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Topological conjugacy of cascades generated by gradient flows on the two-dimensional sphere

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Abstract. This article presents a theorem about the topological conjugacy of a gradient dynamical system with a constant time step and the cascade generated by its Euler method. It is shown that on the two-dimensional sphere $\mathcal{S}^2$ the gradient dynamical flow is, under some natural assumptions, correctly reproduced by the Euler method for a sufficiently small time step. This means that the time-map of the induced dynamical system is globally topologically conjugate to the discrete dynamical system obtained via the Euler method.

1. Introduction. In recent years several papers have been devoted to studying the qualitative properties of discrete-time dynamical systems obtained via discretization methods. The basic question is whether the qualitative properties of continuous-time systems are preserved under discretization. Various concepts of differentiable dynamics were investigated. Results on stability and attraction properties ([KL]), the saddle-point structure about equilibria ([AD], [Bey1], [Bey2]), invariant manifolds ([BL], [Fec1]), averagings ([Fec2]) and algebraic-topological invariants ([MR]) can be mentioned as examples. A number of applications have been studied as well ([Gar4]). The investigations are concerned with both local (see, for instance, [Gar1], [Fec3]) and global conjugacy ([Gar2], [Gar3], [Gar5]).

This paper is devoted to the problem of topological conjugacy between the discretization of a gradient dynamical system and the cascade generated by its Euler method. Similar problems have been solved in recent years for numerical methods of order greater than one (see [Gar2], [Li]).

2. Topological conjugacy of gradient cascades. As mentioned above we consider a gradient differential equation and its Euler method. The time-map of the induced solution is compared to the cascade obtained via the

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Euler method. We show that on the two-dimensional sphere $\mathcal{S}^2$ a gradient dynamical system is, under some natural assumptions, correctly reproduced by the Euler method for a sufficiently small time step. This means that the time-map of the induced dynamical system is globally topologically conjugate to the discrete dynamical system obtained via the Euler method. This fact can be expressed as follows.

THEOREM 2.1. *Let $\mathcal{S}^2$ be the unit sphere in $\mathbb{R}^3$ and let*

$$\phi : \mathcal{S}^2 \times \mathbb{R} \to \mathcal{S}^2$$

be the dynamical system generated by a differential gradient equation

$$\dot{x} = -\operatorname{grad} E(x), \tag{1}$$

where $E \in \mathcal{C}^2(\mathcal{S}^2, \mathbb{R})$, having a finite number of singularities, all hyperbolic. Let, furthermore, the dynamical system ϕ have no saddle-saddle connections. Moreover, let $\phi_h : \mathcal{S}^2 \to \mathcal{S}^2$ be the discretization of ϕ, i.e. $\phi_h(x) := \phi(x,h)$, and let $\psi_h : \mathcal{S}^2 \to \mathcal{S}^2$ be generated by the Euler method for (1). *Then, for sufficiently small $h > 0$, there exists a homeomorphism $\alpha = \alpha_h : \mathcal{S}^2 \to \mathcal{S}^2$ globally conjugating the cascades generated by ϕ_h and ψ_h, i.e.*

$$\phi_h \circ \alpha = \alpha \circ \psi_h. \tag{2}$$

REMARKS. 1. Axiom A and the strong transversality condition are known to be equivalent to the structural stability of a dynamical system (see [PM], p. 171, and [Man]). On the other hand, for gradient dynamical systems, Axiom A implies that the system has only a finite number of singularities, all hyperbolic, whereas the strong transversality condition implies that the system has no saddle-saddle connections. Thus, the structural stability of a dynamical system $(\mathcal{S}^2, \phi)$ implies the assumptions of Theorem 2.1. Moreover, the set of structurally stable systems is open and dense in the space of gradient dynamical systems (see [PM], p. 116).

2. A dynamical system generated by the equation (1), having only a finite number of singularities, all hyperbolic, without saddle-saddle connections is called a *gradient Morse–Smale system*.

3. Estimation of the Euler method on $\mathcal{S}^n$. Let $n_0 \in \mathbb{N}$ and let $a = n_0 h$ denote the length of the time interval on which the error $e_n := \varrho_{\mathcal{M}}(\phi_h^{n_0}(x), \psi_h^{n_0}(x))$ is estimated ($\varrho_{\mathcal{M}}$ denotes the Riemannian metric on $\mathcal{M}$). We will show that on the n-dimensional sphere

$$e_n < \xi(a)h, \tag{3}$$

where, for a given problem, $\xi(a)$ is a constant value which only depends on a.

Consider the problem

$$\dot{x} = f(x), \quad x(0) = x_0, \quad 0 \le t \le a, \tag{4}$$

on a compact manifold $\mathcal{M}$, where f is a vector field on $\mathcal{M}$. The Euler iterative rule for the problem (4) is of the form

$$x_n = \exp_{x_n}(-hf(x_{n-1})), \tag{5}$$

where $f(x_{n-1})$ is a vector of the tangent space $T_{x_{n-1}}\mathcal{M}$.

The Euler method in $\mathbb{R}^n$ is a first order numerical method ([Kru], p. 31). For $x, y \in \mathcal{M}$ in the domain of a chart ϑ of a manifold $\mathcal{M}$, we have

$$m_1\varrho(\vartheta(x), \vartheta(y)) \le \varrho_{\mathcal{M}}(x, y) \le m_2\varrho(\vartheta(x), \vartheta(y)), \tag{6}$$

where m_1, m_2 are constant for a given chart ϑ (see [Rob], p. 453, formula (2.2)); ϱ is the euclidian metric on $\mathbb{R}^n$. By (6), the Euler method on a compact manifold is also a first order method.

The error in a single step of a numerical method is defined by

$$r(x, h) := \varrho_{\mathcal{M}}(\psi_h(x), \phi_h(x))$$

and is a continuous function of x for a constant h. Therefore, on a compact manifold, it reaches its maximum. Set

$$r(h) := \max_{x\in\mathcal{M}} r(x, h).$$

Let us estimate the error of the Euler method for a gradient equation on the sphere $\mathcal{S}^n$ in $\mathbb{R}^{n+1}$. By the assumptions of Theorem 2.1 a gradient dynamical system on a compact manifold has at least one attracting singularity. Change coordinates in $\mathbb{R}^{n+1}$ so that this fixed point is the south pole of the sphere. The sphere can be covered by two charts ϑ_1, ϑ_2 with

$$\vartheta_1 : \mathcal{S}^n \setminus p_{\text{north}} \to \mathbb{R}^n, \qquad \vartheta_2 : \mathcal{S}^n \setminus p_{\text{south}} \to \mathbb{R}^n, \tag{7}$$

where $p_{\text{north}}, p_{\text{south}}$ are the poles of the sphere.

If the south pole is the starting point of the Euler method then

$$\varrho_{\mathcal{M}}(\phi_h^n(x), \psi_h^n(x)) = 0$$

for each $n \in \mathbb{N}$ as the gradient is zero at the south pole. In this case the error is zero.

In the other case, if the starting point x is different from p_{south} then $\phi_h^n(x) \ne p_{\text{south}}$ for each $n \in \mathbb{N}$, because ψ_h is invertible for h sufficiently small. Thus, for each $n \le n_0$, both $\phi_h^n(x)$ and $\psi_h^n(x)$ lie in the domain of the chart ϑ_2. Therefore, the point x and the systems ϕ_h and ψ_h can be transformed into $\mathbb{R}^n$ in order to perform the iterations $\widetilde{\psi}_h^{n_0}(\vartheta_2(x))$ and $\widetilde{\phi}_h^{n_0}(\vartheta_2(x))$. Afterwards, we return to the sphere via ϑ_2^{-1}. The dynamical systems $\widetilde{\psi}_h$ and $\widetilde{\phi}_h$ are the systems ψ_h and ϕ_h transformed from the sphere to $\mathbb{R}^n$ by the maps of the atlas. Notice that if ψ_h is generated by the Euler method of the gradient equation generating ϕ_h on the sphere then $\widetilde{\psi}_h$ is generated by the Euler method of the equation generating $\widetilde{\phi}_h$ in $\mathbb{R}^n$ because the atlas preserves the differential structure.

The error estimate with step h for the equation

$$\dot{x} = f(x),$$

where f is lipschitzian with a constant L in $\mathbb{R}^n$, is given by the formula (see [Kru], p. 32, formula 1.22)

$$e_n \leq \left(e_0 + \frac{rh}{L}\right)e^{La}, \tag{8}$$

where $a = n_0h$ is the time interval on which the solution is considered, rh^2 is the maximal error in a single step and e_0 is the initial error. Since the right side of (1), being a differentiable map on a compact space, is a lipschitzian map, so is the right side of the equation transformed to $\mathbb{R}^n$, by (6). Therefore, the estimate (8) can be applied to the systems $\widetilde{\psi}_h$ and $\widetilde{\phi}_h$. The flow and its Euler method start from the same point, thus $e_0 = 0$ and

$$\varrho(\widetilde{\psi}_h^{n_0}(\vartheta_2(x)), \widetilde{\phi}_h^{n_0}(\vartheta_2(x))) \leq \frac{rh}{L}e^{La}.$$

By (6) we get the following estimate on $\mathcal{S}^n$:

$$\varrho_{\mathcal{M}}(\psi_h^{n_0}(x), \phi_h^{n_0}(x)) \leq m_2\frac{rh}{L}e^{La}, \tag{9}$$

which is of the same form as in $\mathbb{R}^n$. Thus we have obtained (3).

4. Lemmas. One of the key points in the proof of Theorem 2.1 is a construction of proper homeomorphisms conjugating the cascades ϕ_h and ψ_h in a neighbourhood of an attracting singularity (see Lemma 4.6). This construction is based on the following geometric lemma.

LEMMA 4.1. *Let γ_1, γ_2, δ_1, δ_2 be curves in $\mathbb{R}^2$, each parametrized by $\tau \in [0,1]$, homeomorphic to a line segment and such that $\gamma_1(0) = \delta_1(0)$, $\gamma_1(1) = \delta_2(0)$, $\gamma_2(0) = \delta_1(1)$ and $\gamma_2(1) = \delta_2(1)$. Assume that their union is the boundary of a simply connected domain $\mathcal{D}$. Then there exists a homeomorphism $\Lambda : \mathcal{D} \to [0,1]^2$ such that $\Lambda(\gamma_1) = [0,1] \times \{1\}$, $\Lambda(\gamma_2) = [0,1] \times \{0\}$, $\Lambda(\delta_1) = \{1\} \times [0,1]$, $\Lambda(\delta_2) = \{0\} \times [0,1]$.*

The proof of this well known fact is omitted. It can be found in [Bie].

A great number of theorems concerning the topological conjugacy near hyperbolic singularities are known. We will need the following theorem (see [Bey], [Gar2], [Gar3]).

THEOREM 4.2. *For each equilibrium point x_0 of the cascade ϕ_h and for sufficiently small h there exists an equilibrium point x_h of the cascade ψ_h, a neighbourhood V_{x_0} of x_0, and a homeomorphism $\alpha_h : V_{x_0} \to \alpha_h(V_{x_0})$ such that $\alpha_h(x_0) = x_h$ and $(\alpha_h \circ \phi_h)(x) = (\psi_h \circ \alpha_h)(x)$, whenever x and $\phi_h(x)$ are in V_{x_0}.*

In our case, since x_h is an equilibrium point of ψ_h iff $\operatorname{grad} E(x_h) = 0$, we have $x_h = x_0$ for every $h > 0$.

Throughout this section $\mathcal{M}$ denotes a compact differentiable manifold of dimension greater than one, ϕ denotes a Morse–Smale gradient system on $\mathcal{M}$, and ϕ_h and ψ_h are the cascades defined as in Theorem 2.1. Let, furthermore,

$$g_{c_i} : V_{c_i} \to V^*_{c_i}$$

be a homeomorphism locally conjugating ϕ_h and ψ_h in the neighbourhood V_{c_i} of the singularity c_i and $V^*_{c_i} = g(V_{c_i})$.

LEMMA 4.3. *Let c_i be an attracting or saddle singularity. Then there exists $\overline{\overline{r}}_i > E(c_i)$ such that for every $x \neq c_i$ in $W^{\mathrm{s}}_\phi(c_i) \cap V_{c_i}$ the following implication holds: if $E(x) \le \overline{\overline{r}}_i$ then there exists $t > 0$ such that*

$$E(\phi(x,-t)) > \overline{\overline{r}}_i \quad \textit{and} \quad \phi(x,-t) \in W^{\mathrm{s}}_\phi(c_i) \cap V_i.$$

Furthermore the sets

$$K_{c_i,0} := \{x \in W^{\mathrm{s}}_\phi(c_i) : E(x) \le \overline{\overline{r}}_i\},$$

$$K_{c_i,1} := \{x \in W^{\mathrm{s}}_\phi(c_i) : E(x) \le \overline{r}_i\}, \qquad \textit{where } E(c_i) < \overline{r}_i < \overline{\overline{r}}_i,$$

are nonempty.

Proof. The point c_i is attracting on the stable manifold $W^{\mathrm{s}}_\phi(c_i)$. Choose

$$R < \min\{\varrho_{\mathcal{M}}(c_i, c_j) : i \neq j,\ c_j \text{ is a fixed point}\}.$$

Let $B_{\mathrm{rel}}(c_i, R)$ be the closed ball and $S_{\mathrm{rel}}(c_i, R)$ the sphere in $W^{\mathrm{s}}_\phi(c_i)$. By compactness there exists $x_0 \in S_{\mathrm{rel}}(c_i, R)$ such that

$$E(x_0) = \inf\{E(x) : x \in S_{\mathrm{rel}}(c_i, R)\} =: e_i.$$

The following proposition is necessary to complete the proof of Lemma 4.3. Its proof follows simply from the Morse lemma and therefore is omitted (it can be found in [Bie]).

PROPOSITION 4.4. *Let $e_i > r^* > E(c_i)$. Then the set*

$$B_{\mathrm{lev}}(c_i, r^*) := \{x \in W^{\mathrm{s}}_{\phi,c_i} : E(x) \le r^*\}$$

and the ball $B_{\mathrm{rel}}(c_i, R)$ are homeomorphic.

Since the potential E decreases along each orbit, every $\overline{\overline{r}}_i < r^*$ (where r^* is defined in Corollary 4.4) satisfies the implication in Lemma 4.3. That implies that the sets $K_{c_i,m}$, $m = 0, 1$, are nonempty. ∎

LEMMA 4.5. *The sets $K_{c_i,m}$, $m = 1, 2$, have the following properties*:

(i) $c_i \in \operatorname{int}_{\mathrm{rel}} K_{c_i,m}$,

(ii) $\partial_{\mathrm{rel}} K_{c_i,0} = \{x \in W^{\mathrm{s}}_{\phi,c_i} : E(x) = \overline{\overline{r}}\}$ *and* $\partial_{\mathrm{rel}} K_{c_i,1} = \{x \in W^{\mathrm{s}}_{\phi,c_i} : E(x) = \overline{r}\}$,

(iii) $\{\phi(x,t) : t > 0\} \subset K_{c_i,m}$ *for every* $x \in K_{c_i,m}$,

(iv) $K_{c_i,m}$ *is arcwise connected*,
(v) $\inf\{\varrho_{\mathcal{M}}(x,y) : x \in K_{c_i,1},\ y \in K_{c_i,0}\} > 0$.

The boundaries and interiors are considered in the relative topology of $W^{\mathrm{s}}_{\phi,c_i}$ *and therefore are marked by the subscript "*rel*".*

Proof. (i) This is clear since $K_{c_i,m}$ is homeomorphic to a ball $B_{\mathrm{rel}}(c_i,r)$.

(ii) Let $x_0 \in \partial_{\mathrm{rel}} K_{c_i,0}$. Then, for every neighbourhood V_{x_0} of x_0,

$$V_{x_0} \cap \{x \in W^{\mathrm{s}}_{\phi,c_i} : E(x) \le \overline{\overline{r}}\} \ne \emptyset \quad \text{and} \quad V_{x_0} \cap \{x \in W^{\mathrm{s}}_{\phi,c_i} : E(x) > \overline{\overline{r}}\} \ne \emptyset.$$

Since the map E is continuous, $E(x_0)$ is equal to $\overline{\overline{r}}$.

Let $x_0 \in W^{\mathrm{s}}_{\phi,c_i}$. The definition of $K_{c_i,m}$ implies that $x_0 \in K_{c_i,0}$. Assume, by contradiction, that $x_0 \in \mathrm{int}_{\mathrm{rel}}\, K_{c_i,0}$. Then there exists $t_0 > 0$ such that $\phi(x_0,-t) \in K_{c_i,0}$ for every $t \le t_0$. However, for those t,

$$E(\phi(x_0,-t)) > E(\phi(x_0,0)) = E(x_0) = \overline{\overline{r}}.$$

This means that $\phi(x_0,t) \notin K_{c_i,0}$, a contradiction.

The property of $K_{c_i,1}$ can be proved in the same way.

(iii) Let $x_0 \in \partial_{\mathrm{rel}} K_{c_i,0}$. Since the potential decreases along orbits, $E(\phi(x_0,t)) < E(x_0)$ for every $t > 0$. Hence $\phi(x_0,t) \in K_{c_i,0}$ by the definition of $K_{c_i,m}$.

If $x_0 \in \mathrm{int}_{\mathrm{rel}}\, K_{c_i,0}$, then $E(x_0) < \overline{\overline{r}}$. Since the set $K_{c_i,0} = B_{\mathrm{lev}}(c_i,\overline{\overline{r}})$ is homeomorphic to a ball $B_{\mathrm{rel}}(c_i,R)$ such that $K_{c_i,0} \subset B_{\mathrm{rel}}(c_i,R)$ (see Corollary 4.4), the set $\{\phi(x_0,t) : t > 0\}$ would intersect the boundary $\partial_{\mathrm{rel}} K_{c_i,0}$ if the semiorbit $\phi(x_0,t), t > 0$ were not included in $K_{c_i,0}$. However this is impossible because the potential on the boundary of the set $K_{c_i,0}$ is greater than in the interior.

The property of $K_{c_i,1}$ can be shown in the same way.

(iv) Define $\tau : \overline{\mathbb{R}} \ni t \mapsto \tau(t) \in [-1,1]$ by

$$\tau(t) = \begin{cases} -1 & \text{if } t = -\infty, \\ t/(1+|t|) & \text{if } t \in \mathbb{R}, \\ 1 & \text{if } t = \infty. \end{cases}$$

Let $x_1, x_2 \in K_{c_i,m}$, where m equals 1 or 2. Then the formulas

$$\gamma\big(\tfrac{1}{2}\tau(t)\big) = \phi(x_1,t) \quad \text{and} \quad \gamma\big(1 - \tfrac{1}{2}\tau(t)\big) = \phi(x_2,t)$$

define an arc from x_1 to x_2; the arc is included in $K_{c_i,m}$ by Lemma 4.5(ii).

(v) The assertion follows from the compactness of $\mathcal{M}$. Assume, by contradiction, that $\inf\{\varrho_{\mathcal{M}}(x,y) : x \in K_{c_i,1},\ y \in K_{c_i,0}\} = 0$. Then there exists a sequence $\{x_n\} \subset K_{c_i,1}$ which converges to a point $x_0 \in \partial_{\mathrm{rel}} K_{c_i,0}$. But $E(x_0) = \overline{\overline{r}}$ and $E(x_n) \le \overline{r}$, which is a contradiction because $\overline{r} < \overline{\overline{r}}$ and E is continuous. ∎

The following lemma is necessary for the construction of a global homeomorphism conjugating ϕ_h and ψ_h.

LEMMA 4.6. *Let a be an attracting fixed point of the flow ϕ on a two-dimensional manifold $\mathcal{M}$. Let, furthermore, $s_k,\ k \in \{1,\ldots,K\}$, be saddle points whose stable manifolds intersect the stable manifold of a. Then there exists a neighbourhood V_a of a and a homeomorphism g_a defined on this neighbourhood locally conjugating the flows $(\mathcal{M},\phi_h)$ and $(\mathcal{M},\psi_h)$ such that for every $k \in \{1,\ldots,K\}$,*

$$g_a(W^{\mathrm{u}}_{\phi_h}(s_k) \cap V_a) \subset W^{\mathrm{u}}_{\psi_h}(s_k). \tag{10}$$

Proof. We first record some facts and introduce a few definitions. As $\mathcal{M}$ is two-dimensional, the components of the unstable manifolds $W^{\mathrm{u}}_{\phi_h}(s_k)$ of the saddle points are curves with a common end at a. Choose a level set of E on which $E > E(a)$ and such that its connected component is included both in $W^{\mathrm{s}}_{\phi_h}(a_i)$ and in a sufficiently small neighbourhood of a. Denote this component by F_1. Since near an attracting singularity a level is homeomorphic to a circle (see Corollary 4.4), this level can be parametrized as $F_1 = F_1(\tau)$ with $F_1(0) = F_1(1),\ \tau \in [0,1]$, where the map F_1 is continuous. The set $W^{\mathrm{u}}_{\phi}(s_k)\setminus\{s_k,a\}$ has two connected components, each being an orbit of ϕ. As already mentioned, the curve F_1 is homeomorphic to a circle and the value of the potential decreases along an orbit. Thus, every connected component of $\bigcup_{k=1}^{K}(W^{\mathrm{u}}_{\phi_h}(s_k) \setminus \{s_k,a\})$ intersects F_1 in exactly one point. Let there be M such components and let they be numbered from 0 to $M-1$ according to the increasing τ, say $W^{\mathrm{u}}_{\phi_h,m},\ m = 0,\ldots,M-1$.

Set $F_2 := \phi_h(F_1)$. The curve F_2 does not intersect F_1 and is homeomorphic to F_1. The point a lies in the interior of the domain bounded by F_2. Therefore the set

$$P_{F_1,F_2} := \{\phi(x,t) : x \in F_1,\ t \in [0,h]\}$$

is homeomorphic to a closed annulus $B(a,R) \setminus \operatorname{int} B(a,r)$, where $r < R$. Define

$$\begin{gathered}\delta_m := W^{\mathrm{u}}_{\phi_h,m} \cap P_{F_1,F_2}, \qquad m = 0,\ldots,M-1, \\ \Lambda : [0,1]\times[0,h] \ni \{\tau,t\} \mapsto \Lambda(\tau,t) = \phi(F_1(\tau),t) \in P_{F_1,F_2}.\end{gathered} \tag{11}$$

The mapping Λ is a homeomorphism because it is a superposition of homeomorphisms and there exists a finite sequence $\tau_0 = 0, \tau_1, \ldots, \tau_{M-1}$ such that $\Lambda(\tau_m,[0,h]) = \delta_m$.

Now, we begin the proof of Lemma 4.6.

STEP 1: *Construction on fragments of $W^{\mathrm{u}}_{\phi_h}(s_k)$*. Let g_{s_k} denote a homeomorphism locally conjugating ϕ_h and ψ_h near s_k and let ϕ_{-h} be the inverse of ϕ_h. The formula

$$g_{s_k,a}(x) := (\psi_h^{n_0(x)} \circ g_{s_k} \circ \phi_{-h}^{n_0(x)})(x) \tag{12}$$

defines a homeomorphism conjugating ϕ_h and ψ_h on δ_m. The natural number

$n_0(x)$ is chosen in such a way that $\phi_{-h}^{n_0(x)}(x) \in K_{s_k,1}$ and $\phi_{-h}^{n_0(x)-1}(x) \in \mathrm{int}_{\mathrm{rel}}(K_{s_k,0} \backslash K_{s_k,1})$. Note that the parameter m uniquely identifies the saddle point s_k.

The mapping

$$g_a := g_{s_k,a}, \quad x \in \delta_m,$$

defines a homeomorphism conjugating ϕ_h and ψ_h on $\bigcup_{m=0}^{M-1} \delta_m$. The mappings ψ_h, g_{s_k} and g_a are continuous in h and each is the identity for $h = 0$.

Step 2: *Construction on a fragment of the annulus.* Set

$\gamma_{1,m} := \{x = F_1(\tau) : \tau \in [\tau_m, \tau_{m+1}]\}$, $\gamma_{2,m} := \{x = F_2(\tau) : \tau \in [\tau_m, \tau_{m+1}]\}$,

where τ_m is the value of τ for which

$$F_j(\tau_m) = F_j \cap W^{\mathrm{u}}_{\phi_h,m} =: x_j(m), \quad j \in \{1, 2\}.$$

Note that $\gamma_{j,m}$, $j = 1, 2$, $m = 0, \ldots, M-1$, is a curve with end points $x_j(m)$, $x_j(m+1)$ whereas δ_m is a curve with end points $x_1(m)$, $x_2(m)$. All these curves are homeomorphic to line segments. This implies that $\gamma_{1,m} \cup \gamma_{2,m} \cup \delta_m \cup \delta_{m+1}$ bounds a simply connected domain in $\mathbb{R}^2$. By the Riemann Theorem this domain is homeomorphic to a ball and hence to a rectangle. By Lemma 4.1 there exists a homeomorphism

$$\Lambda_m := \Lambda|_{[\tau_m,\tau_{m+1}]\times[0,h]}$$

from $[\tau_m, \tau_{m+1}] \times [0, h]$ onto the closed domain $\mathrm{Sq}_{\gamma_{1,m},\gamma_{2,m},\delta_m,\delta_{m+1}}$ bounded by the curves $\gamma_{1,m}$, $\gamma_{2,m}, \delta_m$ and δ_{m+1}. The mapping Λ has the following properties:

$$\Lambda_m(\tau_m, [0, h]) = \delta_m,$$
$$\Lambda_m([\tau_m, \tau_{m+1}], 0) = \gamma_{1,m},$$
$$\Lambda_m([\tau_m, \tau_{m+1}], h) = \gamma_{2,m}.$$

Perform a similar construction for the cascade ψ_h. Set

$$x_j^*(m) := g_a(x_j(m)).$$

Let a curve $F_1^* := g_a(F_1)$ parametrized by τ be homeomorphic to a circle and such that a is in the interior of the simply connected domain bounded by F_1^*. Let, furthermore, the points $x_1^*(m)$, $m \in \{0, \ldots, M-1\}$, lie in F_1^* and let the parametrization have the property

$$F_1^*(\tau_m) = x_1^*(m), \quad m = 0, \ldots, M-1.$$

Let $F_2^* := \psi_h(F_1^*)$ and suppose $\psi_h(F_1^*(\tau)) = F_2^*(\tau)$ for all τ. Set

$$\delta_m^* := g_a(\delta_m),$$

and let δ_m^* be parametrized in such a way that

$$\delta_m^*(h_0) := g_a(\delta_m(h_0)), \quad h_0 \in [0, h].$$

The homeomorphism g_a is a continuous function of h and is the identity for $h = 0$, so for sufficiently small h, the curve δ^*_m lies near δ_m and does not intersect any other $\delta_{m'}$, $m' \neq m$. Thus, if m increases then so does the value of τ which parametrizes F^*_1. Therefore, F^*_1 can be reparametrized in such a way that the values of τ at $x_j(m)$ equal those at $x^*_j(m)$, $m = 0, \ldots, M-1$. Define

$$\gamma^*_{1,m} := \{x \in F^*_1(\tau) : \tau \in [\tau_m, \tau_{m+1}]\},$$
$$\gamma^*_{2,m} := \{x \in F^*_2(\tau) : \tau \in [\tau_m, \tau_{m+1}]\}.$$

Arguing as for ϕ_h, we can show that the domain bounded by F^*_1 and F^*_2 is homeomorphic to an annulus whereas the domain $\mathrm{Sq}_{\gamma^*_{1,m},\gamma^*_{2,m},\delta^*_m,\delta^*_{m+1}}$ bounded by $\gamma^*_{1,m}$, $\gamma^*_{2,m}$, δ^*_m and δ^*_{m+1} is homeomorphic to a rectangle. By Lemma 4.1 the homeomorphism can be chosen to map the vertices of the rectangle to the points $x^*_1(m)$, $x^*_2(m)$, $x^*_1(m+1)$ and $x^*_2(m+1)$. Denote it by

$$\xi_m : \mathrm{Sq}_{\gamma^*_{1,m},\gamma^*_{2,m},\delta^*_m,\delta^*_{m+1}} \to [\tau_m, \tau_{m+1}] \times [0, h].$$

It has the following properties:

$$\xi_m(\gamma^*_{1,m}) = [\tau_m, \tau_{m+1}] \times \{0\},$$
$$\xi_m(\gamma^*_{2,m}) = [\tau_m, \tau_{m+1}] \times \{h\},$$
$$\xi_m(\delta^*_m) = \{\tau_m\} \times [0, h].$$

Furthermore, ξ_m can be constructed in such a way that on the curves δ^*_m it is consistent with the mapping $g_{m,a}$ (the parameter m determines the saddle point s_k uniquely):

$$g_{m,a}(\phi(x_m, t)) = \xi_m(\tau_m, t), \qquad t \in [0, h], \tag{13}$$

and

$$g_{m+1,a}(\phi(x_m + 1, t)) = \xi_m(\tau_{m+1}, t), \qquad t \in [0, h]. \tag{14}$$

This can be shown in the following way. Equations (13) and (14) imply that on the vertical sides $\tau_m \times [0, h]$ and $\tau_{m+1} \times [0, h]$ the parametrization is settled by increasing homeomorphisms $f_1 : [0, h] \to [0, h]$ and $f_2 : [0, h] \to [0, h]$. We will show that the mapping

$$g : [\tau_m, \tau_{m+1}] \times [0, h] \ni \{\tau, t\} \mapsto g(\tau, t) = (g_1(\tau, t), g_2(\tau, t)) \in [\tau_m, \tau_{m+1}] \times [0, h]$$

defined by

$$g_1(\tau, t) = \tau, \qquad g_2(\tau, t) = \frac{\tau_{m+1} - \tau}{\tau_{m+1} - \tau_m} \cdot f_1(t) + \frac{\tau - \tau_m}{\tau_{m+1} - \tau_m} \cdot f_2(t)$$

is a homeomorphism with the required properties. We will prove that it transforms the rectangle $[\tau_m, \tau_{m+1}] \times [0, h]$ onto itself. Fix $\tau_0 \in [\tau_m, \tau_{m+1}]$

and $t_0 \in [0,h]$. Without losing generality we can assume that $f_1(t_0) \le f_2(t_0)$. Then

$$g_2(\tau_0,t_0) \ge \frac{\tau_{m+1}-\tau_0}{\tau_{m+1}-\tau_m}\cdot f_1(t_0) + \frac{\tau_0-\tau_m}{\tau_{m+1}-\tau_m}\cdot f_1(t_0) = f_1(t_0) \in [0,h]$$

and

$$g_2(\tau_0,t_0) \le \frac{\tau_{m+1}-\tau_0}{\tau_{m+1}-\tau_m}\cdot f_2(t_0) + \frac{\tau_0-\tau_m}{\tau_{m+1}-\tau_m}\cdot f_2(t_0) = f_2(t_0) \in [0,h].$$

Furthermore, $g_2(\tau_m,\cdot) = f_1(\cdot)$ and $g_2(\tau_{m+1},\cdot) = f_2(\cdot)$. The jacobian

$$\operatorname{jac} g = \begin{pmatrix} \partial g_1/\partial\tau & \partial g_1/\partial t \\ \partial g_2/\partial\tau & \partial g_2/\partial t \end{pmatrix} = \frac{\tau_{m+1}-\tau}{\tau_{m+1}-\tau_m}\cdot\frac{df_1(t)}{dt} + \frac{\tau-\tau_m}{\tau_{m+1}-\tau_m}\cdot\frac{df_2(t)}{dt}$$

is positive at each point because both components are nonnegative (f_1 and f_2 are increasing) and do not equal zero simultaneously.

The superposition

$$\Upsilon_m = \Lambda_m \circ g \circ \xi_m$$

is a homeomorphism transforming the closure of $\mathrm{Sq}_{\gamma^*_{1,m},\gamma^*_{2,m},\delta^*_m,\delta^*_{m+1}}$ onto the closure of $\mathrm{Sq}_{\gamma_{1,m},\gamma_{2,m},\delta_m,\delta_{m+1}}$ in such a way that for each $x \in \delta_m \cup \delta_{m+1}$,

$$\Upsilon_m(x) = g_{m,a}(x). \tag{15}$$

Step 3: *Construction on the annulus.* By (13) and (14), $\Upsilon_m(x)=\Upsilon_{m+1}(x)$ for each $x \in \delta_{m+1}$ and $\Upsilon_0(x) = \Upsilon_{M-1}(x)$ for $x \in \delta_0$. Therefore the mapping defined as

$$\Upsilon(x) = \Upsilon_m(x) \quad \text{for } x \in \mathrm{Sq}_{\gamma^*_{1,m},\gamma^*_{2,m},\delta^*_m,\delta^*_{m+1}} \text{ and } m \in \{0,1,\ldots,M-1\}$$

is a homeomorphism transforming the "annulus" $P_{F^*_1,F^*_2}$ onto the "annulus" P_{F_1,F_2}.

Step 4: *Construction on the neighbourhood of a.* Extend Υ to the whole neighbourhood of a. Let $y \in P_{F^*_1,F^*_2}$, and define $x \in V_a$ by

$$x = \psi_h^{k(x)}(y), \qquad k \in \mathbb{N}.$$

Set

$$\widetilde{\Upsilon}_a(x) := \begin{cases} (\psi_h^{-k(x)} \circ \Upsilon \circ \phi_h^{k(x)})(x) & \\ \qquad\qquad \text{if } \exists y \in P_{F^*_1,F^*_2} : x = \psi_h^{k(x)}(y),\ k \in \mathbb{N}, \\ a \qquad\qquad \text{for } x = a. \end{cases} \tag{16}$$

For points of the curve F^*_2 the mapping $\widetilde{\Upsilon}_a$ is defined in two ways. First, $F^*_2 \subset P_{F^*_1,F^*_2}$ so we can take *zero* as the value of k. On the other hand we can take $k = 1$ because every point of F^*_2 is the image of a point of F^*_1. However, both ϕ_h and ψ_h preserve values of τ so for each $x \in F^*_2$ both the ways give the same image. This also implies that $\widetilde{\Upsilon}_a$ is continuous on F^*_2.

To prove the continuity of $\widetilde{\Upsilon}_a$ at a consider a sequence $\{x_n\}$ converging to a. For each n there exists a natural number k_n such that $\psi_h^{-k_n}(x_n) \in P_{F_1^*,F_2^*}$. Furthermore, $k_n \to \infty$. Thus $\widetilde{\Upsilon}_a(x_n) \to a$. Hence, the function

$$\widetilde{\Upsilon}_a : P_{F_1^*,F_2^*} \to P_{F_1,F_2}$$

is a homeomorphism.

Since on the sets $\delta_m,\ m = 0, \ldots, M-1$ (see (11)) the mapping Υ is defined by the local conjugating homeomorphism g_{s_k} (see (12) and (15)) which transforms $W^{\mathrm{u}}_{\phi_h}(s_k)$ onto $W^{\mathrm{u}}_{\psi_h}(s_k)$, the inclusion (10) is satisfied. ■

Introduce the following notations:

$\mathcal{D}_{\phi_h}$ — the set of all repelling points,

$\mathcal{S}_{\phi_h}$ — the set of all saddle points and the attracting points which are not contained in the closure of the unstable manifold of any saddle point,

$\mathcal{P}_{\phi_h}$ — the set of attracting points which are contained in the closure of the unstable manifold of a saddle point.

Define

$$W^{\mathrm{s}}_{\phi_h}(\mathcal{S}_{\phi_h}) := \bigcup_{c\in\mathcal{S}_{\phi_h}} W^{\mathrm{s}}_{\phi_h}(c), \qquad W^{\mathrm{s}}_{\phi_h}(\mathcal{P}_{\phi_h}) := \bigcup_{c\in\mathcal{P}_{\phi_h}} W^{\mathrm{s}}_{\phi_h}(c),$$

and $\Theta : W^{\mathrm{s}}_{\phi_h}(\mathcal{S}_{\phi_h}) \to \mathcal{M}$ by

$$(17) \qquad \Theta(x) = \begin{cases} (\psi^{n_0}_{-h} \circ g_c \circ \phi^{n_0}_h)(x) & \text{for } x \in W^{\mathrm{s}}_{\phi_h}(c) \setminus \{c\}, \\ x & \text{for } x = c, \end{cases}$$

where $c \in \mathcal{S}_{\phi_h},\ g_c$ is a local homeomorphism conjugating the flows ϕ_h and ψ_h on a neighbourhood of c and ψ_{-h} is the inverse of ψ_h. If x is in one of the sets $K_{c,1}$, then the natural number $n_0 = n_0(x)$ is zero. In the other case it is chosen in such a way that $\phi_h^{n_0}(x) \in K_{c,1}$ and $\phi_h^{n_0-1}(x) \in \mathrm{int}_{\mathrm{rel}}(K_{c,0} \setminus K_{c,1})$.

Define $\alpha : \mathcal{M} \to \mathcal{M}$ as follows:

$$(18) \qquad \alpha(x) = \begin{cases} \Theta(x) & \text{for } x \in W^{\mathrm{s}}_{\phi_h}(\mathcal{S}_{\phi_h}), \\ \widetilde{\Upsilon}_a(x) & \text{for } x \in W^{\mathrm{s}}_{\phi_h}(\mathcal{P}_{\phi_h}),\ a \in \mathcal{P}_{\phi_h}, \\ x & \text{for } x \in \mathcal{D}_{\phi_h}. \end{cases}$$

Every nonrepelling point on a compact manifold is contained in a stable manifold of an attracting or saddle singularity. Thus (18) defines α on the whole $\mathcal{M}$. In the next section it is shown that α is a homeomorphism globally conjugating the flows $(\mathcal{M}, \phi_h)$ and $(\mathcal{M}, \psi_h)$.

LEMMA 4.7. *For all positive constants ε, there exists a positive constant h_0 such that for all $x \in \mathcal{M},\ t \in \mathbb{R}$ and $0 < h < h_0$,*

$$\varrho_{\mathcal{M}}(\phi(x,t), \phi(x,t+h)) < \varepsilon.$$

The proof of this simple lemma is omitted (it can be found in [Bie]).

LEMMA 4.8. *Let a be an attracting fixed point of the flow ϕ, and p_1, p_2 repelling fixed points, $p_1 \neq p_2$. Assume that $p_1, p_2 \in \operatorname{cl} W^{\mathrm{s}}_{\phi}(a)$. Then there exist saddle points q_1, q_2, not necessarily different, such that $p_1 \in \operatorname{cl} W^{\mathrm{s}}_{\phi}(q_1)$ and $p_2 \in \operatorname{cl} W^{\mathrm{s}}_{\phi}(q_2)$.*

Proof. Since $p_1, p_2 \in \partial W^{\mathrm{s}}_{\phi}(a)$, there exist y_1 and y_2 such that

$$\lim_{t\to\infty} \phi(y_1, t) = a, \qquad \lim_{t\to-\infty} \phi(y_1, t) = p_1,$$
$$\lim_{t\to\infty} \phi(y_2, t) = a, \qquad \lim_{t\to-\infty} \phi(y_2, t) = p_2.$$

Let $z_1 \in \operatorname{orb}(y_1) \cap V_a$ and $z_2 \in \operatorname{orb}(y_2) \cap V_a$, where $\operatorname{orb}(y)$ denotes the orbit of y, and let $V_a \subset W^{\mathrm{s}}_{\phi,a}$ be a neighbourhood of a. The manifold $\mathcal{M}$ is locally arcwise connected, so let

$$l : [0,1] \ni \tau \mapsto l(\tau) \in X$$

be a closed arc included in V_a, avoiding a and such that $l(0) = z_1$, $l(1) = z_2$. Since z_1 and z_2 lie in the unstable manifolds of p_1 and p_2 respectively, the boundaries of the unstable manifolds intersect l in points w_1 and w_2, not necessarily different. As the unstable manifolds of repelling points are open and $\mathcal{M}$ is compact, the points w_1, w_2 are contained in the unstable manifolds of saddle points q_1, q_2, not necessarily different. From the λ-lemma it follows that for every neighbourhood V_{q_i} of q_i there exists $t > 0$ such that

$$\phi(l \cap W^{\mathrm{u}}_{\phi}(p_1), -t) \cap V_{q_1} \neq 0.$$

We have

$$(W^{\mathrm{s}}_{\phi}(q_1) \setminus \{q_1\}) \cap V_{q_1} \neq 0.$$

Thus, for every $\varepsilon > 0$ there exists $t > 0$ such that $\phi(l \cap W^{\mathrm{u}}_{\phi}(p_1), -t)$ intersects the ε-envelope of $W^{\mathrm{s}}_{\phi}(q_1)$. This implies that there exists $u \in W^{\mathrm{s}}_{\phi}(q_1)$ such that

$$\lim_{t\to-\infty} \phi(u, t) = p_1.$$

Therefore, the point $p_1 \in \operatorname{cl} W^{\mathrm{u}}_{\phi}(q_1)$. The same can be said about the points q_2 and p_2. ■

COROLLARY 4.9. *Let p be a repelling point which is not contained in the closure of the stable manifold of any saddle point. Then there exists only one attracting point a such that $p \in \operatorname{cl} W^{\mathrm{s}}_{\phi}(a)$.*

This follows easily from Lemma 4.8.

COROLLARY 4.10. *Let $\mathcal{M} = \mathcal{S}^n$, $n > 1$. Then, for every $h > 0$, the cascade $(\mathcal{S}^n, \phi_h)$ has no saddle-saddle connections and, for sufficiently small h, the cascade $(\mathcal{S}^n, \psi_h)$ has no such connections either.*

Proof. By the assumptions of Theorem 2.1, the flow $(\mathcal{S}^n, \phi)$ has no saddle-saddle connections. Thus each saddle point q_i, $i = 1, \ldots, I$, has a neighbourhood U_{q_i} such that every point $x_0 \in U_{q_i}$ lies in $W^{\mathrm{s}}_{\phi}(q_i)$ or in $W^{\mathrm{s}}_{\phi}(a_j)$ for an attracting point a_j. For every $x \in \mathcal{S}^n$ and $h > 0$,

$$\lim_{t\to\infty} \phi(x,t) = \lim_{n\to\infty} \phi_h^n(x),$$

hence $x_0 \in W^{\mathrm{s}}_{\phi_h}(q_i)$ or $x_0 \in W^{\mathrm{s}}_{\phi_h}(a_j)$.

Define

$$P_i := \{x \in U_{q_i} : \forall_{n\in\{1,2,\ldots\}}\ \psi_h^n(x) \notin U_{q_i}\}$$

and let $U_{q_i} \subset \operatorname{int} V_{q_i}$, where V_{q_i} is a neighbourhood on which the cascades ϕ_h and ψ_h are locally conjugate. Choose real numbers R and h_i such that $B(q_i,R) \subset \operatorname{int} U_{q_i}$ and $P_i \cap B(q_i,R/2) \neq \emptyset$ for every $h \in [0,h_i]$.

Step 1: We will show that $P_i \cap W^{\mathrm{s}}_{\phi}(q_i) = \emptyset$ for sufficiently small h_i. Define

$$t_i := \sup_{x\in U_{q_i}\cap W^{\mathrm{s}}_{\phi}(q_i)} \{t : \phi(x,t) \in B(q_i,R/4) \setminus B(q_i,R/8)\}.$$

It is obvious that $t_i \in (0,\infty)$. Suppose h_i is such that there exists a natural number n_1 with $t_i = n_1 h_i$ and

$$\xi h_i < R/4,$$

where ξ is the constant from (3). Since there are only a finite number of singularities, we can take ξ as the maximum value of the constants of all saddle points.

Let $x \in U_{q_i} \cap W^{\mathrm{s}}_{\phi}(q_i)$. Then

$$\phi_{h_i}^{n_1}(x) \in B(q_i,R/4)$$

and by (3) and the choice of h_i,

$$\varrho_{\mathcal{M}}(\phi_{h_i}^{n_1}(x), \psi_{h_i}^{n_1}(x)) < R/4,$$

or equivalently

$$\psi_{h_i}^{n_1}(x) \in B(q_i,R/2).$$

Since $P_i \cap B(q_i,R/2) = \emptyset$, the point x is not in P_i.

Step 2: *Proof of lemma.* Let $x \in P_i$. The first step implies that $x \notin W^{\mathrm{s}}_{\phi}(q_i)$. Since there are no saddle-saddle connections, the point x is in $W^{\mathrm{s}}_{\phi}(a_j)$ for an attracting point a_j. Let V_{a_j} be the neighbourhood of a_j on which ϕ_h and ψ_h are topologically conjugate. Decompose the set P_i into disjoint components in the following way:

$$P_{ij} := \{x \in P_i : x \in W^{\mathrm{s}}_{\phi}(a_j)\}.$$

Let, furthermore,

$$t_{ij} := \sup\{t : \phi(x,t) \in B(a_j,r/2) \setminus B(a_j,r/4),\ x \in P_{ij}\},$$

where r is chosen in such a way that $B(a_j,r) \subset V_{a_j}$. It is obvious that $t_{ij} \in (0,\infty)$. Denote by $\overline{e}_n$ the error after the nth step of the Euler method.

Choose h_{ij} so small that ϕ_h and ψ_h are topologically conjugate on V_{a_j} and $\xi h_{ij} < r/2$. Moreover, let $t_{ij} = n_2 h_{ij}$, $n_2 \in \{1,2,\ldots\}$. Then $\phi(x,t_{ij}) \in B(a_j, r/2)$ for every $x \in P_{ij}$. By (3) and the choice of h_{ij},

$$\overline{e}_n := \varrho_{\mathcal{M}}(\phi^{n_2}_{h_{ij}}(x), \psi^{n_2}_{h_{ij}}(x)) < r/2.$$

Hence

$$\psi^{n_2}_{h_{ij}}(x) \in B(a_j, r).$$

Assuming that there are I saddle points and J attracting points, set

$$h_0 := \min_{i\in\{1,\ldots,I\},\, j\in\{1,\ldots,J\}} \{h_{ij}\}. \tag{19}$$

As $V_{a_j} \subset W^{\mathrm{s}}_{\phi_{h_0}}(a_j)$ and the cascades are locally topologically conjugate, we have $V_{a_j} \subset W^{\mathrm{s}}_{\psi_h}(a_j)$ for every $h \in [0, h_0]$. Hence

$$\psi^{n_2}_h(x) \in W^{\mathrm{s}}_{\psi_h}(a_j).$$

Therefore, every $x \in U_{q_i}$ is either in $W^{\mathrm{s}}_{\psi_h}(q_i)$ or in $W^{\mathrm{s}}_{\psi_h}(a_j)$, where a_j is an attracting point. This implies that ψ_h has no saddle-saddle connections. ∎

Remark. By Lemma 4.10 and Theorem 4.2, if $(\mathcal{S}^n, \phi)$ is a gradient Morse–Smale system, then so are the cascade $(\mathcal{S}^n, \phi_h)$ (for every positive h) and $(\mathcal{S}^n, \psi_h)$ (for sufficiently small h).

Corollary 4.11. *Let U_{q_i} be a neighbourhood of saddle point q_i on which the cascades ϕ_h and ψ_h are conjugate by a homeomorphism $g_{q_i} : U_{q_i} \to g_{q_i}(U_{q_i})$. Then there exists a constant $h_0 > 0$ such that for every $h \in (0, h_0)$ and $x \in U_{q_i} \cap W^{\mathrm{u}}_{\phi_h}(q_i)$ if $x \in W^{\mathrm{s}}_{\phi_h}(a_j)$, where a_j is an attracting fixed point, then $g_i(x) \in W^{\mathrm{s}}_{\psi_h}(a_j)$.*

Proof. Step 1. Define

$$H_{ij} := W^{\mathrm{u}}_{\phi_h}(q_i) \cap (B(q_i, r_i) \setminus \operatorname{int} B(q_i, r_i/2)) \cap W^{\mathrm{s}}_{\phi_h}(a_j). \tag{20}$$

First, it will be shown that there exists $h_0 > 0$ so small that for every $h \in (0, h_0)$ and every $x \in H_{ij}$ we have $g_i(x) \in W^{\mathrm{s}}_{\psi_h}(a_j)$. The radius r_i is chosen such that $B(q_i, r_i) \subset U_{q_i}$.

The definition (20) implies that H_{ij} is closed in $\mathcal{M}$, hence compact. The stable manifold $W^{\mathrm{s}}_{\phi_h}(a_j)$ is open and $H_{ij} \subset W^{\mathrm{s}}_{\phi_h}(a_j)$. Thus, for every $x \in H_{ij}$ there exists $r_x > 0$ such that $B(x, r_x) \subset W^{\mathrm{s}}_{\phi_h}(a_j)$. The set

$$\mathcal{K} := \{B(x, r_x) \cap W^{\mathrm{u}}_{\phi_h}(q_i)\}$$

is a covering of H_{ij}. Thus, we can choose a finite subcovering $\mathcal{K}^*$. Let r_{ij} be the smallest radius of the balls $B(x, r_x)$ used to construct $\mathcal{K}^*$. Take $h_{0ij} > 0$ such that $g_{q_i}(x) \in B(x, r_{ij})$ for every point $x \in H_{ij}$. Then $g_{q_i}(x) \in W^{\mathrm{s}}_{\phi_h}(a_j)$. It can be shown (in the same way as in the second step of the proof of Corollary 4.10) that there exists h_{1ij} such that $g(H_{ij}) \subset W^{\mathrm{s}}_{\psi_h}(a_j)$ for every $h \in (0, h_{1ij})$. Set $h_{ij} := \min\{h_{0ij}, h_{1ij}\}$.

STEP 2. Suppose, by contradiction, that there exists $y \in W^{\mathrm{u}}_{\phi_h}(q_i) \cap \operatorname{int} B(q_i, r_i/2)$ such that $y_1 \in W^{\mathrm{s}}_{\phi_h}(a_j)$ and $z_1 := g_i(y_1) \in W^{\mathrm{s}}_{\phi_h}(a_k)$, where a_k is an attracting fixed point and $k \neq j$. Corollary 4.7 implies that, for sufficiently small h, there exists a natural m_0 such that

$$\phi_h^{m_0}(y_1) \in B(q_i, r_i) \setminus \operatorname{int} B(q_i, r_i/2).$$

As $y_1 \in W^{\mathrm{s}}_{\phi_h}(a_j)$, the point $y_2 := \phi_h^{m_0}(y_1)$ is in $W^{\mathrm{s}}_{\phi_h}(a_j)$. Therefore, by Step 1, $z_2 := g_i(y_2) \in W^{\mathrm{s}}_{\psi_h}(a_j)$ as $y_2 \in P_{ij}$. However

$$\begin{aligned} z_2 := g(y_2) &= g(\phi_h^{m_0}(y_1)) = (g \circ \phi_h)(\phi_h^{m_0-1}(y_1)) \\ &= (\psi_h \circ g \circ \phi_h)(\phi_h^{m_0-2}(y_1)) \\ &= (\psi_h \circ \psi_h \circ g)(\phi_h^{m_0-2}(y_1)) = \ldots = \psi_h^{m_0}(g(y_1)) = \psi_h^{m_0}(z_1), \end{aligned}$$

and $\psi_h^{m_0}(z_1) \in W^{\mathrm{s}}_{\psi_h}(a_k)$ as $z_1 \in W^{\mathrm{s}}_{\psi_h}(a_k)$, which leads to a contradiction. ∎

5. Proof of Theorem. Firstly, we prove that the map α defined in (18) is a bijection. Then we prove that it is continuous on $\mathcal{M}$. This implies that α is a homeomorphism. The map α conjugates the cascades ϕ_h and ψ_h, which follows directly from its definition (see (16)–(18)).

Injectivity. Let $x_1, x_2 \in \mathcal{S}^2$. If at least one of them is a repelling singularity of the system ϕ, then $\alpha(x_1) \neq \alpha(x_2)$ by the definition of α. Otherwise the following two cases have to be considered:

CASE 1: *The points lie in different orbits.* Then

$$y_1 := \phi_h^{n_0(x_1)}(x_1) \neq \phi_h^{n_0(x_2)}(x_2) =: y_2.$$

The system ϕ has only a finite number of singularities and every singularity c_i is a fixed point of g_i. The sphere $\mathcal{S}^2$ is compact so every point is contained in the stable manifold of a stable point: $x_1 \in W^{\mathrm{s}}_{\phi}(c_i)$, $x_2 \in W^{\mathrm{s}}_{\phi}(c_j)$.

If $i = j$, then $z_1 := g_i(y_1) \neq g_i(y_2) = z_2$ as g_i is a bijection. Since g_i is a local conjugating homeomorphism, the images of different orbits do not intersect. Thus $\psi_{-h}^{n_0(x_1)}(z_1) \neq \psi_{-h}^{n_0(x_2)}(z_2)$, which implies that $\alpha(x_1) \neq \alpha(x_2)$.

If $i \neq j$, then we can choose the domains V_i of the homeomorphisms g_i in such a way that $V_i \cap V_j = \emptyset$ and $g_i(V_i) \cap g_j(V_j) = \emptyset$ for $i \neq j$. Thus $z_1 := g_i(y_1) \neq g_j(y_2) =: z_2$. As $x_1 \in W^{\mathrm{s}}_{\phi_h, a_i}$ and $x_2 \in W^{\mathrm{s}}_{\phi_h, a_j}$, the points z_1 and z_2 lie in the disjoint manifolds $W^{\mathrm{s}}_{\psi_h, a_i}$ and $W^{\mathrm{s}}_{\psi_h, a_j}$ respectively. This implies that $\alpha(x_1) \neq \alpha(x_2)$.

CASE 2: *The points lie in the same orbit.* We can assume that $E(x_2) > E(x_1)$. Then $x_1 = \phi_h^m(x_2)$ for some positive natural number $m = m(x_1, x_2)$.

Thus

$$\alpha(x_2) = (\psi_{-h}^{n_0(x_1)+m} \circ g_{c_i} \circ \phi_h^{n_0(x_1)+m})(x_2)$$
$$= (\psi_{-h}^{m} \circ \psi_{-h}^{n_0(x_1)} \circ g_{c_i} \circ \phi_h^{n_0(x_1)})(x_1) = \psi_{-h}^{m}(\alpha(x_1)) \neq \alpha(x_1).$$

Surjectivity. If $y \in \mathcal{S}^2$ is a repelling fixed point, then $y = \alpha(y)$ (see definition of α). Otherwise $y \in W^{s}_{\psi_h, c_i}$ for some fixed point c_i (attracting or saddle). Let $n_0 = n_0(y)$ be such that

$$y_{n_0-1} := \psi_h^{n_0-1}(y) \in \operatorname*{int}_{\mathrm{rel}}(g(K_{c_i,0}) \setminus g(K_{c_i,1})) \quad \text{and} \quad y_{n_0} := \psi_h^{n_0}(y) \in g(K_{c_i,0}).$$

As g is a conjugating homeomorphism,

$$(\phi_h^{-1} \circ g^{-1} \circ \psi_h)(y_{n_0-1}) = g^{-1}(y_{n_0-1}).$$

Furthermore, since $\psi_h(y_{n_0-1}) = y_{n_0}$, we have

$$(\phi_h^{-1} \circ g^{-1})(y_{n_0}) = g^{-1}(y_{n_0-1}).$$

Set

$$x_{n_0} := g^{-1}(y_{n_0}) \quad \text{and} \quad x_{n_0-1} := g^{-1}(y_{n_0-1}) = \phi_h^{-1}(x_{n_0}).$$

Since g_{c_i} is a homeomorphism, $x_{n_0-1} \in (\mathrm{int}_{\mathrm{rel}}\, K_{c_i,0} \setminus K_{c_i,1})$ and $x_{n_0} \in K_{c_i,1}$. This means that for $x^* = \phi_{-h}^{n_0}(x_{n_0})$ the natural number n_0 is the same as in the definition of α, which implies that $y = (\psi_h^{-n_0} \circ g \circ \phi_h^{n_0})(x^*) = \alpha(x^*)$.

Continuity

Case 1: *Continuity on the stable manifold of an attracting point.* Let x lie on the stable manifold of an attracting point a_i. There exists $n_0 = n_0(x)$ such that

$$\phi_h^{n_0}(x) \in K_{a_i,1} \quad \text{and} \quad \phi_h^{n_0-1}(x) \in \mathrm{int}(K_{a_i,0} \setminus K_{a_i,1}).$$

If $\phi_h^{n_0}(x) \in \mathrm{int}\, K_{a_i,1}$, then as $\mathrm{int}\, K_{a_i,1}$ and $\mathrm{int}(K_{a_i,0} \setminus K_{a_i,1})$ are open and the map $\phi(\cdot, t)$ is continuous, there exists a neighbourhood U_x of x such that

$$\phi_h^{n_0}(U_x) \subset \mathrm{int}\, K_{a_i,1} \quad \text{and} \quad \phi_h^{n_0-1}(U_x) \subset \mathrm{int}(K_{a_i,o} \setminus K_{a_i,1}).$$

As g_i, $\psi_{-h}(\cdot, t)$ and $\phi_h(\cdot, t)$ are continuous, the map $\alpha = \psi_{-h}^{n_0} \circ g_i \circ \phi_h^{n_0}$ is continuous at x.

If $y := \phi_h^{n_0}(x) \in \partial K_{a_i,1}$, then every neighbourhood U_y of y intersects $\mathrm{int}(K_{a_i,0} \setminus K_{a_i,1})$. Let $(y_n)_{n=1}^{\infty} \subset \mathrm{int}(K_{a_i,0} \setminus K_{a_i,1})$ converge to y. Then there exists N such that for every natural n greater than N, $\phi_h(y_n) \in \mathrm{int}\, K_{a_i,1}$. Indeed, suppose otherwise. Then there exists a subsequence (y_{n_k}) such that $\phi_h((y_{n_k})_{k=1}^{\infty}) \not\subset \mathrm{int}\, K_{a_i,1}$. However, $\phi_h(y) \in \mathrm{int}\, K_{a_i,1}$ because the potential E is constant on $\partial K_{a_i,1}$ and decreases along a trajectory. This means that for every k, $\varrho_{\mathcal{M}}(\phi_h(y), \phi_h(y_{n_k})) > \inf\{\varrho_{\mathcal{M}}(\phi_h(y), w) : w \in \partial K_{a_i,1})\} > 0$. This is a contradiction because ϕ is continuous.

We have demonstrated that for every $y \in \partial K_{a_i,1}$ there exists a neighbourhood $U_y^\star$ with $\phi_h(U_y^\star) \subset K_{a_i,1}$. The neighbourhood $U_y^\star$ has two disjoint components: the first one, $U_y^{\star 1}$, is included in $\operatorname{int} K_{a_1,1}$ whereas the second, $U_y^{\star 2}$, is not. The point x is transformed by the map α in such a way that $U_y^{\star 1}$ is transformed by g_i, and $U_y^{\star 2}$ by $\psi_{-h} \circ g_i \circ \phi_h$. Afterwards $U_y^{\star 1} \cup U_y^{\star 2}$ is transformed by the map $\psi_{-h}^{n_0}$. But g_i is a conjugating homeomorphism so

$$(\psi_{-h} \circ g_i \circ \phi_h)(U_y^{\star 2}) = (\psi_{-h} \circ \psi_h \circ g_i)(U_y^{\star 2}) = g_i(U_y^{\star 2})$$

and therefore we can say that the whole neighbourhood $U_y^\star$ is transformed by g_i. The continuity of ϕ_h, g_i and ψ_{-h} implies that α is continuous at x.

CASE 2: *Continuity at repelling fixed points.* There are two possibilities in this case: either

(i) the repelling point, say p, is in the closure of the stable manifold of a saddle point q, or

(ii) p is in the closures of the stable manifolds of attracting points.

Suppose first that (i) holds.

STEP 1: *Restriction to the stable manifold.* Let $\{x_k\}_{k\in\mathbb{N}} \subset W^{\mathrm{s}}_{\phi,q}$, $p \in \operatorname{cl} W^{\mathrm{s}}_\phi(q)$ and $\lim_{k\to\infty} x_k = p$.

Let V_q be a neighbourhood of q such that the homeomorphism $g_{h,q}$ conjugating ϕ_h and ψ_h is defined on V_q. For sufficiently small h and every natural N almost all elements of the sequence $\{x_k\}_{k\in\mathbb{N}}$ have the following property:

$$\phi_h^{n_k}(x_k) \in \operatorname{int} K_{q,1}, \qquad \phi_h^{n_k-1}(x_k) \in \operatorname{int}(K_{q,0} \setminus K_{q,1}), \qquad n_k > N,$$

which follows from Lemma 4.7 (for the definition of the sets $K_{q,0}$ and $K_{q,1}$, see Lemma 4.3). The same lemma also implies that there exists $r > 0$ such that $y_k = \phi(x_k, n_k h) \notin B(q,r) \cap W^{\mathrm{s}}_\phi(q) = K_{q,2}$ for all k. In other words, since the step on the manifold $\mathcal{M}$ is small, all the y_k lie near $\partial K_{q,1}$. Since q is a fixed point of $g_{h,q}$, $g_{h,q}(V_q)$ is a neighbourhood of q and $g_{h,q}(V_q) \cap W^{\mathrm{s}}_{\phi_h}(q) \subset W^{\mathrm{s}}_{\psi_h,q}$ as $g_{h,q}$ locally conjugates ϕ_h and ψ_h. Since p is also a repelling point of ψ_h, it lies in $W^{\mathrm{s}}_{\psi_h}(q)$ (by Corollary 4.11). As $g_{h,q}$ is a homeomorphism, $g_q(y_k) \notin g_{h,q}(K_{q,2})$ for every k. Let $W^{p,\mathrm{s}}_{\psi_h}(q)$ be the maximal connected component of $W^{\mathrm{s}}_{\psi_h}(q) \setminus \{q\}$ containing p. Then, for every $\varepsilon > 0$, there exists N such that

$$\psi_{-h}^n(W^{p,\mathrm{s}}_{\psi_h}(q) \setminus K_{q,2}) \subset B(p,\varepsilon) \quad \text{for all } n > N.$$

Since almost all n_k are greater than N and every y_k is in $W^{p,\mathrm{s}}_{\psi_h}(q) \setminus K_{q,2}$, almost all $\psi_{-h}^{n_k}(y_k)$ lie in $B(p,\varepsilon)$. This means that

$$\lim_{k\to\infty} \psi_{-h}^{n_k}(g_q(y_k)) = p.$$

Thus, we have shown that the map α restricted to $W^{\mathrm{s}}_{\phi_h}(q) \cup \{p\}$ is continuous at p.

STEP 2: *Continuity at the repelling point.* Let $\varepsilon > 0$. Since α is continuous on the stable manifold of every saddle point (see case 3), for every x^* in such a manifold and $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ such that if $x \in B(x^*, \delta_1)$ then $\alpha(x) \in B(x^*, \varepsilon_1)$. On the other hand, p is a fixed point of α, and $\alpha | W^{\mathrm{s}}_{\phi}(q)$ is continuous at p (see step 1). Thus, for every $\varepsilon_2 > 0$ there exists $\delta_2 > 0$ such that if $x^* \in B(p, \delta_2)$, then $\alpha(x^*) \in B(p, \varepsilon_2)$. Let ε_1 and ε_2 be such that $\varepsilon_1 + \varepsilon_2 = \varepsilon$. Then the triangle inequality implies that if $x \in B(p, \delta_1 + \delta_2)$ then $\alpha(x) \in B(p, \varepsilon_1 + \varepsilon_2)$.

Suppose now that (ii) holds. Corollary 4.9 implies that there exists only one attracting singularity a such that $p \in W^{\mathrm{s}}_{\phi}(a)$, whereas Lemma 4.8 implies that only one repelling point can lie in $\operatorname{cl} W^{\mathrm{s}}_{\phi}(a)$. This means that there exists a neighbourhood V_p of p such that

$$\lim_{t \to \infty} \phi(x, t) = a$$

for all $x \in V_p \setminus \{p\}$ and there exists a neighbourhood V_a of a such that

$$\lim_{t \to \infty} \psi(y, -t) = p \quad \text{for all } y \in V_a \setminus \{a\}.$$

We can now repeat the first step of subcase (i) to show that the map α is continuous at p.

CASE 3: *Continuity on the stable manifold of a saddle point.* Let q_i be a saddle point. If the restriction $\phi_h | W^{\mathrm{s}}_{\phi_h}(q_i)$ is considered, then q_i is an attracting singularity and, repeating the argument from case 1, we have continuity of α in the relative topology on the stable manifold.

Thus, let $x_0 \in W^{\mathrm{s}}_{\phi_h}(q_i)$ and $x \in U_{x_0} \setminus W^{\mathrm{s}}_{\phi_h}(q_i)$, where U_{x_0} is a neighbourhood of x_0. Since there are no saddle-saddle connections, there exist attracting points a_j, a_k, not necessarily different, such that $W^{\mathrm{u}}_{\phi_h}(q_i)_1 \subset W^{\mathrm{s}}_{\phi_h}(a_j)$ and $W^{\mathrm{u}}_{\phi_h}(q_i)_2 \subset W^{\mathrm{s}}_{\phi_h}(a_k)$, where $W^{\mathrm{u}}_{\phi_h}(q_i)_1$ and $W^{\mathrm{u}}_{\phi_h}(q_i)_2$ are the connected components of the manifold $W^{\mathrm{u}}_{\phi_h}(q_i)$.

For every $x_0 \in W^{\mathrm{s}}_{\phi}(q_i)$, each neighbourhood V_{q_i} of q_i and sufficiently small $h > 0$, there exists $\delta_1 > 0$ and n_1 such that

$$\phi_h^{n_1}(B(x, \delta_1)) \subset V_{q_i}.$$

Suppose $V_{q_i} \cap W^{\mathrm{s}}_{\phi}(q_i) = \operatorname{int} K_{q_1,1}$, and h and n_1 are such that

$$\phi_h^{n_1 - 1}(B(x_0, \delta_1) \cap W^{\mathrm{s}}_{\phi}(q_i)) \subset \operatorname{int}(K_{q_i,0} \setminus K_{q_i,1}).$$

Thus $x^{n_1} := \phi_h^{n_1}(x)$ lies in the local stable manifold $W^{\mathrm{s}}_{\phi,\mathrm{loc}}(q_i)$. Denote by D^{u}_{ϕ} a disc transversal to $W^{\mathrm{s}}_{\phi,\mathrm{loc}}(q_i)$, containing x^{n_0}, $x^{n_1} := \phi_h^{n_1(x)}$ and embedded in V_{q_i}. According to the λ-lemma, for every $\delta_2 > 0$ and sufficiently small h, there exists n_2 such that

$$D^{\mathrm{u}}_{n_2} := \phi_h^{n_2}(D^{\mathrm{u}}_{\phi})$$

is δ_2-close to $W^{\mathrm{u}}_{\phi,loc}(q_i)$. Thus, for every $y \in D^{\mathrm{u}}_{n_2}$ there exists a point $\overline{y}$ in one of the connected components of $W^{\mathrm{u}}_{\phi}(q_i)$, for instance in $W^{\mathrm{u}}_{\phi}(q_i)_1 \subset W^{\mathrm{s}}_{\phi}(a_j)$, such that

$$\varrho_{\mathcal{M}}(y, \overline{y}) < \delta_2.$$

For every neighbourhood $U_{q_i} \subset V_{q_i}$ of q_i we can choose x so close to x_0 that $y := \phi_h^{n_1+n_2}(x) \in V_{q_i}$.

We will trace the behaviour of the point y lying near the $W^{\mathrm{u}}_{\phi}(q_i)$ using a "spying point" $\overline{y}$ whose behaviour is known because it lies in $W^{\mathrm{u}}_{\phi,\mathrm{loc}}(q_i)_1 \subset W^{\mathrm{s}}_{\phi}(a_j)$, where a_j is an attracting point. Since $\overline{y}$ is in the attracting basin of a_j, there exists a neighbourhood $V_{\overline{y}} \subset W^{\mathrm{s}}_{\phi}(a_j)$. For every $\delta_2 > 0$, by the λ-lemma and continuity of ϕ_h, we can choose x so close to x_0 that $y \in B(\overline{y}, \delta_2)$. According to Lemmas 4.7 and 4.5 we can choose n_3 such that

$$\phi_h^{n_3}(\overline{y}) =: \overline{y^{n_3}} \in \operatorname{int} K_{a_j,1}, \qquad \phi_h^{n_3-1}(\overline{y}) \in \operatorname{int}(K_{a_j,0} \setminus K_{a_j,1})$$

and

$$\phi_h^{n_3}(y) =: y^{n_3} \in \operatorname{int} K_{a_j,1}, \qquad \phi_h^{n_3-1}(y) \in \operatorname{int}(K_{a_j,0} \setminus K_{a_j,1}).$$

The map ϕ_h is continuous so, for every $\delta_3 > 0$, there exists $\delta_2 > 0$ such that if $\varrho_{\mathcal{M}}(\overline{y}, y) < \delta_2$, then $\varrho_{\mathcal{M}}(\overline{y^{n_3}}, y^{n_3}) < \delta_3$. Define

$$\overline{z^{n_3}} := (\psi_{-h}^{n_3} \circ g_{a_j})(\overline{y^{n_3}}), \qquad z^{n_3} := \psi_{-h}^{n_3}(g_{a_j}(y^{n_3})),$$

where g_{a_j} is a local conjugating homeomorphism constructed in Lemma 4.6. Then the definition of g_{q_i,a_j} (see formula (12)) implies that

$$g_{q_i}(\overline{y}) = \overline{z^{n_3}},$$

where g_{q_i} is a local homeomorphism conjugating ϕ_h and ψ_h in a neighbourhood of q_i. This follows from the definition since

$$\overline{z^{n_3}} := (\psi_{-h}^{n_3} \circ g_{q_i,a_j} \circ \phi_{-h}^{n_3})(\overline{y}) = (\psi_{-h}^{n_3} \circ (\psi_h^{n_3} \circ g_{q_i} \circ \phi_{-h}^{n_3}) \circ \phi_h^{n_3})(\overline{y}) = g_{q_i}(\overline{y}).$$

Thus, for every neighbourhood U_{q_i} of q_i, if $\varrho_{\mathcal{M}}(x, x_0)$ is sufficiently small, then the points y, $\overline{y}$ and $y_0 := g_{q_i}(\phi_h^{n_1+n_2}(x_0))$ all lie in U_{q_i}. The continuity of the map g_{q_i} and the equality $q_{q_i}(q_i) = q_i$ imply that for every $r > 0$ there exists a neighbourhood U_{q_i} such that $\overline{z^{n_3}}, g_{q_i}(y_0) \in B(q_i, r)$. Furthermore, z^{n_3} is also in this ball by the continuity of $\psi_{-h}^{n_3} \circ g_{q_i,a_j} \circ \phi_{-h}^{n_3}$ and the equality $g_{q_i}(\overline{y}) = \overline{z^{n_3}}$. The continuity of ψ_{-h} assures that for every $r_1 > 0$ there exists a radius r such that

$$\varrho_{\mathcal{M}}(\psi_{-h}^{n_2}(y), \psi_{-h}^{n_2}(y_0)) < r_1.$$

On the other hand $g_{q_i}(x_0^{n_1}) = \psi_{-h}^{n_2}(y_0)$ because

$$\begin{aligned}\psi_{-h}^{n_2}(y_0) &= (\psi_{-h}^{n_2} \circ g_{q_i} \circ \phi_h^{n_2})(x_0^{n_1}) = (\psi_{-h}^{n_2-1} \circ \psi_{-h} \circ g_{q_i} \circ \phi_h \circ \phi_h^{n_2-1})(x_0^{n_1}) \\ &= (\psi_{-h}^{n_2-1} \circ \psi_{-h} \circ \psi_h \circ g_{q_i} \circ \phi_h^{n_2-1})(x_0^{n_1}) \\ &= (\psi_{-h}^{n_2-1} \circ g_{q_i} \circ \phi_h^{n_2-1})(x_0^{n_1}) = \ldots = (\psi_{-h} \circ g_{q_i} \circ \phi_h)(x_0^{n_1}) \\ &= (\psi_{-h} \circ \psi_h \circ g_{q_i})(x_0^{n_1}) = g_{q_i}(x_0^{n_1}).\end{aligned}$$

Recapitulating, for every $r_1 > 0$ we can choose the radius r of the ball $B(q_i, r)$ such that

$$\varrho_{\mathcal{M}}(\psi_{-h}^{n_2}(y), g_{q_i}(x_0^{n_1})) < r_1.$$

The continuity of ψ_h implies that for every $\varepsilon > 0$ there exists r_1 such that

$$\varrho_{\mathcal{M}}(\psi_{-h}^{n_1}(\psi_{-h}^{n_2}(y)), \psi_h^{n_1}(g_{q_i}(x_0^{n_1}))) = \varrho_{\mathcal{M}}(\alpha(x), \alpha(x_0)) < \varepsilon.$$

This completes the proof. ∎

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