(d,\lambda)\cos d_2\,(\cos d_3 - \sin d_3) + r_3(d,\lambda)(\cos 2d_3 - \sin 2d_3)].$$

By (3.7)–(3.9), (2.17) and (2.15) applied to d_1, d_2, d_3 and the first and second terms of the above equality, we get

$$\frac{\partial w_3}{\partial d_3}(d, r(d,\lambda)) = \frac{\partial w_2}{\partial d_3}(d_2, d_3, r(d_2, d_3, \lambda)),$$

which proves the result for $\frac{\partial w_3}{\partial d_3}(d, r(d,\lambda))$.

Now fix $n \in \mathbb{N}$, $n \geq 4$. Observe that

$$\frac{\partial w_n}{\partial d_i}(d, r(d,\lambda)) = 4\lambda^2 r_i(d,\lambda)\Big(\sum_{j=1}^{i-1} 2r_j(d,\lambda)\cos d_j\,(\cos d_i - \sin d_i) + r_i(d,\lambda)(\cos 2d_i - \sin 2d_i) - \sum_{j>i} r_j(d,\lambda) 2\sin d_i\,(\cos d_j + \sin d_j)\Big). \tag{3.14}$$

Hence to prove (3.11) and (3.12) for $i = 2, \ldots, n-2$, apply (2.14) to d_{n-1} and d_n in the last two terms of the sum in (3.14) and the induction hypothesis for $n-1$ and $d_1, \ldots, d_{n-1}$. To prove (3.12) for $i = n-1$ and (3.13), apply (2.15) to d_1, d_2 in the first and second terms of $\sum_{j=1}^{i-1} 2r_j(d,\lambda)\cos d_j\,(\cos d_i - \sin d_i)$ and the induction hypothesis for $n-1$ and $d_2, \ldots, d_n$.

Now note that, since we consider $d_1, \ldots, d_n$ belonging to $(0, \pi/4)$, by (3.7)–(3.13) the system of equations

$$\frac{\partial w_n}{\partial r_i}(d, r(d,\lambda)) = 0, \qquad \frac{\partial w_n}{\partial d_i}(d, r(d,\lambda)) = 0, \qquad \sum_{i=1}^{n} r_i = 1,$$

has for $n = 1, 2, \ldots$ in $\mathrm{int}(D_n)$ the only solution

$$d^0 = \left(\frac{\pi}{8n}, \ldots, \frac{(2n-1)\pi}{8n}\right), \qquad r^0 = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right). \tag{3.15}$$

Hence if we prove that the function g_n does not attain a global maximum on the boundary of D_n then it has to attain it at (d^0, r^0). This will also be shown by induction on n. If $n = 1$, then $g_1(d) = \cos d\,(\cos d + \sin d)$. Hence $g_1''(d) = -2(\cos 2d + \sin 2d)$, which shows that g_1 attains a global maximum at $d^0 = \pi/8$. Now take any $n \geq 2$. Since D_n is a compact set, g_n attains a global maximum at some point $(d, r) \in D_n$. If $r_i > 0$ and $d_i < d_{i+1}$ for $i = 1, \ldots, n-1$, and $d_1 = 0$ or $d_n = \pi/4$, then, by easy calculations,

$g_n(d,r) = \max\{g_n(w,z) : (w,z) \in D_n^\varepsilon\}$, where

$$D_n^\varepsilon = \Big\{(d,r) \in \mathbb{R}^{2n} : r_i > 0,\ \sum_i r_i = 1,\ -\varepsilon < d_1 < \ldots < d_n < \pi/4 + \varepsilon\Big\}$$

and $0 < \varepsilon < \min\{d_2 - d_1, d_n - d_{n-1}\}$. But D_n^ε is an open set and by (3.7)–(3.13) the maximum on this set can be attained at (d^0, r^0) given by (3.15); a contradiction. So suppose that $d_i = d_{i+1}$ for some $i \in \{1, \ldots, n-1\}$. Then

$$\begin{aligned} g_n(d,r) &= g_{n-1}(d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n, r_1, \ldots, r_{i-1}, r_i + r_{i+1}, \ldots, r_n) \\ &< g_{n-1}\bigg(\frac{\pi}{8(n-1)}, \ldots, \frac{(2n-3)\pi}{8(n-1)},\ \frac{1}{n-1}, \ldots, \frac{1}{n-1}\bigg) \\ &\qquad\qquad\qquad\qquad\text{(by the induction hypothesis)} \\ &= \cos\frac{\pi}{8(n-1)} \cdot \frac{\sum_{j=1}^{n-1}\big(\cos\frac{(2j-1)\pi}{8(n-1)} + \sin\frac{(2j-1)}{8(n-1)}\big)}{n-1} \\ &= \frac{\cos\frac{\pi}{8(n-1)}}{2(n-1)\sin\frac{\pi}{8(n-1)}} \qquad \text{(by (2.16)).} \end{aligned}$$

Note that the function $f(x) = (4/x)\cos(\pi/x)/\sin(\pi/x)$ is strictly increasing for $x > 0$. Hence

$$\frac{\cos\frac{\pi}{8(n-1)}}{2(n-1)\sin\frac{\pi}{8(n-1)}} < \frac{\cos\frac{\pi}{8n}}{2n\sin\frac{\pi}{8n}} = g_n(d^0, r^0) \qquad \text{(by (2.16)),}$$

which shows that g_n does not attain its maximum on the boundary of D_n. The same reasoning applies to the case $r_i = 0$ for some $i \in \{1, \ldots, n\}$. Hence

$$\max_{(d,r)\in D_n} g_n(d,r) = g_n(d^0, r^0), \tag{3.16}$$

as required.

Observe that the unit ball of the space generated by (d^0, r^0) is a regular $8n$-polygon, so this space is isometric to V_n. To finish the proof of our theorem we apply the idea given in Remark 2.10 twice. Note that by Lemma 2.7,

$$g_n(d,r) \ge f_n(d,r) = \bigg(\sum_{i=1}^n \frac{r_i}{\sum_{j=1}^n r_j a_{ij}(d)}\bigg)^{-1}.$$

By Lemma 2.5, $f_n(d,r)$ is the norm of the projection $P_{[a,b]}$ defined in Lemma 2.5 (we use polar coordinates). Note that, by (3.6),

$$f_n(d^0, r^0) = g_n(d^0, r^0).$$

Now we show that the projection $P_{[a,b]}$ associated with (d^0, r^0) is a minimal projection. To do this, by Theorem 1.2, Remarks 1.3 and 1.4, we have to show that for $i = 1, \ldots, n$,

$$(a_i, b_i)/\|(a_i, b_i)\| = (r_i^0 \cos d_i^0, r_i^0 \sin d_i^0)/\|(r_i^0 \cos d_i^0, r_i^0 \sin d_i^0)\|$$

is the only norming point for the functional associated with (a_i, b_i) (see Th. 1.2). But by Lemma 2.6,

$$\|(a_i, b_i)\| = \sum_{j=1}^{n} r_j^0 a_{ij}(d^0).$$

By (3.6) and Corollary 2.2, all the extreme points of the unit ball of $V_{[a,b]}$ lie on the same Euclidean sphere. This shows that $(a_i, b_i)/\|(a_i, b_i)\|$ is the only norming point for the functional (a_i, b_i). To finish the proof of the theorem, note that, by Theorems 1.5 and 1.6 and Remark 2.4, for any two-dimensional real symmetric Banach space with $8n$ extreme points,

$$\lambda(V) = \lambda(V_{[a,b]}) = \lambda(V_{[a,b]}, l^1) \le \|P_{[a,b]}\| \le \lambda(V_n, l^1) = \lambda(V_n).$$

Moreover, by Remark 2.4 and the above reasoning, if $\lambda(V) = \lambda(V_n)$, then V has to be linearly isometric to V_n.

The proof of Theorem 3.1 is complete.

Now we apply Theorem 3.1 to reprove in a simple way a result of [1] and [5] concerning arbitrary two-dimensional real symmetric spaces.

THEOREM 3.2 [1, 5]. *For any two-dimensional symmetric real Banach space V,*

$$\lambda(V) \le \lambda(l_2^{(2)}).$$

Proof. By Theorem 1.6, we can assume that $V \subset L_1[-\pi/2, \pi/2]$. Hence we can approximate V, in the sense of the Banach–Mazur distance, by subspaces $V_{[a^n, b^n]} \subset l^1$ whose unit balls have exactly $8n$ extreme points. Since the function $V \mapsto \lambda(V)$ is continuous with respect to the Banach-Mazur distance, by Theorems 1.5 and 1.6,

$$\begin{aligned}\lambda(V) &= \lim_n \lambda(V_{[a^n, b^n]}) \le \lim_n \lambda(V_n)\\ &= \lim_n \frac{\cos\frac{\pi}{8n}}{2n \sin\frac{\pi}{8n}} \qquad \text{(by (2.17))}\\ &= 4/\pi = \lambda(l_2^{(2)}),\end{aligned}$$

as required.

References

[1] B. L. Chalmers, C. Franchetti and M. Giaquinta, *On the self-length of two-dimensional Banach spaces*, Bull. Austral. Math. Soc. 53 (1996), 101–107.

[2] B. L. Chalmers and F. T. Metcalf, *The determination of minimal projections and extensions in L^1*, Trans. Amer. Math. Soc. 329 (1992), 289–305.

[3] —, —, *A characterization and equations for minimal projections and extensions*, J. Operator Theory 32 (1994), 31–46.

[4] B. L. Chalmers and F. T. Metcalf, *A simple formula showing L^1 is a maximal overspace for two-dimensional real spaces*, Ann. Polon. Math. 56 (1992), 303–309.
[5] H. Koenig, *Projections onto symmetric spaces*, Quaestiones Math. 18 (1995), 199–220.
[6] J. Lindenstrauss, *On the extension of operators with a finite-dimensional range*, Illinois J. Math. 8 (1964), 488–499.
[7] N. Tomczak-Jaegermann, *Banach–Mazur Distances and Finite-Dimensional Operator Ideals*, Wiley, New York, 1989.
[8] D. Yost, *L_1 contains every two-dimensional normed space*, Ann. Polon. Math. 49 (1988), 17–19.

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