Continuous solutions of a polynomial-like iterative equation with variable coefficients

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Abstract. Using the fixed point theorems of Banach and Schauder we discuss the existence, uniqueness and stability of continuous solutions of a polynomial-like iterative equation with variable coefficients.

I. Introduction. Let $I = [a, b]$ be a given closed bounded interval. Given a continuous $F : I \rightarrow I$ such that $F(a) = a$ and $F(b) = b$, and given continuous functions $\lambda_1, \ldots, \lambda_n : I \rightarrow [0, 1]$ such that $\sum_{i=1}^{n} \lambda_i(x) = 1$ for all $x \in I$, we wish to find continuous functions $f : I \rightarrow I$ such that

$$\lambda_1(x)f(x) + \lambda_2(x)f^2(x) + \ldots + \lambda_n(x)f^n(x) = F(x) \quad \text{for all } x \in I. \quad (1)$$

Here $f^i$ denotes the $i$th iterate of $f$ (i.e., $f^0(x) = x$ and $f^{i+1}(x) = f(f^i(x))$ for all $x \in I$ and all $i = 0, 1, \ldots$). We suppose that $n \geq 2$.

The case in which the $\lambda_i$’s are constant was considered in [4]–[7] and [9]–[11] for special choices of $F$ and/or $n$. Similar equations are discussed on pages 237–240 of [5]. Such problems are related both to problems concerning iterative roots (see [1], [3] and [8]), e.g. finding a function $f$ such that

$$f^n(x) = F(x), \quad \forall x \in I,$$

and to the theory of invariant curves for mappings (see Chapter XI of [5]).

Note that we may assume without loss of generality that $a = 0$ and $b = 1$. Indeed, if $[a, b] \neq [0, 1]$ and (1) holds, define

$$h(t) = a + t(b - a) \quad \text{for } 0 \leq t \leq 1$$

and let

$$g = h^{-1} \circ f \circ h, \quad G = h^{-1} \circ F \circ h, \quad \mu_i = \lambda_i \circ h \quad \text{for } 1 \leq i \leq n.$$

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where $\circ$ denotes composition. Since $h$ and $h^{-1}$ are affine and $\sum_{i=1}^{n} \lambda_i(x) = 1$ for all $x \in I$, it follows that

$$\sum_{i=1}^{n} \mu_i(t)g^i(t) = G(t) \quad \text{for all } t \in [0,1].$$

Conversely, if (2) holds so does (1). Thus assume that $I = [0,1]$.

For economy of exposition we adopt the following notation. Let $C(I)$ denote the real Banach algebra consisting of all continuous maps of $I$ into $\mathbb{R}$ with respect to the uniform norm; for $f \in C(I)$, $\|f\| = \max\{|f(t)| : t \in I\}$.

Let $X = \{f \in C(I) : 0 = f(0) \leq f(t) \leq f(1) = 1 \text{ for all } t \in I\}$. Note that $X$ is closed under composition and hence under iteration. For $0 \leq m \leq 1 \leq M$ let

$$X(m,M) = \{f \in X : m(y-x) \leq f(y) - f(x) \leq M(y-x) \text{ whenever } 0 \leq x \leq y \leq 1\}.$$ 

II. Some lemmas

**Lemma 1.** Suppose $0 \leq m \leq 1 \leq M$. Then $X(m,M)$ is a compact convex subset of $C(I)$. Moreover, if $f,g \in X(m,M)$ then

$$\|f^{\nu} - g^{\nu}\| \leq \sum_{j=0}^{\nu-1} M^j \|f - g\| \quad \text{for all } \nu = 1,2,\ldots$$

**Proof.** It is clear that $X(m,M)$ is a closed, bounded and convex subset of $C(I)$. It is also clear that $X(m,M)$ is uniformly equicontinuous. Thus, by the Ascoli–Arzelà lemma, $X(m,M)$ is a compact convex subset of $C(I)$.

If $\nu = 1$ then the inequality is trivial. Suppose it holds when $1 \leq \nu \leq k$ for some $k \geq 1$. Then, for all $x \in I$,

$$|f^{k+1}(x) - g^{k+1}(x)| = |f(f^k(x)) - g(g^k(x))|$$

$$\leq |f(f^k(x)) - f(g^k(x))| + |g(g^k(x)) - g(g^k(x))|$$

$$\leq M\|f - g\| + \|f - g\|$$

$$\leq M \left( \sum_{j=0}^{k-1} M^j \right) \|f - g\| + \|f - g\|$$

$$= \left( \sum_{j=0}^{k} M^j \right) \|f - g\|.$$ 

Thus, by induction, the inequality is true for all $\nu \geq 1$. ■
Lemma 2. Suppose $0 < m \leq 1 \leq M$ and $f, g \in X(m, M)$. Then

(i) $f^{-1} \in X(M^{-1}, m^{-1})$,
(ii) $\|f - g\| \leq M\|f^{-1} - g^{-1}\|$, and
(iii) $\|f^{-1} - g^{-1}\| \leq m^{-1}\|f - g\|$.

Proof. Since $m > 0$, $f$ is a strictly increasing homeomorphism of $I$ onto itself and, for $0 \leq x < y \leq 1$,

$$M^{-1} \leq \frac{y' - x'}{f(y') - f(x')} \leq m^{-1}$$

where $y' = f^{-1}(y)$ and $x' = f^{-1}(x)$. Thus (i) holds.

To prove (ii) note that for all $x \in I$,

$$|f(x) - g(x)| = |f(x) - f((f^{-1} \circ g)(x))| \leq M|x - f^{-1}(g(x))|$$
$$= M|g^{-1}(g(x)) - f^{-1}(g(x))| \leq M\|g^{-1} - f^{-1}\|.$$

It follows that $\|f - g\| \leq M\|g^{-1} - f^{-1}\| = M\|f^{-1} - g^{-1}\|$.

Property (iii) follows easily from (i) and (ii).

These lemmas are essentially Lemmas 2.2 and 2.5 of [11]. Also note that, by (iii), the inversion map $I : X(m, M) \to X(M^{-1}, m^{-1})$ (defined by $If = f^{-1}$ for $f \in X(m, M)$) is a Lipschitz mapping.

Lemma 3. If $f \in X(m, M)$ and $g \in X(s, S)$ with $0 \leq m \leq 1 \leq M$ and $0 \leq s \leq 1 \leq S$, then $f \circ g \in X(ms, MS)$ and

$$f^k \in X(m^k, M^k) \quad \text{for all } k = 0, 1, \ldots$$

Proof. It suffices to note that, for $0 \leq x \leq y \leq 1$,

$$f(g(y)) - f(g(x)) \leq M(g(y)) - g(x)) \leq MS(y - x)$$

and, similarly,

$$f(g(y)) - f(g(x)) \geq ms(y - x).$$

III. Existence. Our main result is the following

Theorem 1. Suppose that $\lambda_1(x) \geq c$ for all $x \in I$ and

$$\text{Lip} \lambda_k := \sup \left\{ \frac{\lambda_k(y) - \lambda_k(x)}{y - x} : 0 \leq x < y \leq 1 \right\} \leq \beta \quad \text{for } k = 1, 2, \ldots$$

where $c$ and $\beta$ are real constants such that

$$0 < c < 1 \quad \text{and} \quad 0 \leq n\beta \leq 1.$$

Also suppose that $F \in X(\delta, M)$ with

$$n\beta \leq \delta \leq 1 \leq M.$$ 

Then (1) has a solution $f$ in $X(0, (M + n\beta)/c)$. 

Proof. Let $L = (M + n\beta)/c$ and note that $L > 1$ since $0 < c < 1 \leq M$. For $x \in I$ and $f \in X(0, L)$ define $f_x : I \to \mathbb{R}$ by

$$f_x(t) = \sum_{i=1}^{n} \lambda_i(x)f^{i-1}(t) \quad \text{for } t \in I.$$  

Our task is to prove that, for some $f \in X(0, L)$,

(1)' 

$$f_x(f(x)) = F(x) \quad \text{for all } x \in I.$$  

The idea behind our proof is based on the observation that if every $f_x$ were a bijection of $I$ then (1)' would be equivalent to

(1)'' 

$$f(x) = (f_x)^{-1}(F(x)) \quad \text{for all } x \in I;$$  

i.e. recasting the problem as a fixed point problem.

Suppose $f \in X(0, L)$ and $x \in I$. Then $f_x(0) = 0$, $f_x(1) = 1$, $f_x(t) \in I$ for all $t \in I$ and $f_x$ is continuous. Moreover, if $0 \leq t \leq u \leq 1$ then, by Lemma 3,

$$f_x(u) - f_x(t) = \sum_{i=1}^{n} \lambda_i(x)(f^{i-1}(u) - f^{i-1}(t))$$

$$\leq \sum_{i=1}^{n} \lambda_i(x)L^{i-1}(u - t) \leq \left(\sum_{i=1}^{n} L^{i-1}\right)(u - t)$$

and

$$f_x(u) - f_x(t) \geq \lambda_1(x)(u - t) \geq c(u - t).$$

Thus

(3) 

$$f_x \in X(c, C) \quad \text{for } x \in I \text{ and } f \in X(0, L)$$

where $C = \sum_{i=1}^{n} L^{i-1}$.

If $f \in X(0, L)$, $0 \leq x < y \leq 1$ and $t \in I$ then

$$|f_y(t) - f_x(t)| = \left|\sum_{i=1}^{n} (\lambda_i(y) - \lambda_i(x))f^{i-1}(t)\right| \leq n\beta(y - x).$$

Thus

(4) 

$$\|f_y - f_x\| \leq n\beta|y - x| \quad \text{for } f \in (0, L) \text{ and } x, y \in I.$$

Now suppose that $f \in X(0, L)$, $0 \leq x < y \leq 1$ and $t \in I$. By (3) and (4),

$$0 = t - t = f_y(f_y^{-1}(t)) - f_x(f_x^{-1}(t))$$

$$= f_y(f_y^{-1}(t)) - f_y(f_x^{-1}(t)) + f_y(f_x^{-1}(t)) - f_x(f_x^{-1}(t))$$

$$\geq c(f_y^{-1}(t) - f_x^{-1}(t)) - n\beta(y - x)$$

and, similarly,

$$0 \leq C(f_y^{-1}(t) - f_x^{-1}(t)) + n\beta(y - x)$$

so that

(5) 

$$-n\beta C^{-1} \leq (f_y^{-1}(t) - f_x^{-1}(t))/(y - x) \leq n\beta c^{-1}.$$
Thus, for $f \in X(0, L)$,
\[
\|f^{-1}_y - f^{-1}_x\| \leq n\beta c^{-1}|y - x| \quad \text{for all } x, y \in I
\]
since $0 < c < 1 < L < C$.

Now for $f \in X(0, L)$ define $Tf : I \to \mathbb{R}$ by
\[
Tf(x) = f^{-1}_x(F(x)) \quad \text{for } x \in I;
\]
notice that $Tf(0) = 0$, $Tf(1) = 1$ and $Tf(x) \in I$ for all $x \in I$.

Suppose that $f \in X(0, L)$ and $0 < x < y \leq 1$. By (5) and (i) of Lemma 2,
\[
Tf(y) - Tf(x) = f^{-1}_y(F(y)) - f^{-1}_x(F(x))
= f^{-1}_y(F(y)) - f^{-1}_x(F(y)) + f^{-1}_x(F(y)) - f^{-1}_x(F(x))
\leq n\beta c^{-1}(y - x) + c^{-1}(f(y) - F(x))
\leq (n\beta + M)c^{-1}(y - x) = L(y - x).
\]
Similarly,
\[
Tf(y) - Tf(x) = f^{-1}_y(F(y)) - f^{-1}_x(F(y)) + f^{-1}_x(F(y)) - f^{-1}_x(F(x))
\geq (-n\beta C^{-1})(y - x) + C^{-1}(f(y) - F(x))
\geq (-n\beta + \delta)C^{-1}(y - x) \geq 0
\]
since $n\beta \leq \delta \leq 1$. Thus $Tf \in X(0, L)$. We conclude that $T$ maps $X(0, L)$ into itself.

Aiming to prove that $T$ is continuous, suppose that $f, g \in X(0, L)$. By the lemmas, for any $x \in I$ we have
\[
|Tf(x) - Tg(x)| = |f^{-1}_x(F(x)) - g^{-1}_x(F(x))| \leq \|f^{-1}_x - g^{-1}_x\|
\leq c^{-1}\|f_x - g_x\| \leq c^{-1}\max_{t \in I} \sum_{i=2}^n \lambda_i(x)|f^{i-1}(t) - g^{i-1}(t)|
\leq c^{-1}\sum_{i=2}^n \lambda_i(x)|f^{i-1} - g^{i-1}|
\leq c^{-1}\sum_{i=2}^n \lambda_i(x)\left(\sum_{j=0}^{i-2} L^j\right)\|f - g\|
\leq c^{-1}\left(\sum_{j=0}^{n-2} L^j\right)\left(\sum_{i=2}^n \lambda_i(x)\right)\|f - g\|
= c^{-1}\left(\sum_{j=0}^{n-2} L^j\right)(1 - \lambda_1(x))\|f - g\|
\leq c^{-1}(1 - c)\left(\sum_{j=0}^{n-2} L^j\right)\|f - g|;
recall that $0 < c < 1$ and $c \leq \lambda_1(x)$ for all $x \in I$. We have proved that
\begin{equation}
\|Tf - Tg\| \leq \gamma \|f - g\| \quad \text{for all } f, g \in X(0, L)
\end{equation}
where
\begin{equation}
\gamma = c^{-1}(1 - c) \sum_{j=0}^{n-2} L^j.
\end{equation}
Thus $T$ is continuous. By Schauder’s fixed point theorem $T$ has a fixed point, i.e., $(1)''$ holds for some $f \in X(0, L)$. ■

IV. Uniqueness and stability. If $\gamma < 1$ then $T$ is a contraction, in which case Banach’s fixed point theorem implies that our problem has a unique solution.

**Theorem 2.** If, in addition to the assumptions of Theorem 1, $c$ is so close to 1 that
\[(1 - c) \sum_{j=1}^{n-1} (M + n\beta)^{j-1}/c^j < 1\]
then (1) has a unique solution $f$ in $X(0, (M + n\beta)/c)$.

**Proof.** It suffices to note (7) and (8) and recall that $L = (M + n\beta)/c$.

Under the assumptions of Theorem 2, the solution to our problem depends continuously upon the given data in the sense of

**Theorem 3.** In addition to the assumptions of Theorem 2, suppose that $\mu_1, \ldots, \mu_n : I \to I$ are continuous, $\sum_{i=1}^{n} \mu_i(x) = 1$ for all $x \in I$, $\mu_1(x) \geq c$ for all $x \in I$,
\[|\mu_k(y) - \mu_k(x)| \leq \beta |y - x| \quad \text{for } x, y \in I \text{ and } 1 \leq k \leq n\]
and $G \in X(\delta, M)$. Let $g$ be that member of $X(0, L)$ satisfying
\begin{equation}
\sum_{k=1}^{n} \mu_k(x)g^k(x) = G(x) \quad \text{for all } x \in I
\end{equation}
(whose existence and uniqueness is guaranteed by Theorem 2). Then
\begin{equation}
\|f - g\| \leq (1 - \gamma)^{-1} c^{-1} \left( \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + \|F - G\| \right).
\end{equation}

**Proof.** To indicate the dependence of the relevant operators on the given data, let us write $\lambda_x \varphi$ instead of $\varphi_x$ for $\varphi \in X(0, L)$ and write $T_\lambda$ instead of $T$. For $\varphi \in X(0, L)$ and $x \in I$ define $\mu_x \varphi(t) = \sum_{i=1}^{n} \mu_i(x)\varphi^{i-1}(t)$ for $t \in I$. For $\varphi \in X(0, L)$ let
\[T_\mu \varphi(x) = (\mu_x \varphi)^{-1}(G(x)) \quad \text{for } x \in I.\]
Suppose then that \( f, g \in X(0, L) \), (1) and (9) hold and \( x \in I \). Then

\[
|f(x) - g(x)| = |(\lambda_x f)^{-1}(F(x)) - (\mu_x g)^{-1}(G(x))|
\]

\[
\leq |(\lambda_x f)^{-1}(F(x)) - (\mu_x g)^{-1}(F(x))| + |(\mu_x g)^{-1}(F(x)) - (\mu_x g)^{-1}(G(x))|
\]

\[
\leq ||(\lambda_x f)^{-1} - (\mu_x g)^{-1}|| + c^{-1}|F(x) - G(x)|
\]

\[
\leq c^{-1}\|\lambda_x f - \mu_x g\| + \|F - G\|
\]

by Lemma 2 since \( \lambda_x f, \mu_x g \in X(c, C) \). By using Lemma 1 several times we find that, for all \( t \in I \),

\[
|\lambda_x f(t) - \mu_x g(t)| = \left| \sum_{i=1}^{n} \lambda_i(x) f_i^{-1}(t) - \mu_i(x) g_i^{-1}(t) \right|
\]

\[
\leq \sum_{i=1}^{n} |\lambda_i(x) - \mu_i(x)| |f_i^{-1}(t)| + \sum_{i=1}^{n} \mu_i(x) |f_i^{-1}(t) - g_i^{-1}(t)|
\]

\[
\leq \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + \sum_{i=2}^{n} \mu_i(x) \|f_i^{-1} - g_i^{-1}\|
\]

\[
\leq \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + \sum_{i=2}^{n} \mu_i(x) \left( \sum_{j=0}^{i-2} L^j \right) \|f - g\|
\]

\[
\leq \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + \sum_{i=2}^{n} \mu_i(x) \left( \sum_{j=0}^{n-2} L^j \right) \|f - g\|
\]

\[
= \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + (1 - \mu_1(x))c(1 - c)^{-1} \|f - g\|
\]

\[
\leq \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + (1 - c)\gamma c(1 - c)^{-1} \|f - g\|
\]

by the definition (8) of \( \gamma \). It follows that

\[
\|f - g\| \leq c^{-1}\left\{ \sum_{i=1}^{n} \|\lambda_i - \mu_i\| + \gamma c\|f - g\| + \|F - G\| \right\},
\]

i.e., (10) holds. ■

V. Remarks and questions. The normalization assumption that \( \sum_{i=1}^{n} \lambda_i(x) = 1 \) is not severe. Instead one could suppose that \( \lambda_i : I \to [0, \infty) \) is continuous for \( 1 \leq i \leq n \) and \( \sum_{i=1}^{n} \lambda_i(x) > 0 \) for all \( x \in I \). Then the equation can be normalized by dividing by \( \sum_{i=1}^{n} \lambda_i(x) \); of course, the assumptions on \( F \) would have to be altered appropriately.
We conclude the paper with some questions for possible future discussion.
1. How can (1) be treated without the assumption that $\lambda_1(x) \geq c > 0$ for all $x \in I$?
2. What more can be said in case the given functions $\lambda_1, \ldots, \lambda_n$ and $F$ are smooth?
3. What can be said in case $F(0) = 1$ and $F(1) = 0$?

References

[9] W. N. Zhang, Discussion on the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$, Chinese Sci. Bull. 32 (1987), 1444–1451.
[10] —, Stability of the solution of the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$, Acta Math. Sci. 8 (1988), 421–424.
[11] —, Discussion on the differentiable solutions of the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$, Nonlinear Anal. 15 (1990), 387–398.

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