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Applications of the Carathéodory theorem to PDEs

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Abstract. We discuss and exploit the Carathéodory theorem on existence and uniqueness of an absolutely continuous solution $x : \mathcal{J}\ (\subset \mathbb{R}) \to X$ of a general ODE $\dot{x} \overset{(\star)}{=} \mathcal{F}(t,x)$ for the right-hand side $\mathcal{F} : \operatorname{dom}\mathcal{F}\ (\subset \mathbb{R}\times X) \to X$ taking values in an arbitrary Banach space X, and a related result concerning an extension of x. We propose a definition of solvability of $(\star)$ admitting all connected $\mathcal{J}$ and unifying the cases "$\operatorname{dom}\mathcal{F}$ is open" and "$\operatorname{dom}\mathcal{F} = \mathcal{J}\times\Omega$ for some $\Omega \subset X$". We show how to use the theorems mentioned above to get approximate solutions of a nonlinear parabolic PDE and exact solutions of a linear evolution PDE with distribution data.

1. Introduction. We start with reviewing the original Carathéodory result in the simplest case of $X=\mathbb{R}$. Let $\mathcal{F}$ be a real function on the rectangle $[a,b]\times[c,d]\subset\mathbb{R}^2$. A real function x on an open interval $]\alpha,\beta[\subset[a,b]$ is said to be a *solution* of the ordinary differential equation (ODE)

$$x' = \mathcal{F}(t,x) \tag{1}$$

if x is absolutely continuous in $]\alpha,\beta[$ (i.e. it is absolutely continuous in any $[\alpha',\beta'] \subset\ ]\alpha,\beta[$) and $x'(t) = \mathcal{F}(t,x(t))$ for almost all $t \in\ ]\alpha,\beta[$.

We say that the right-hand side $\mathcal{F}$ *satisfies the Carathéodory conditions* (see Carathéodory [3], Opial [13], Pelczar–Szarski [14], Sansone [18]) if:

(i) for all $\mathsf{x} \in [c,d]$ the function $\mathcal{F}(\cdot,\mathsf{x}) : [a,b] \ni t \mapsto \mathcal{F}(t,\mathsf{x}) \in \mathbb{R}$ is measurable;

(ii) for all $t \in [a,b]$ the function $\mathcal{F}(t,\cdot) : [c,d] \ni \mathsf{x} \mapsto \mathcal{F}(t,\mathsf{x}) \in \mathbb{R}$ is continuous;

(iii) there exists a measurable function $M : [a,b] \to \mathbb{R}_+ := [0,\infty[$ such that $\int_a^b M(t)\,dt < \infty$ and

$$\forall (t,\mathsf{x}) \in [a,b]\times[c,d] : \quad |\mathcal{F}(t,\mathsf{x})| \le M(t).$$

These assumptions allow proving the existence of a solution of the Cauchy problem $\{(1),\ x(t_0) = \mathsf{x}_0\}$ for each $(t_0,\mathsf{x}_0) \in\ ]a,b[\ \times\ ]c,d[$ (see Carathéodory

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[3], Chap. XI, §582). Such a solution is given on a certain neighbourhood of t_0. In the fifties G. Aguaro [1] introduced the condition

(iv) for any continuous function $z : [a, b] \to [c, d]$ the function $[a, b] \ni t \mapsto \mathcal{F}(t, z(t)) \in \mathbb{R}$ is summable and the family of functions

$$\Big\{\mathsf{x} \mapsto \int_a^{\mathsf{x}} \mathcal{F}(s, z(s))\, ds \;\Big|\; z : [a, b] \to [c, d] \text{ is continuous}\Big\}$$

is absolutely equicontinuous.

He proved that under assumptions (i), (ii), (iv) the assertion of the Carathéodory existence theorem remains true. Next, Z. Opial [13] observed that the systems of conditions {(i), (ii), (iii)} and {(i), (ii), (iv)} are equivalent.

C. Carathéodory tackled the existence theorem, straightaway, in the general case of $X = \mathbb{R}^n$. Replacing $[a, b]$ and $[c, d]$ above by $]a, b[$ and $\mathbb{R}^n$, respectively, he proved that for any $(t_0, \mathsf{x}_0) \in \,]a, b[\, \times \mathbb{R}^n$ the Cauchy problem $\{(1),\ x(t_0) = \mathsf{x}_0\}$ has a solution defined on the whole $]a, b[$. He also proved ([3], Chap. XI, §583) that the generalized Lipschitz condition

$$|\mathcal{F}(t, \mathsf{x}_1) - \mathcal{F}(t, \mathsf{x}_2)| \le L(t)|\mathsf{x}_1 - \mathsf{x}_2| \tag{2}$$

for $(t, \mathsf{x}_j) \in \,]a, b[\, \times \mathbb{R}^n$ $(j = 1, 2)$ with some summable function L on $]a, b[$ is a sufficient condition for the solution to be unique.

From among a variety of later generalizations of the Carathéodory theorem we want to mention only two which are related to the version of the theorem on existence and uniqueness (Th. 2.6 below) used in this paper. P. S. Bondarenko [2] generalized the result of Carathéodory to the case of $\operatorname{dom} \mathcal{F}$ being a domain in $\mathbb{R} \times \mathbb{R}^n$. Next, W. N. Everitt and D. Race [5] showed that for linear ODEs the local summability of the coefficients is sufficient for local existence. They also discussed when this assumption is necessary, and mentioned a similar result for certain nonlinear equations.

In the present paper we consider absolutely continuous solutions of a general ODE taking values in an abstract Banach space. We propose a definition of solvability (Def. 2.2) which takes into account solutions defined in a both-sided or one-sided neighbourhood of an initial moment and, in particular, unifies two basic cases mentioned in the abstract. It enables us to solve (3) even if $\operatorname{dom} \mathcal{F}$ is neither open nor a Cartesian product (Ex. 2.4). The assumptions of Theorem 2.6 (on existence and uniqueness) resemble those made by Carathéodory and by Bondarenko. The well known Picard–Banach method of successive approximations, exploited originally for right-hand sides satisfying (2) in $\mathbb{R}^n$, works without essential modifications also in the case of $\dim X = \infty$; so, we confine ourselves to give a sketch of proof of Theorem 2.5. However, in our opinion the second fundamental result, Theorem 2.8 (on extension of solutions), should be proved most scrupulously.

Unlike other authors (for example Hartman [6]), we impose no assumptions ensuring uniqueness and local existence, but, simply, we assume that (3) is uniquely solvable. We do not exclude discontinuity of $\mathcal{F}$ and $\dot{x}$, so condition (ii) differs considerably from extension theorems for $\mathcal{C}^1$-solutions. From our experience, Theorem 2.8 is, despite an abstract undertone, very useful in applications.

In the literature concerning ODEs (in both classical and Carathéodory formulations) the dominating case is that of $\operatorname{dom}\mathcal{F}$ being open in $\mathbb{R}\times X$. Nevertheless, applications to PDEs seem to be more connected with the case of $\operatorname{dom}\mathcal{F} = \mathcal{J}\times\Omega$, especially for PDEs of parabolic type. In the present paper we try to redress the balance and emphasize the last case, both in examples (Sections 3, 4) and in the general theory (Sec. 2). That case distinguishes itself by an evident set-theoretic simplicity, so the versions 2.5 and 2.7 of the fundamental theorems have transparent statements and relatively simple proofs.

In Sections 3 and 4 we show how Theorems 2.5 and 2.7 work in the theory of partial differential equations (PDEs). We regard the examples considered there (important in PDE theory) as being *indispensable* to understand the essence of these theorems. The applications of ODEs in Sections 3 and 4 are not elementary; however, in comparison with examples which can be found in handbooks, they are an authentic origin of Theorems 2.6 and 2.8.

Specifically, in Section 3 we show how, in the case of an abstract nonlinear parabolic equation, the Faedo–Galerkin method works, and in Section 4 how to realize the old Fourier idea of reduction of a linear evolution partial differential equation to an ordinary differential equation. By "reduction" we mean a constructive procedure which gives an exact solution in analytic form. In Section 3 we quote the fragments of the paper [9] and, in Section 4, we announce the results of preprint [7].

2. Short theory of ODEs of the Carathéodory type. Let X be a Banach space with the norm $|\cdot|$, over a scalar field $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$). A curve $x\subset\mathbb{R}\times X$, i.e. an X-valued function defined on an interval $\operatorname{dom}x$ ($\subset\mathbb{R}$), is said to be *absolutely continuous* if the restriction $x|_{[a,b]}$ is absolutely continuous for any $a,b\in\operatorname{dom}x$. The absolute continuity of $x|_{[a,b]}$ means, as in the theory of real functions, that

$$\forall\varepsilon>0\ \exists\delta>0:\quad \sum_{i=1}^{k}|x(\beta_i)-x(\alpha_i)|<\varepsilon$$

for any family $]\alpha_1,\beta_1[,\dots,]\alpha_k,\beta_k[$ of pairwise disjoint subintervals of $[a,b]$ satisfying $\sum_{i=1}^{k}|\beta_i-\alpha_i|<\delta$.

Let a "right-hand side" $\mathcal{F}\subset(\mathbb{R}\times X)\times X$ be any mapping defined on an arbitrary set $\operatorname{dom}\mathcal{F}\subset\mathbb{R}\times X$. An absolutely continuous curve $x\subset\operatorname{dom}\mathcal{F}$

is called a *solution* of the ordinary differential equation

$$\dot{x} = \mathcal{F}(t,x) \tag{3}$$

if $\operatorname{dom} x$ is an infinite interval in $\mathbb{R}$ and there exists a set Z $(\subset \operatorname{dom} x)$ of zero Lebesgue measure such that for every $t \in \operatorname{dom} x \backslash Z$, $\dot{x}(t)$ exists and

$$\dot{x}(t) = \mathcal{F}(t, x(t)). \tag{4}$$

Here $\dot{x}(t)$ $(\in X)$ denotes the derivative of the curve x at time t, that is,

$$\lim_{h\to 0} \left| \frac{x(t+h) - x(t)}{h} - \dot{x}(t) \right| = 0.$$

In particular, each solution $x \subset \mathbb{R} \times X$ of (3) satisfies

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} \dot{x}(t)\,dt \quad \text{ for any } t_1, t_2 \in \operatorname{dom} x.$$

This follows from Theorem 5.1 (see Appendix), which, together with other properties of vector-valued absolutely continuous functions (which will be needed here), can be achieved as a generalization of appropriate properties of real functions, using, without essential modifications, the methods described e.g. in Saks [17] (Chap. III, §§ 12, 13) or Łojasiewicz [11] (Chap. 7, §§ 7.3, 7.4). This only requires some familiarity with the Bochner integral and a counterpart of the Luzin theorem for Banach spaces.

As in the classical theory of differential equations we say that the equation (3) has the *uniqueness property* if for arbitrary solutions x_1, x_2 of (3) such that $x_1 \cap x_2 \neq \emptyset$ we have $x_1 = x_2$ in $\operatorname{dom} x_1 \cap \operatorname{dom} x_2$.

The notion of solvability of (3) at a fixed point $(t_0, \mathsf{x}_0) \in \operatorname{dom} \mathcal{F}$ must include the topological structure of the domain of our right-hand side in a neighbourhood of (t_0, x_0). We begin with the important particular case of $\operatorname{dom} \mathcal{F}$ being a Cartesian product.

Definition 2.1. Assume that $\operatorname{dom} \mathcal{F} = \mathcal{J} \times \Omega$ for some $\mathcal{J} \subset \mathbb{R}$ and $\Omega \subset X$. We say that the differential equation (3) is *solvable at a point* $(t_0, \mathsf{x}_0) \in \mathcal{J} \times \Omega$ if there exists a solution $x \subset \mathbb{R} \times X$ of (3) such that

$$x(t_0) = \mathsf{x}_0 \tag{5}$$

and $\operatorname{dom} x$ is a neighbourhood of t_0 in $\mathcal{J}$ (with relative topology).

Exactly this type of solvability appears in 2.5 and 2.7. In the general case, when $\operatorname{dom} \mathcal{F}$ is not necessarily a Cartesian product, we propose the following universal

Definition 2.2. The differential equation (3) is *solvable at a point* $(t_0, \mathsf{x}_0) \in \operatorname{dom} \mathcal{F}$ if there exists a solution $x \subset \mathbb{R} \times X$ of (3) and a neighbourhood G of (t_0, x_0) (in $\mathbb{R} \times X$) such that (5) is satisfied and

$$\text{(the projection of } (G \cap \operatorname{dom} \mathcal{F}) \text{ onto the time axis)} \subset \operatorname{dom} x.$$

This is a generalization of Definition 2.1. Indeed, we have the following elementary

PROPOSITION 2.3. *Suppose* $\operatorname{dom} \mathcal{F}$ *is open in* $\mathbb{R} \times X$ (*respectively*, $\operatorname{dom} \mathcal{F} = \mathcal{J} \times \Omega$ *for some* $\mathcal{J} \subset \mathbb{R}$ *and* $\Omega \subset X$). *Fix* $(t_0, \mathsf{x}_0) \in \operatorname{dom} \mathcal{F}$. *Then the following conditions are equivalent*:

(i) *the equation* (3) *is solvable at* (t_0, x_0);
(ii) *there exists a solution* $x \subset \mathbb{R} \times X$ *of* (3) *such that* (5) *holds and* $\operatorname{dom} x$ *is a neighbourhood of* t_0 *in* $\mathbb{R}$ (*in* $\mathcal{J}$, *respectively*).

We say that (3) is *solvable* if it is solvable at any point of $\operatorname{dom} \mathcal{F}$ (in the sense of Def. 2.2).

EXAMPLE 2.4. Let $X = \mathbb{R}$ and

$$\mathcal{F}(t, \mathsf{x}) := \begin{cases} -\mathsf{x}/t & \text{if } t \neq 0, \\ 1 & \text{if } t = 0, \end{cases} \tag{6}$$

for $(t, \mathsf{x}) \in \operatorname{dom} \mathcal{F} := \{(0,0)\} \cup (]0, \infty[\times \mathbb{R})$ (which is, of course, neither open nor a Cartesian product). Then for each $(t_0, \mathsf{x}_0) \in \operatorname{dom} \mathcal{F}$ the equation (3) is solvable at (t_0, x_0) in the sense of Definition 2.2 with

$$G = \begin{cases} \mathbb{R}^2 & \text{if } (t_0, \mathsf{x}_0) = (0,0), \\ ]0, \infty[\times \mathbb{R} & \text{if } (t_0, \mathsf{x}_0) \in]0, \infty[\times \mathbb{R}. \end{cases}$$

Indeed, the functions $[0, \infty[\ni t \mapsto 0$ and $]0, \infty[\ni t \mapsto c/t$ $(c \in \mathbb{R} \setminus \{0\})$ are solutions of (3), i.e. they are absolutely continuous and satisfy (4) for t outside the sets of measure zero $Z = \{0\}$ and $Z = \emptyset$, respectively. So, (3) is (globally) solvable. Furthermore, since $\mathcal{F}|_{]0,\infty[\times\mathbb{R}}$ is of class $\mathcal{C}^1$, the equation (3) has the uniqueness property.

Next, the ODE

$$\dot{x}(t) = -\mathcal{F}(t, x(t)) \qquad (= x(t)/t) \tag{7}$$

is solvable with the uniform $G = \mathbb{R}^2$ (because every linear function $[0, \infty[\ni t \mapsto ct$ $(c \in \mathbb{R})$ is its solution), but it does not have the uniqueness property.

Lastly, the formula (6) defines an extension $\overline{\mathcal{F}}$ of the above right-hand side $\mathcal{F}$ to the whole plane $\operatorname{dom} \overline{\mathcal{F}} = \mathbb{R}^2$. Then for each $\mathsf{x}_0 \in \mathbb{R} \setminus \{0\}$ the equation

$$\dot{x} = \overline{\mathcal{F}}(t, x) \tag{8}$$

has no solution defined on an open interval and satisfying (5) with $t_0 = 0$. So, (8) has the uniqueness property but it is not solvable. ∎

In what follows, we assume the following

Convention. By a *measurable* function we mean a Borel measurable function; *summability* is understood in the Bochner sense (in particular the range of values of a summable function is supposed to be separable).

The old idea of successive approximations originating from Liouville [10] and Picard [15] (see historic comments in Hartman [6]) gave inspiration to formulate the following

Theorem 2.5. *Assume that* $\operatorname{dom}\mathcal{F} = \mathcal{J} \times \mathcal{O}$ *for some connected infinite* $\mathcal{J} \subset \mathbb{R}$ *and open* $\mathcal{O} \subset X$. *Let* $L : \mathcal{J} \to \mathbb{R}_+$ *be locally summable. Suppose that*

(i) *for any* $\mathsf{x} \in \mathcal{O}$ *the curve* $\mathcal{J} \ni \tau \mapsto \mathcal{F}(\tau, \mathsf{x}) \in X$ *is locally summable*,
(ii) $\forall t \in \mathcal{J}\ \forall \mathsf{x}_1, \mathsf{x}_2 \in \mathcal{O} : |\mathcal{F}(t, \mathsf{x}_1) - \mathcal{F}(t, \mathsf{x}_2)| \le L(t)|\mathsf{x}_1 - \mathsf{x}_2|$.

Then the differential equation (3) *is solvable and has the uniqueness property.*

The condition (ii) (cf. (2)) comes from Picard [15], while (i) from Carathéodory [3]. The proof resembles the reasoning in proofs of standard theorems on existence and uniqueness of classical ($\mathcal{C}^1$-) solutions, so we only give the following

Sketch of proof. First, we show that

(9) $\mathcal{J} \ni \tau \mapsto \mathcal{F}(\tau, \gamma(\tau)) \in X$ is locally summable whenever $\gamma : \mathcal{J} \to \mathcal{O}$ is continuous.

Let $S : \mathcal{J} \to \mathcal{O}$ be a measurable simple curve. Then $\mathcal{F}(\cdot, S(\cdot))$ is measurable by (i). There is a sequence (γ_n) of measurable simple curves taking values in $\mathcal{O}$ such that $\gamma_n \to \gamma$ pointwise as $n \to \infty$. Since

$$|\mathcal{F}(t, \gamma_n(t)) - \mathcal{F}(t, \gamma(t))| \le L(t)|\gamma_n(t) - \gamma(t)| \xrightarrow{n\to\infty} 0 \quad \text{for every } t \in \mathcal{J},$$

the curve $\mathcal{F}(\cdot, \gamma(\cdot))$ is measurable. For a given $t_0 \in \mathcal{J}$ we choose a compact neighbourhood N of t_0 in $\mathcal{J}$ and, fixing $t \in N$, estimate:

$$\begin{aligned} |\mathcal{F}(t, \gamma(t))| &\le |\mathcal{F}(t, \gamma(t)) - \mathcal{F}(t, \gamma(t_0))| + |\mathcal{F}(t, \gamma(t_0))| \\ &\le L(t)|\gamma(t) - \mathsf{x}_0| + |\mathcal{F}(t, \mathsf{x}_0)| \le (\operatorname{osc}_N \gamma) L(t) + |\mathcal{F}(t, \mathsf{x}_0)|, \end{aligned}$$

where $\mathsf{x}_0 := \gamma(t_0)$ and $(\operatorname{osc}_N \gamma) := \operatorname{diam} \gamma(N)$ $(< \infty)$. Finally, $\mathcal{F}(\cdot, \gamma(\cdot))|_N$ is summable, which, in view of the arbitrariness of t_0, completes the proof of (9).

Fix $(t_0, \mathsf{x}_0) \in \mathcal{J} \times \mathcal{O}$. We now prove the *existence* of a solution of the problem $\{(3), (5)\}$ in a neighbourhood of t_0 in $\mathcal{J}$. There is an $r > 0$ so small that $\overline{K}(\mathsf{x}_0, r) := \{|\mathsf{x} - \mathsf{x}_0| \le r\} \subset \mathcal{O}$ and a connected neighbourhood N of t_0 in $\mathcal{J}$ so small that $\int_N L(\tau)\, d\tau < 1/2$. The class Γ of all continuous curves from N into $\overline{K}(\mathsf{x}_0, r)$ with the metric $(\gamma_1, \gamma_2) \mapsto \sup_{t \in N} |\gamma_1(t) - \gamma_2(t)|$ is a complete metric space. Next, the mapping

$$Q : X \ni \mathsf{x} \mapsto \mathsf{x}_0 + rR\left(\frac{\mathsf{x}-\mathsf{x}_0}{r}\right) \in X,$$

given by means of the retraction R from Lemma 5.2 (see Appendix), is a retraction onto $\overline{K}(\mathsf{x}_0, r)$ and satisfies the Lipschitz condition with constant 2. Finally, by (9) and the Banach contraction principle, the mapping $\Gamma \ni \gamma \mapsto Q(\mathsf{x}_0 + \int_{t_0}^{t} \mathcal{F}(\tau, \gamma(\tau))\, d\tau) \in \Gamma$ is well defined and has a fixed point, which, after restriction to a sufficiently small neighbourhood of t_0 (in $\mathcal{J}$), is the required solution.

Uniqueness may be showed by standard reduction of $\{(3), (5)\}$ to the integral form and use of the following version of the Gronwall inequality:

(10) *if* $t_0, t_1 \in \mathbb{R}$, $C \in \mathbb{R}_+$, $u : I := [t_0, t_1] \to \mathbb{R}_+$ *is continuous*, $a : I \to \mathbb{R}_+$ *is summable and* $u(t) \le C + \int_{[t_0,t]} a(\tau)u(\tau)\, d\tau$ *for each* $t \in I$, *then* $u(t) \le C \exp(\int_{[t_0,t]} a(\tau)\, d\tau)$ *for each* $t \in I$,

which can be derived elementarily from its classical prototype (proved for continuous a e.g. in Rabczuk [16]). ∎

Related results (e.g. Th. 2.6 below) can be obtained in a more general case, when $\operatorname{dom} \mathcal{F}$ need not be a Cartesian product. To prove them, it suffices to use Theorem 2.5 locally.

THEOREM 2.6. *Assume that for every* $(t_0, \mathsf{x}_0) \in \operatorname{dom} \mathcal{F}$ *there are: a connected infinite* $\mathcal{J} \subset \mathbb{R}$, *an open* $\mathcal{O} \subset X$ *and a locally summable* $L : \mathcal{J} \to \mathbb{R}_+$ *such that* $(t_0, \mathsf{x}_0) \in \mathcal{J} \times \mathcal{O} \in \operatorname{top} \operatorname{dom} \mathcal{F}$ *and*

(i) *for any* $\mathsf{x} \in \mathcal{O}$ *the curve* $\mathcal{J} \ni \tau \mapsto \mathcal{F}(\tau, \mathsf{x}) \in X$ *is locally summable*,
(ii) $\forall t \in \mathcal{J}\ \forall \mathsf{x}_1, \mathsf{x}_2 \in \mathcal{O} : |\mathcal{F}(t, \mathsf{x}_1) - \mathcal{F}(t, \mathsf{x}_2)| \le L(t)|\mathsf{x}_1 - \mathsf{x}_2|$.

Then the differential equation (3) *is solvable and has the uniqueness property.*

The symbol $\operatorname{top} \operatorname{dom} \mathcal{F}$ above stands for the topology in $\operatorname{dom} \mathcal{F}$ inherited from $\mathbb{R} \times X$.

Now we show a necessary and sufficient condition for the existence of a solution of the Cauchy problem $\{(3), (5)\}$ defined on the *whole* $\mathcal{J}$, first in the special case of $\operatorname{dom} \mathcal{F} = \mathcal{J} \times \Omega$ (with a not necessarily open $\Omega \subset X$).

THEOREM 2.7. *Assume* $\operatorname{dom} \mathcal{F} = \mathcal{J} \times \Omega$ *for some interval* $\mathcal{J} \subset \mathbb{R}$ *and some* $\Omega \subset X$. *Suppose that the equation* (3) *is solvable and has the uniqueness property. Let* $(t_0, \mathsf{x}_0) \in \mathcal{J} \times \Omega$. *Let* γ *be the maximal* (*with respect to inclusion*) *solution of the initial-value problem* $\{(3), (5)\}$. *Then the following conditions are equivalent*:

(i) $\mathcal{J} \subset \operatorname{dom} \gamma$;

(ii) *if* $x \subset \mathbb{R} \times X$ *is a solution of* $\{(3),\,(5)\}$ *such that* $\overline{\operatorname{dom} x}$ *is a compact subinterval of* $\mathcal{J}$, *then* $\overline{x} \subset \operatorname{dom} \mathcal{F}$ *and* $\int |\dot{x}(t)|\,dt < \infty$.

The symbol $\overline{x}$ stands for the closure of the relation x in $\mathbb{R} \times X$.

Proof. (i)$\Rightarrow$(ii). Suppose $\mathcal{J} \subset \operatorname{dom}\gamma$. Consider a solution x of $\{(3),(5)\}$ such that $K := \overline{\operatorname{dom} x}$ is a compact subinterval of $\mathcal{J}$. From the maximality of γ we infer that $x \subset \gamma$. The restriction $\gamma|_K$ is an absolutely continuous curve defined on a compact interval. So, $(\gamma|_K)'$ is summable and

$$\int\limits_{\operatorname{dom} x} |\dot{x}(t)|\,dt = \int\limits_{\operatorname{dom} x} |\dot{\gamma}(t)|\,dt = \int\limits_{K} |\dot{\gamma}(t)|\,dt < \infty.$$

To show $\overline{x} \subset \operatorname{dom}\mathcal{F}$, fix $(t,\mathsf{x}) \in \overline{x}$. There is a sequence $((t_\nu,\mathsf{x}_\nu))_{\nu\in\mathbb{N}} \in x$ such that $(t_\nu,\mathsf{x}_\nu) \to (t,\mathsf{x})$ as $\nu \to \infty$. Of course $t \in K$. Finally,

$$\gamma(t) \leftarrow \gamma(t_\nu) = x(t_\nu) = \mathsf{x}_\nu \to \mathsf{x},$$

so $(t,\mathsf{x}) = (t,\gamma(t)) \in \gamma \subset \operatorname{dom}\mathcal{F}$, which gives (ii).

(ii)$\Rightarrow$(i). First, we show that

$$\mathcal{J} \cap [t_0,\infty[\subset \operatorname{dom}\gamma. \tag{11}$$

Suppose, contrary to our claim, that there is $\widehat{t} \in \mathcal{J}$ such that $t < \widehat{t}$ for every $t \in \operatorname{dom}\gamma$. Clearly, $T := \sup\operatorname{dom}\gamma \le \widehat{t}$, so, $I := [t_0,T]$ is a compact subinterval of $\mathcal{J}$. The curve $x := \gamma|_I : I \cap \operatorname{dom}\gamma \to X$ is a solution of $\{(3),\,(5)\}$, so, according to (ii), $\overline{x} \subset \operatorname{dom}\mathcal{F}$ and $\dot{x}$ is summable. Of course, $[t_0,T[\, \subset \operatorname{dom} x$ and

$$x(t) = \mathsf{x}_0 + \int\limits_{t_0}^{t} \dot{x}(\tau)\,d\tau \quad \text{for any } t_0 \le t < T.$$

Define $\mathsf{x}_T := \mathsf{x}_0 + \int_{t_0}^T \dot{x}(\tau)\,d\tau$ $(\in X)$. Since $(T,\mathsf{x}_T) \in \overline{x}$ it also belongs to $\operatorname{dom}\mathcal{F}$. The gluing $\widetilde{\gamma} := \gamma \cup \{(T,\mathsf{x}_T)\}$ is an absolutely continuous function from $(\operatorname{dom}\gamma) \cup \{T\}$ into X. Moreover,

$$\widetilde{\gamma}'(t) = \gamma'(t) = \mathcal{F}(t,\gamma(t)) = \mathcal{F}(t,\widetilde{\gamma}(t)) \quad \text{for almost all } t \in \operatorname{dom}\widetilde{\gamma}.$$

Thus, $\widetilde{\gamma}$ is a solution of $\{(3),\,(5)\}$. Consequently, $\widetilde{\gamma} = \gamma$ by maximality. Then $T \in \operatorname{dom}\gamma$ and $\gamma(T) = \mathsf{x}_T$. But our ODE is solvable at (T,x_T), so there is a solution $y \subset \mathbb{R} \times X$ of (3) with the initial condition $y(T) = \mathsf{x}_T$ and $\operatorname{dom} y$ is a neighbourhood of T in $\mathcal{J}$. Consequently, there exists S open in $\mathbb{R}$ and such that $T \in S$ and $\mathcal{J} \cap S \subset \operatorname{dom} y$. Since $T \in \operatorname{dom}\gamma$, we have $T < \widehat{t}$. There is a positive $\varepsilon < \widehat{t} - T$ so small that $[T,T+\varepsilon] \subset S$. Simultaneously, $[T,T+\varepsilon] \subset \mathcal{J}$, and, finally,

$$[T,T+\varepsilon] \subset \mathcal{J} \cap S \subset \operatorname{dom} y.$$

Since $\gamma(T) = \mathsf{x}_T = y(T)$, the curve $\Gamma := \gamma \cup (y|_{[T,T+\varepsilon]})$ is absolutely continuous and

$$\Gamma'(t) = \mathcal{F}(t, \Gamma(t)) \quad \text{for almost all } t \in \operatorname{dom} \Gamma = (\operatorname{dom} \gamma) \cup [T, T+\varepsilon].$$

Thus, Γ is a solution of $\{(3), (5)\}$ and, again from the maximality of γ, we have $\Gamma \subset \gamma$. In particular, $\operatorname{dom} \Gamma \subset \operatorname{dom} \gamma$. Finally, $T + \varepsilon \in \operatorname{dom} \gamma$, so $T+\varepsilon \le \sup \operatorname{dom} \gamma = T$, which is impossible. This contradicts our hypothesis and gives (11).

A symmetric reasoning shows the inclusion $\mathcal{J} \cap]-\infty, t_0] \subset \operatorname{dom} \gamma$, which together with (11) completes the proof. ∎

Let us pass to the case of an arbitrary $\operatorname{dom} \mathcal{F}$. Adding an elementary topological argument to the reasoning above we obtain the following quite general

THEOREM 2.8. *Suppose that the equation* (3) *is solvable and has the uniqueness property. Consider a pair* $(t_0, \mathsf{x}_0) \in \operatorname{dom} \mathcal{F}$ *and an interval* $\mathcal{J}$ *such that* $t_0 \in \mathcal{J}$. *Let* γ *be the maximal* (*with respect to inclusion*) *solution of the initial-value problem* $\{(3), (5)\}$. *Then the following conditions are equivalent*:

(i) $\mathcal{J} \subset \operatorname{dom} \gamma$;

(ii)(a) *for every* $(t, \mathsf{x}) \in (\mathcal{J} \times X) \cap \overline{\gamma} \cap \operatorname{dom} \mathcal{F}$ *and every neighbourhood* G *of* (t, x) *in* $\mathbb{R} \times X$ *we have*

$$\mathcal{J} \cap (\textit{projection of } G \cap \operatorname{dom} \mathcal{F} \textit{ onto the time axis})$$

is a neighbourhood of t *in* $\mathcal{J}$,

(b) *if* $x \subset \mathbb{R} \times X$ *is a solution of* $\{(3), (5)\}$ *such that* $\overline{\operatorname{dom} x}$ *is a compact subinterval of* $\mathcal{J}$, *then* $\overline{x} \subset \operatorname{dom} \mathcal{F}$ *and* $\int |\dot{x}(t)|\, dt < \infty$.

Moreover, (ii)(a) *holds in the following special case*:

(12) $\operatorname{dom} \mathcal{F} \in \operatorname{top}(\mathbb{R} \times X)$ *or* $\operatorname{dom} \mathcal{F} = \mathcal{J} \times \Omega$ *for some* $\Omega \subset X$.

As before, the symbols $\overline{\gamma}$, $\overline{x}$ stand for the closures of γ, x in $\mathbb{R} \times X$.

Lastly, to show how Theorems 2.5 and 2.7 work together, we will examine the existence, uniqueness and continuous dependence (on data) of a solution of a general linear ODE of the Carathéodory type.

EXAMPLE 2.9. Consider another Banach space $\mathcal{A}$, a continuous bilinear mapping $\Lambda : \mathcal{A} \times X \to X$ and locally summable curves $A : \mathcal{J} \to \mathcal{A}$, $C : \mathcal{J} \to X$ on a nondegenerate interval $\mathcal{J} \subset \mathbb{R}$. Define

$$a\mathsf{x} = a \cdot \mathsf{x} := \Lambda(a, \mathsf{x}) \quad \text{for every } (a, \mathsf{x}) \in \mathcal{A} \times X.$$

With a fixed pair $(t_0, \mathsf{x}_0) \in \mathcal{J} \times X$ we may associate the Cauchy problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + C(t), & (13) \\ x(t_0) = \mathsf{x}_0. & (14) \end{cases}$$

The equation (13) is said to be a generalised *linear ODE* with the coefficients A and C. It is an important particular case of (3), corresponding to the right-hand side

$$\mathcal{F} : \mathcal{J} \times X \ni (t, \mathsf{x}) \overset{\text{def}}{\mapsto} A(t)\mathsf{x} + C(t) \in X. \tag{15}$$

Putting $L(t) := |\Lambda| \cdot |A(t)|$ for $t \in \mathcal{J}$, where $|A(t)|$ stands for the norm of the vector $A(t)$ while $|\Lambda| := \sup\{|a\mathsf{x}| : |a| \le 1,\ |\mathsf{x}| \le 1\}$, we can easily check that assumptions (i) and (ii) of Theorem 2.5 hold. So,

(16) the differential equation (13) is solvable and has the uniqueness property.

Let $x \subset \mathbb{R} \times X$ be a solution of $\{(13), (14)\}$ such that $K := \overline{\operatorname{dom} x}$ is a compact subinterval of $\mathcal{J}$. Of course, $\overline{x} \subset K \times X \subset \mathcal{J} \times X = \operatorname{dom} \mathcal{F}$. After integrating (13) and using (14) we estimate:

$$\begin{aligned} |x(t)| &= \Big|\mathsf{x}_0 + \int_{t_0}^{t} A(\tau)x(\tau)\,d\tau + \int_{t_0}^{t} C(\tau)\,d\tau\Big| \\ &\le |\mathsf{x}_0| + \int_{[t_0,t]} L(\tau)|x(\tau)|\,d\tau + \int_K |C(\tau)|\,d\tau \end{aligned}$$

for any fixed $t \in \operatorname{dom} x$. Hence, in virtue of (10),

$$|x(t)| \le \Big(|\mathsf{x}_0| + \int_K |C(\tau)|\,d\tau\Big) \exp\Big(\int_{[t_0,t]} L(\tau)\,d\tau\Big) \quad \text{for } t \in \operatorname{dom} x. \tag{17}$$

So, x is bounded. Finally, since

$$\begin{aligned} |\dot{x}(t)| &= |A(t)x(t) + C(t)| \\ &\le |\Lambda| \cdot |A(t)| \sup_{\tau \in \operatorname{dom} x} |x(\tau)| + |C(t)| \quad \text{for } t \in \operatorname{dom} \dot{x}, \end{aligned}$$

$\dot{x}$ is summable. By Theorem 2.7,

(18) the problem $\{(13), (14)\}$ has a solution γ defined on $\mathcal{J}$.

Now, we deal with the continuous dependence on data of such a solution. Let $t_{0,\nu} \to t_0$ in $\mathcal{J}$ (as $\mathbb{N} \ni \nu \to \infty$), $\mathsf{x}_{0,\nu} \to \mathsf{x}_0$ in X, $A_\nu \to A$ in the Fréchet space $L^1_{\mathrm{loc}}(\mathcal{J}, \mathcal{A})$ (i.e. $\lim_{\nu\to\infty} \int_K |A_\nu(t) - A(t)|dt = 0$ for every compact $K \subset \mathcal{J}$) and $C_\nu \to C$ in $L^1_{\mathrm{loc}}(\mathcal{J}, X)$ as $\mathbb{N} \ni \nu \to \infty$. From (17) and (18), the initial-value problem

$$\begin{cases} \dot{x}(t) = A_\nu(t)x(t) + C_\nu(t), & (13_\nu) \\ x(t_{0,\nu}) = \mathsf{x}_{0,\nu}, & (14_\nu) \end{cases}$$

has a unique (absolutely continuous) solution $\gamma_\nu : \mathcal{J} \to X$. Let $K \subset \mathcal{J}$ be any compact interval containing $\{t_{0,\nu} : \nu \in \mathbb{N}\}$. Then, by (17),

$$\sup_{\nu\in\mathbb{N},\, t\in K} |\gamma_\nu(t)| \le M = \sup_{\nu\in\mathbb{N}} \Big(\Big(|\mathsf{x}_{0,\nu}| + \int_K |C_\nu(\tau)|\,d\tau\Big) e^{|\Lambda| \int_K |A_\nu(\tau)|\,d\tau}\Big) < \infty.$$

Setting $a_\nu := A_\nu - A$, $c_\nu := C_\nu - C$, $y_\nu := \gamma_\nu - \gamma$ and $\mathsf{y}_\nu := y_\nu(t_{0,\nu})$, and remembering (14), (14_ν) we have

$$\lim_{\nu\to\infty} \int_K |a_\nu(t)|\,dt = \lim_{\nu\to\infty} \int_K |c_\nu(t)|\,dt = \lim_{\nu\to\infty} |\mathsf{y}_\nu| = 0. \tag{19}$$

Next, combining (13) with (13_ν) we calculate:

$$\begin{aligned}
|y_\nu(t)| &\le |y_\nu(t) - y_\nu(t_{0,\nu})| + |y_\nu(t_{0,\nu})| \\
&= \Big|\int_{t_{0,\nu}}^{t} \dot y_\nu(\tau)\,d\tau\Big| + |\mathsf{y}_\nu| \\
&\le \Big|\int_{t_{0,\nu}}^{t} a_\nu(\tau)\gamma_\nu(\tau)\,d\tau\Big| + \Big|\int_{t_{0,\nu}}^{t} A(\tau) y_\nu(\tau)\,d\tau\Big| + \Big|\int_{t_{0,\nu}}^{t} c_\nu(\tau) d\tau\Big| + |\mathsf{y}_\nu| \\
&\le \int_K |a_\nu(\tau)\gamma_\nu(\tau)|\,d\tau + \Big|\int_{t_{0,\nu}}^{t_0} A(\tau) y_\nu(\tau)\,d\tau + \int_{t_0}^{t} A(\tau) y_\nu(\tau)\,d\tau\Big| \\
&\quad + \int_K |c_\nu(\tau)| d\tau + |\mathsf{y}_\nu| \\
&\le M|\Lambda| \int_K |a_\nu(\tau)|\,d\tau + 2M|\Lambda| \int_{[t_0, t_{0,\nu}]} |A(\tau)|\,d\tau \\
&\quad + \int_{[t_0,t]} L(\tau) |y_\nu(\tau)|\,d\tau + \int_K |c_\nu(\tau)|\,d\tau + |\mathsf{y}_\nu|.
\end{aligned}$$

In view of the arbitrariness of t $(\in K)$, we may use (10) again to get

$$|y_\nu(t)| \le r_\nu \exp\Big(\int_{[t_0,t]} L(\tau)\,d\tau\Big) \quad \text{for each } t \in K,$$

where

$$r_\nu := M|\Lambda| \Big(\int_K |a_\nu(\tau)|\,d\tau + 2 \int_{[t_0,t_{0,\nu}]} |A(\tau)|\,d\tau\Big) + \int_K |c_\nu(\tau)|\,d\tau + |\mathsf{y}_\nu|.$$

Therefore, $\sup_{t\in K} |y_\nu(t)| \le r_\nu \exp(\int_K L(\tau) d\tau)$ ($\xrightarrow{\nu\to\infty} 0$ due to (19)). Finally,

$$\gamma_\nu \xrightarrow{\nu\to\infty} \gamma \quad \text{in the Fréchet space } \mathcal{C}(\mathcal{J}, X), \tag{20}$$

that is, $\sup_{t\in K} |\gamma_\nu(t) - \gamma(t)| \xrightarrow{\nu\to\infty} 0$ for every compact $K \subset \mathcal{J}$. ∎

3. An application of Theorems 2.5 and 2.7 to a nonlinear parabolic equation. The Faedo–Galerkin method enables us to reduce an initial-boundary problem for a parabolic PDE to the Cauchy problem $\{(3), (5)\}$. We will show it taking as an example the general variational equation (24) below, examined in the paper [9]. The question of convergence of the sequence of approximate solutions (obtained below by the Carathéodory method) to an exact solution of (24) is rather topological in character and it is not our main subject. Nevertheless, we will discuss two algebraic examples (3.1 and 3.2) and they allow us to check certain sensible conditions implying such a convergence.

Consider a Hilbert space $(H, (\cdot|\cdot))$ over a scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Define $|h| := \sqrt{(h|h)}$ for $h \in H$. Let $(V, \|\cdot\|)$ be a reflexive Banach space (over $\mathbb{K}$) with separable dual space V'. Consider also a separable Banach space Φ and assume that

(21) $\Phi \subset V \subset H$, Φ is dense in V, V is dense in H,

(22) the injections $\Phi \hookrightarrow V$, $V \hookrightarrow H$ are linear and continuous.

Let $0 < T < \infty$. Suppose that a family $b(t) : V \to \Phi'$ $(0 \le t \le T)$ of (perhaps nonaffine) mappings fulfils the following conditions:

(A) for every $\varphi \in \Phi$ and every measurable curve $v : [0,T] \to V$ satisfying

$$\int_0^T \|v(t)\|^2\,dt + \sup_{0\le t\le T} |v(t)| < \infty, \tag{23}$$

the function $[0,T] \ni t \mapsto (b(t)v(t))\varphi \in \mathbb{K}$ is measurable; moreover, there is a continuous function $\vartheta : (\mathbb{R}_+)^2 \to \mathbb{R}_+$ vanishing on $\mathbb{R}_+ \times \{0\}$ such that for every $\varphi \in \Phi$ and every V-measurable v with property (23) we have

$$\int_0^T |(b(t)v(t))\varphi|\,dt \le \vartheta(\|v\|_{L^2(0,T;V)} + \sup_t |v(t)|, |\varphi|_\Phi);$$

(C) for every $(t_\star, \varphi_\star) \in [0,T] \times \Phi$ there exist $\mathcal{J} \in \mathrm{top}[0,T]$, $\mathcal{O} \in \mathrm{top}\,\Phi$ and a summable function $L : \mathcal{J} \to \mathbb{R}_+$ such that $(t_\star, \varphi_\star) \in \mathcal{J} \times \mathcal{O}$ and

$$|b(t)\varphi_1 - b(t)\varphi_2|_{\Phi'} \le L(t)|\varphi_1 - \varphi_2|_\Phi$$

for every $t \in \mathcal{J}$ and all $\varphi_1, \varphi_2 \in \mathcal{O}$;

(D) there exists a constant $\mu > 0$ and summable functions $C, a : [0,T] \to \mathbb{R}_+$ such that for all $t \in [0,T]$ and all $\varphi \in \Phi$,

$$\mathrm{Re}(b(t)\varphi)\varphi \ge \mu\|\varphi\|^2 - C(t) - a(t)|\varphi|^2.$$

For a given vector $u_0 \in H$ we look for a weakly continuous curve $u : [0,T] \to H$ such that

(a) $u^{-1}(V)$ has Lebesgue measure T and $\int_0^T \|u(t)\|^2\,dt < \infty$,
(b) for every $t \in [0,T]$ and every $\varphi \in \Phi$,

$$(\varphi|u(t)) - (\varphi|u_0) = -\int_0^t (b(\tau)u(\tau))\varphi\,d\tau. \tag{24}$$

(Note that the weak continuity of u as an H-valued curve makes the conditions (a) and (b) meaningful. Indeed, since H is separable, u is a (strongly) measurable H-valued curve; next, by the Banach–Alaoglu theorem and Mazur separation theorem (used as in the elementary Lemma 2.7 of [9]), $u^{-1}(V)$ is a Borel set and, finally, the restriction $u|_{u^{-1}(V)}$ is measurable as a V-valued mapping). Such a curve u is called a *solution* of the variational equation (24) or an evolution of the "initial state" u_0 in the "velocity field" b. The equation (24) can be called *parabolic* because of the generalized Gårding inequality (D).

As we have previously announced, we will show how to find, by the Faedo–Galerkin method, an approximate solution of the equation (24). Suppose that

(25) there exists a Banach space $\mathcal{H}$ and a sequence $(V_N)_{N=1}^{\infty}$ of finite-dimensional linear subspaces of $\mathcal{H}$ such that $\mathcal{H}$ is a dense linear subspace of Φ, the canonical injection $\mathcal{H} \hookrightarrow \Phi$ is continuous and $\lim_{N\to\infty} |\varphi - P_N(\varphi)|_{\mathcal{H}} = 0$ for any $\varphi \in \mathcal{H}$,

where $P_N : H \to V_N$ is the $(\cdot|\cdot)$-orthogonal projection onto V_N. Fix $N \in \mathbb{N}$ and consider a measurable curve $u_N : [0,T] \to V_N$ such that $\sup_t |u_N(t)| < \infty$. Then $\int_0^T \|u_N(t)\|^2\,dt < \infty$ because the restrictions of the norms $|\cdot|, \|\cdot\|$ to V_N are equivalent. According to (A), the function $[0,T] \ni t \mapsto (b(t)u_N(t))\varphi \in \mathbb{K}$ is summable for each $\varphi \in \Phi$. Suppose that u_N is a solution of the following *Galerkin equation* in V_N corresponding to (24):

$$(\varphi|u_N(t)) - (\varphi|u_0) = -\int_0^t (b(\tau)u_N(\tau))\varphi\,d\tau \tag{24_N}$$

for every $(t,\varphi) \in [0,T] \times V_N$. Putting $t = 0$ and $\varphi = u_N(0) - P_N(u_0)$ we get

$$u_N(0) = P_N(u_0). \tag{26}$$

For a fixed $\varphi \in V_N$ the function $[0,T] \ni t \mapsto (\varphi|u_N(t)) \in \mathbb{K}$ is absolutely continuous and

$$\frac{d}{dt}(\varphi|u_N(t)) = -(b(t)u_N(t))\varphi \quad \text{for almost every } t \in [0,T].$$

Hence, the curve $u_N : [0,T] \to V_N$ itself is absolutely continuous (indeed, $u_N(t) = \sum_k \overline{(\varphi_k|u_N(t))}\varphi_k$ for each $0 \le t \le T$, for an arbitrary

$(\cdot|\cdot)$-orthonormal basis (φ_k) in V_N). Moreover,

$$\frac{d}{dt}u_N(t) = \mathcal{F}_N(t, u_N(t)) \quad \text{for a.e. } t \in [0,T], \tag{27}$$

where $\mathcal{F}_N : [0,T] \times V_N \to V_N$ and, for a fixed $(t,w) \in [0,T] \times V_N$, the vector $\mathcal{F}_N(t,w)$ is defined as the Riesz representation of the linear functional

$$V_N \ni \varphi \mapsto -(b(t)w)\varphi \in \mathbb{K},$$

in the sense of $(\cdot|\cdot)$. In other words,

$$-(b(t)w)\varphi = (\varphi|\mathcal{F}_N(t,w)) \quad \text{for all } (t,w,\varphi) \in [0,T] \times V_N \times V_N. \tag{28}$$

The Cauchy problem $\{(27), (26)\}$ is an equivalent form of the Galerkin variational equation (24_N). We will study it from the viewpoint of Theorems 2.5 and 2.7.

For a fixed $w \in V_N$ the curve $[0,T] \ni t \mapsto \mathcal{F}_N(t,w) \in V_N$ is weakly summable (in virtue of assumption (A)) and therefore it is summable. So, condition (i) of Theorem 2.5 holds for $\mathcal{J} = [0,T]$, $\mathcal{F} = \mathcal{F}_N$, $\mathcal{O} = V_N$. Since any two norms in V_N are equivalent, the condition (ii) of Theorem 2.5 is satisfied, in virtue of assumption (C) (cf. p. 107 in [9]). Thus, the equation (27) is solvable in the sense of Definition 2.1 and it has the uniqueness property.

Let γ denote the maximal (with respect to inclusion) solution of the initial value problem $\{(27), (26)\}$. Now, our aim is to show that

$$\mathcal{J} := [0,T] \subset \operatorname{dom}\gamma. \tag{29}$$

Let $x \subset [0,T] \times V_N$ be any solution of the Cauchy problem $\{(27), (26)\}$. It is clear that $\overline{x} \subset \operatorname{dom}\mathcal{F}_N$. Thus, to see that condition (ii) of Theorem 2.7 holds, it remains to show that $\dot{x}$ is summable. Let $\tau \in \operatorname{dom}\dot{x}$ satisfy $\dot{x}(\tau) = \mathcal{F}_N(\tau, x(\tau))$. By (28),

$$\begin{aligned} \frac{d}{d\tau}\frac{1}{2}|x(\tau)|^2 &= \operatorname{Re}(x(\tau)|\dot{x}(\tau)) = \operatorname{Re}(x(\tau)|\mathcal{F}_N(\tau, x(\tau))) \\ &= -\operatorname{Re}(b(\tau)x(\tau))x(\tau). \end{aligned} \tag{30}$$

Using (D) we estimate

$$\frac{d}{d\tau}\frac{1}{2}|x(\tau)|^2 \le -\mu\|x(\tau)\|^2 + C(\tau) + a(\tau)|x(\tau)|^2. \tag{31}$$

Let $t \in \operatorname{dom} x$. We integrate the inequality (31) over $[0,t]$ and estimate:

$$\frac{1}{2}|x(t)|^2 - \frac{1}{2}|P_N(u_0)|^2 \le -\mu\int_0^t \|x(\tau)\|^2\,d\tau + \int_0^T C(\tau)\,d\tau + \int_0^t a(\tau)|x(\tau)|^2\,d\tau.$$

Hence, after a rearrangement, we get

$$|x(t)|^2 + 2\mu\int_0^t \|x(\tau)\|^2\,d\tau \le |u_0|^2 + 2\|C\|_{L^1} + \int_0^t 2a(\tau)|x(\tau)|^2\,d\tau. \tag{32}$$

In particular, for every $t \in \operatorname{dom} x$,

$$|x(t)|^2 \le |u_0|^2 + 2\|C\|_{L^1} + \int_0^t 2a(\tau)|x(\tau)|^2\,d\tau$$

and so, by the Gronwall inequality (see (10) above),

$$\forall t \in \operatorname{dom} x: \quad |x(t)|^2 \le (|u_0|^2 + 2\|C\|_{L^1})\exp\Big(\int_0^t 2a(\tau)\,d\tau\Big).$$

Therefore,

$$\sup_{t\in\operatorname{dom} x} |x(t)|^2 \le M_0 := (|u_0|^2 + 2\|C\|_{L^1})\exp\Big(\int_0^T 2a(\tau)\,d\tau\Big).$$

Coming back to (32) we estimate

$$\int_{\operatorname{dom} x} \|x(\tau)\|^2\,d\tau \le \frac{1}{2\mu}(|u_0|^2 + 2\|C\|_{L^1} + 2M_0\|a\|_{L^1}) =: M_1. \tag{33}$$

Consider the measurable curve $v : [0,T] \to V$ given by the formula

$$v(t) := \begin{cases} x(t) & \text{if } t \in \operatorname{dom} x, \\ 0 & \text{if } t \notin \operatorname{dom} x. \end{cases}$$

It is clear that

$$\sup_{0\le t\le T} |v(t)| = \sup_{t\in\operatorname{dom} x} |x(t)| \le \sqrt{M_0},$$

$$\int_0^T \|v(\tau)\|^2\,d\tau = \int_{\operatorname{dom} x} \|x(\tau)\|^2\,d\tau \le M_1.$$

According to assumption (A), the curve $[0,T] \ni t \mapsto (b(t)v(t))\varphi \in \mathbb{K}$ is summable for each $\varphi \in \Phi$, so, by (28), the curve $[0,T] \ni t \mapsto \mathcal{F}_N(t, v(t)) \in V_N$ is weakly summable. In fact, it is (strongly) summable because $\dim V_N < \infty$. In particular, its restriction $\operatorname{dom} x \ni t \mapsto \mathcal{F}_N(t, x(t)) \in V_N$ is also summable. Finally, $\dot{x}(t) = \mathcal{F}_N(t, x(t))$ for a.e. $t \in \operatorname{dom} x$, and therefore, $\dot{x}$ is summable, as required. This way, condition (ii) of Theorem 2.7 is satisfied, so (29) holds.

The curve $u_N := \gamma : [0,T] \to V_N$ is called the Nth *approximation* of an (exact) solution u of (24). This terminology is justified, because we add the assumption:

(B) there are a separable Banach space E, a normed space B, a linear completely continuous operator $\imath : V \to B$ and a continuous $\mathbb{R}$-linear operator $\sigma_0 : B \to E'$ such that

(i) E is a dense linear subspace of Φ, the injection $E \overset{\text{can}}{\hookrightarrow} \Phi$ is continuous;

(ii) $\ker \sigma_0 = 0$, $\forall \varphi \in E : \sup_{\mathsf{x} \in E} |(\sigma_0 \circ \imath)(\mathsf{x})\varphi| / (1 + |(\mathsf{x}|\varphi)|) < \infty$;

(iii) if $v_\nu, v : [0, T] \to V$ are square-summable ($\nu \in \mathbb{N}$), $\sup_{\nu,\tau} |v_\nu(\tau)| < \infty$, $v_\nu \to v$ weakly in $L^2(0, T; V)$, $v_\nu(t) \to v(t)$ weakly in H and $(\imath \circ v_\nu)(t) \to (\imath \circ v)(t)$ in B (strongly) for any $t \in [0, T]$, then

$$\int_0^T (b(t)v_\nu(t))\varphi\, dt \xrightarrow{\nu\to\infty} \int_0^T (b(t)v(t))\varphi\, dt$$

for any $\varphi \in E$.

Now one can prove, replacing Φ by E, that (25) holds and there exists a subsequence of $(u_N)_{N\in\mathbb{N}}$ which is weakly (in a certain sense) convergent to the required solution u (see Holly–Wiciak [9]). Finally, the full system of assumptions A)–(D) guarantees the existence of a solution of (24) (satisfying additionally an "energy" inequality, i.e. a relation of the type (32)) for an arbitrary initial value $u_0 \in H$.

To show when the assumption (B) works, we give two examples of $E, B, \imath, \sigma_0$ satisfying conditions (i), (ii) of (B).

Example 3.1. Suppose that the canonical injection $\imath : V \hookrightarrow H$ is completely continuous. Then we put $E = \Phi$, $B = H$, and

$$\sigma_0 : H \ni h \overset{\text{def}}{\mapsto} (\cdot|h)|_\Phi \in \Phi'.$$

Note that $(\sigma_0 \circ \imath)(\mathsf{x})\varphi = (\varphi|\mathsf{x})$ for any $\varphi, \mathsf{x} \in \Phi$. (Exactly such a scheme was used to construct a solution u of the initial-boundary problem for a system of nonlinear diffusion equations—see Holly–Danielewski [8].) ■

Example 3.2. Consider a normed space Y with separable dual space Y' such that H is a dense linear subspace of Y and the canonical injection $V \hookrightarrow Y$ is completely continuous. Suppose that the linear subspace

$$E := \{\varphi \in \Phi \mid \text{the functional } (\cdot|\varphi) : H \to \mathbb{K} \text{ is } |\cdot|_Y\text{-continuous}\}$$

(of Φ) is dense in Φ. Then E, with the norm

$$|\varphi|_E := |\varphi|_\Phi + |\overline{(\cdot|\varphi)}|_{Y'},$$

is a separable Banach space. Here $\overline{(\cdot|\varphi)}$ denotes the unique continuous extension of the functional $(\cdot|\varphi)$ ($\in H'$) to Y. The $\mathbb{R}$-linear operator $\pi : H \ni h \overset{\text{def}}{\mapsto} (\cdot|h)|_E \in E'$ is $|\cdot|_Y$-continuous, so there exists a unique continuous extension $\overline{\pi} : Y \to E'$ of π to Y. Since $\ker \overline{\pi}$ is a $\mathbb{K}$-linear closed subspace of

Y, the quotient space $B := Y/\ker \overline{\pi}$ with the norm $|[y]|_B := \mathrm{dist}_Y(y, \ker \overline{\pi})$, where $[y] := y + \ker \overline{\pi}$, is a normed space. The linear operator

$$\imath : V \ni \psi \overset{\mathrm{def}}{\mapsto} [\psi] \in B$$

is completely continuous. Finally, the operator

$$\sigma_0 : B \ni [y] \overset{\mathrm{def}}{\mapsto} \overline{\pi}(y) \in E'$$

is a continuous $\mathbb{R}$-monomorphism and

$$(\sigma_0 \circ \imath)(\mathsf{x})\varphi = (\varphi|\mathsf{x}) \quad \text{ for any } \varphi, \mathsf{x} \in E.$$

(The scheme above was used to construct a solution u of the nonstationary Navier–Stokes equations in an arbitrary open $\Omega \subset \mathbb{R}^n$ for any $n \geq 2$—see Sec. 4 in Holly–Wiciak [9].) ■

4. Example of the use of Theorems 2.5 and 2.7 when $\dim X = \infty$**.** In Section 3 we have solved an ordinary differential equation of the Carathéodory type in a finite-dimensional Banach space V_N, getting an approximate solution of the original nonlinear PDE. In the present section we will find an *exact* solution of a linear PDE in $\mathbb{R}^n$ using the theory of ODEs of the Carathéodory type for an infinite-dimensional Banach space X.

The approach to evolution PDEs which is announced (and, to a large extent, examined) in this section is, from some point of view, less general than the theory of semigroups of operators. Namely, we need the coefficients of differential operators to be independent of the space variable. Instead, we admit any distribution initial values and any t-summable distribution exterior forces.

Let B be a complex Banach space and $\mathcal{S}$ a semimetrizable locally convex space over $\mathbb{C}$. Consider an arbitrary nonnegative measure μ on a measurable space $(Z, \mathrm{dom}\, \mu)$. The symbol $\mathcal{L}(\mathcal{S}, B)$ will stand for the linear space of all continuous linear operators of $\mathcal{S}$ into B. With a fixed continuous seminorm q on $\mathcal{S}$ we may associate a linear space

$$\mathcal{L}((\mathcal{S}, q), B) := \{T \in \mathcal{L}(\mathcal{S}, B) : |T|_q := \sup_{q(\varphi) \leq 1} |T(\varphi)| < \infty\},$$

where $|T(\varphi)|$ denotes the norm of the vector $T(\varphi)$ in B. Note that $\mathcal{L}((\mathcal{S}, q), B)$, equipped with the norm $|\cdot|_q$, is a Banach space. We say that a mapping $\kappa : Z \to \mathcal{L}(\mathcal{S}, B)$ is (strongly) *summable* if there exists a continuous seminorm $q : \mathcal{S} \to \mathbb{R}_+$ such that

1° $\kappa(Z) \subset \mathcal{L}((\mathcal{S}, q), B)$;
2° $\kappa(Z)$ is a separable metric subspace of $\mathcal{L}((\mathcal{S}, q), B)$;
3° $\forall \mathcal{O} \in \mathrm{top}\, \mathcal{L}((\mathcal{S}, q), B) : \kappa^{-1}(\mathcal{O}) \in \mathrm{dom}\, \mu$;
4° $\int_Z |\kappa(t)|_q \, d\mu < \infty$.

Briefly, this means that κ is summable in the Bochner sense, as a function taking values in $\mathcal{L}((\mathcal{S},q),B)$. Its integral,

$$\int_Z \kappa\, d\mu \qquad (\in \mathcal{L}(\mathcal{S},B)), \tag{34}$$

does not depend on the choice of a seminorm q satisfying 1°–4° (see Chap. I, Sec. 3 of [7]).

With any continuous linear operator $L : \mathcal{S} \to \mathcal{S}$ we may associate the conjugate operator

$$L^\star : \mathcal{L}(\mathcal{S},B) \ni T \overset{\text{def}}{\mapsto} T \circ L \in \mathcal{L}(\mathcal{S},B).$$

For every continuous seminorm $q : \mathcal{S} \to \mathbb{R}_+$ the composition $q \circ L$ is again a continuous seminorm, so that $L^\star$ maps $\mathcal{L}((\mathcal{S},q),B)$ continuously into $\mathcal{L}((\mathcal{S},q\circ L),B)$. Therefore,

(35) if $\kappa : Z \to \mathcal{L}(\mathcal{S},B)$ is summable, then $L^\star \circ \kappa : Z \to \mathcal{L}(\mathcal{S},B)$ is summable and

$$\int_Z (L^\star \circ \kappa) d\mu = L^\star\Big(\int_Z \kappa\, d\mu\Big).$$

We say that a curve $u : [a,b] \to \mathcal{L}(\mathcal{S},B)$ is (strongly) *absolutely continuous* iff there is a summable curve $\kappa : [a,b] \to \mathcal{L}(\mathcal{S},B)$ such that

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} \kappa(t)\, dt \qquad \text{for all } t_1,t_2 \in [a,b].$$

This time $Z = [a,b]$, where $a,b \in \mathbb{R}$, and μ is the linear Lebesgue measure on the σ-algebra $\operatorname{dom}\mu$ of all Borel subsets of $[a,b]$. Properties of the integral (34) and $\mathcal{L}(\mathcal{S},B)$-valued absolutely continuous curves are examined in Holly [7]. We say that a curve $u : \mathcal{J} \to \mathcal{L}(\mathcal{S},B)$, defined on a connected $\mathcal{J} \subset \mathbb{R}$, is *absolutely continuous* if the restriction $u|_{[a,b]}$ is absolutely continuous for any $a,b \in \mathcal{J}$. Then for almost all $t \in \mathcal{J}$ the limit

$$\dot u(t) := \lim_{h\to 0} \frac{u(t+h)-u(t)}{h} \qquad (\in \mathcal{L}(\mathcal{S},B))$$

exists in the inductive topology of $\mathcal{L}(\mathcal{S},B)$ (that is, in the strongest locally convex topology such that the canonical injection $\mathcal{L}((\mathcal{S},q),B) \hookrightarrow \mathcal{L}(\mathcal{S},B)$ is continuous for all continuous seminorms $q : \mathcal{S} \to \mathbb{R}_+$).

From now on, let $\mathcal{S}$ be the Schwartz space, i.e.

$$\mathcal{S} := \{\varphi : \mathbb{R}^n \to \mathbb{C} \mid \varphi \text{ is a rapidly decreasing function}\}.$$

With the vector topology induced by the family of norms

$$q_N : \mathcal{S} \ni \varphi \overset{\text{def}}{\mapsto} \sup\{(1+|\mathsf{x}|^2)^N |(D^\alpha\varphi)(\mathsf{x})| : \mathsf{x} \in \mathbb{R}^n, \alpha \in \mathbb{N}^n, |\alpha| \le N\}$$

($N = 0, 1, \ldots$), where $|\alpha| := \alpha_1 + \ldots + \alpha_n$, $\mathcal{S}$ is a Fréchet space (i.e. it is a locally convex space, metrizable and complete). Next, let $\mathcal{D}(\mathbb{R}^n)$ stand for the space of all complex test functions on $\mathbb{R}^n$. So, every tempered distribution $T : \mathcal{D}(\mathbb{R}^n) \to B$ has a unique continuous extension $\overline{T} : \mathcal{S} \to B$. A curve $u : \mathcal{J} \to \mathcal{D}'_{\text{temp}}$ with values in the space $\mathcal{D}'_{\text{temp}} := \mathcal{D}'_{\text{temp}}(\mathbb{R}^n; B)$ of all tempered distributions $\mathcal{D}(\mathbb{R}^n) \to B$ is said to be *absolutely continuous* if the curve

$$\overline{u} : \mathcal{J} \ni t \overset{\text{def}}{\mapsto} \overline{u(t)} \in \mathcal{L}(\mathcal{S}, B)$$

is absolutely continuous. Then, in particular, for almost all $t \in \mathcal{J}$ the limit

$$\frac{d}{dt} u(t) = \dot{u}(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h} \qquad (\in \mathcal{D}'_{\text{temp}})$$

exists in the sense of pointwise convergence.

Consider the initial-value problem

$$\frac{d}{dt} u(t) = \sum_{\alpha} a_\alpha(t) D^\alpha u(t) + f(t), \tag{36}$$

$$u(t_0) = u_0, \tag{37}$$

where the abbreviation $\sum_\alpha$ means $\sum_{\alpha \in \mathbb{N}^n, |\alpha| \le m}$ for some fixed $m \in \mathbb{N}$, and the "initial state" $u_0 \in \mathcal{D}'_{\text{temp}}$, "external forces" $f : \mathcal{J} \to \mathcal{D}'_{\text{temp}}$ and coefficients $a_\alpha : \mathcal{J} \to \mathbb{C}$ ($\alpha \in \mathbb{N}^n, |\alpha| \le m$) are given. We assume that $\mathcal{J}$ is an interval of the time axis, $t_0 \in \mathcal{J}$, the function a_α is locally summable for any $|\alpha| \le m$, and, finally, the curve f is also *locally summable*, that is, for all $t_1, t_2 \in \mathcal{J}$ the curve $[t_1, t_2] \ni t \mapsto \overline{f(t)} \in \mathcal{L}(\mathcal{S}, B)$ is summable. (Then the integral

$$\int_{t_1}^{t_2} f(t)\, dt := \Big(\int_{t_1}^{t_2} \overline{f(t)}\, dt \Big) \Big|_{\mathcal{D}(\mathbb{R}^n)} : \mathcal{D}(\mathbb{R}^n) \to B$$

is a tempered distribution.)

Suppose that an absolutely continuous curve $u : \mathcal{J} \to \mathcal{D}'_{\text{temp}}$ is a *solution* of the problem $\{(36), (37)\}$, that is, u satisfies (37) and (36) for almost all $t \in \mathcal{J}$. Then, according to (35), the curve

$$\widehat{u} : \mathcal{J} \ni t \overset{\text{def}}{\mapsto} \widehat{u(t)} \in \mathcal{D}'_{\text{temp}}$$

is absolutely continuous and

$$\frac{d}{dt} \widehat{u}(t) = \sum_{\alpha} a_\alpha(t) (i\xi)^\alpha \widehat{u}(t) + \widehat{f}(t) \qquad \text{for a.e. } t \in \mathcal{J},$$

where

$$\widehat{u}(t) = \widehat{u(t)} = \Im(u(t)) : \mathcal{D}(\mathbb{R}^n) \ni \varphi \overset{\text{def}}{\mapsto} \overline{u(t)}(\widehat{\varphi}) \in B$$

is the Fourier transform of the distribution $u(t)$.

Let $t_\star \in \mathcal{J}$, $\psi \in \mathcal{D}(\mathbb{R}^n)$. Then, again by (35), the curve

$$x : \mathcal{J}_\star := [t_0, t_\star] \ni t \overset{\text{def}}{\mapsto} \psi\widehat{u}(t) \in \mathcal{D}'_{\text{temp}}$$

is absolutely continuous and

$$\frac{d}{dt}x(t) = \sum_\alpha a_\alpha(t)(i\xi)^\alpha x(t) + \psi\widehat{f}(t) \quad \text{for a.e. } t \in \mathcal{J}_\star. \tag{38}$$

There exists an integer $N \in \mathbb{N}$ so large that

(a) the curve $\mathcal{J}_\star \ni t \mapsto \overline{\psi\widehat{f}(t)} \in \mathcal{L}((\mathcal{S}, q_N), B)$ is well defined and summable (in the Bochner sense, as a curve with values in $\mathcal{L}((\mathcal{S}, q_N), B)$);

(b) the curve $\overline{x} : \mathcal{J}_\star \to \mathcal{L}((\mathcal{S}, q_N), B)$ is well defined, differentiable almost everywhere and absolutely continuous (in the usual sense).

Consider the Banach space

$$X := \{T \in \mathcal{D}'_{\text{temp}} : \operatorname{supp} T \subset \operatorname{supp}\psi,\ \overline{T} \text{ is } q_N\text{-continuous}\}$$

with the norm

$$|T| := |\overline{T}|_{q_N} = \sup\{|\overline{T}(\varphi)| : \varphi \in \mathcal{S},\ q_N(\varphi) \le 1\}.$$

Because of (38), the absolutely continuous curve $x : \mathcal{J}_\star \to X$ is a solution (in the Carathéodory sense) of the equation (3) with the right-hand side

$$\mathcal{F} : \mathcal{J}_\star \times X \ni (s, T) \overset{\text{def}}{\mapsto} \Big(\sum_\alpha a_\alpha(s)(i\xi)^\alpha\Big)^{\sim}(T) + \psi\widehat{f}(s) \in X,$$

where

$$\widetilde{P} : X \ni T \overset{\text{def}}{\mapsto} PT \in X$$

for any smooth function $P : \mathbb{R}^n \to \mathbb{C}$ (cf. (15)). Moreover, x satisfies (5) with $\mathsf{x}_0 := \psi|\widehat{u}_0|$ ($\in X$). As proved in Holly [7] (Chap. II, Sec. 4), the transformation $P \mapsto \widetilde{P}$ is a continuous homomorphism from the Fréchet algebra $\mathcal{C}^\infty(\mathbb{R}^n)$ (with multiplication of functions) into the Banach algebra $\operatorname{End}X := \mathcal{L}(X, X)$ (with composition of operators); in particular,

$$\{\widetilde{P} : P \in \mathcal{C}^\infty(\mathbb{R}^n)\} \tag{39}$$

is a commutative subalgebra of $\operatorname{End}X$. Of course, X is an infinite-dimensional linear space whenever $\psi \not\equiv 0$ and $B \neq 0$ (e.g. in the case of $B = \mathbb{C}^k$, $k \in \mathbb{N} \setminus \{0\}$). Nevertheless, we will find an *analytic* formula for the solution of (38). It will allow us to get rid of the factor ψ (on which both x and X depend) and to write an original solution u of $\{(36), (37)\}$ as an explicit functional of the data u_0 and f. The reasoning (known sometimes as *analysis of ancients*), leading from any solution of a problem to a concrete formula which determines this solution is of course an ocular proof of uniqueness. Uniqueness itself can usually be obtained in a simpler manner, e.g. by examining the kernel of a suitable linear operator (in the case of linear equations). However, the explicit formula for a solution is, above all, a hint how to *construct* this solution.

Let us consider an abstract Banach space M and a commutative Banach algebra $\mathcal{A}$ such that M is a module over $\mathcal{A}$ and the external multiplication $\mathcal{A} \times M \to M$ is continuous. Consider the following abstract equivalent of (38):

$$\frac{d}{dt}y(t) = A(t)y(t) + C(t). \tag{40}$$

The coefficients $A : \mathcal{J}_\star \to \mathcal{A}$ and $C : \mathcal{J}_\star \to M$ are assumed to be summable, while an absolutely continuous solution $y : \mathcal{J}_\star \to M$ is meant in the Carathéodory sense. By (17) and (18), equation (40) together with the initial condition

$$y(t_0) = y_0 \quad \text{(any fixed vector from } M) \tag{41}$$

has a unique solution defined on the whole $\mathcal{J}_\star$. We will compute it, copying the Euler method, well known in the case of $M = \mathcal{A} = \mathbb{R}$.

Let $\Gamma : \mathcal{J}_\star \to \mathcal{A}$ be an absolutely continuous and almost everywhere differentiable curve with values in the group of invertible elements. The operation of inversion (defined on this group) is of class $\mathcal{C}^1$, so the curve $t \mapsto \Gamma(t)^{-1}$ is also absolutely continuous, a.e. differentiable and

$$0 = \frac{d}{dt}(\Gamma(t)^{-1}\Gamma(t)) = \left(\frac{d}{dt}\Gamma(t)^{-1}\right)\Gamma(t) + \Gamma(t)^{-1}\left(\frac{d}{dt}\Gamma(t)\right)$$

for almost all $t \in \mathcal{J}_\star$. Thus,

$$\frac{d}{dt}\Gamma(t)^{-1} = -\Gamma(t)^{-1}\dot{\Gamma}(t)\Gamma(t)^{-1} \quad \text{for a.e. } t \in \mathcal{J}_\star. \tag{42}$$

We multiply (40) by $\Gamma(t)^{-1}$ and, using (42), compute:

$$\begin{aligned}
\Gamma(t)^{-1}C(t) &= \Gamma(t)^{-1}\dot{y}(t) - \Gamma(t)^{-1}A(t)y(t) \\
&= \Gamma(t)^{-1}\dot{y}(t) + (\frac{d}{dt}\Gamma(t)^{-1})y(t) \\
&\quad + \Gamma(t)^{-2}\dot{\Gamma}(t)y(t) - \Gamma(t)^{-1}A(t)y(t) \\
&= \frac{d}{dt}(\Gamma(t)^{-1}y(t)) + \Gamma(t)^{-2}(\dot{\Gamma}(t) - A(t)\Gamma(t))y(t)
\end{aligned}$$

for almost all $t \in \mathcal{J}_\star$. Hence, taking as Γ the unique absolutely continuous solution of the problem

$$\begin{cases} \dot{a}(t) = A(t)a(t), & (43) \\ a(t_0) = I \ (:= \text{the identity of the algebra } \mathcal{A}), & (44) \end{cases}$$

(see again (17) and (18)) we get

$$\frac{d}{d\tau}(\Gamma(\tau)^{-1}y(\tau)) = \Gamma(\tau)^{-1}C(\tau) \quad \text{for almost all } \tau \in \mathcal{J}_\star. \tag{45}$$

The conclusion is correct because Γ takes values in the group of invertible elements; indeed, denoting by $\widetilde{\Gamma}$ the unique absolutely continuous solution of the equation

$$\dot{a}(t) = -A(t)a(t) \tag{46}$$

with the initial condition (44), we have

$$\frac{d}{dt}(\Gamma(t)\widetilde{\Gamma}(t)) = 0 \quad \text{for almost all } t \in \mathcal{J}_\star$$

and, consequently, $\Gamma(t)\widetilde{\Gamma}(t) = I$ for all $t \in \mathcal{J}_\star$ (the equation (46) was suggested here by the relation (42)). For a fixed $t \in \mathcal{J}_\star$ we integrate (45) from t_0 to t and get

$$\widetilde{\Gamma}(t)y(t) - \widetilde{\Gamma}(t_0)y(t_0) = \int_{t_0}^{t} \widetilde{\Gamma}(\tau)C(\tau)\,d\tau,$$

or

$$y(t) = \Gamma(t)(y_0 + \int_{t_0}^{t} \widetilde{\Gamma}(\tau)C(\tau)\,d\tau). \tag{47}$$

The element $\Gamma(t)$ of the algebra $\mathcal{A}$ is denoted by $\prod_{t_0}^{t} e^{A(\tau)d\tau}$ and called the *product integral* of the curve Γ from t_0 to t (see Dollard–Friedman [4]). From (20), it is easy to infer the continuous dependence of the product integral on the itegrand, namely

$$\prod_{t_0}^{t} e^{A_\nu(\tau)d\tau} \xrightarrow{\nu\to\infty} \prod_{t_0}^{t} e^{A(\tau)d\tau} \text{ in } \mathcal{A} \text{ whenever } A_\nu \xrightarrow{\nu\to\infty} A \text{ in } L^1(\mathcal{J}_\star, \mathcal{A}). \tag{48}$$

Thanks to commutativity of $\mathcal{A}$, the integral $\prod_{t_0}^{t} e^{A(\tau)d\tau}$ can be "computed". We will do this in three steps.

First, assume that $A = \text{const}$, say $A \equiv \overline{A} \in \mathcal{A}$, and that a family $\mathcal{L}$ of non-zero multiplicative linear functionals on $\mathcal{A}$ separates points in $\mathcal{A}$. Then for any $l \in \mathcal{L}$ the function $\mathcal{J}_\star \ni t \mapsto l(\prod_{t_0}^{t} e^{A(\tau)d\tau}) \in \mathbb{C}$ is a (classical) solution of the problem

$$\dot{z}(t) = l(\overline{A})z(t), \qquad z(t_0) = 1 \ (\in \mathbb{C}),$$

so, for a fixed $t \in \mathcal{J}_\star$,

$$\begin{aligned} l\Big(\prod_{t_0}^{t} e^{A(\tau)d\tau}\Big) &= e^{(t-t_0)l(\overline{A})} = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}(l((t-t_0)\overline{A}))^\nu \\ &= l\Big(\sum_{\nu=0}^{\infty} \frac{((t-t_0)\overline{A})^\nu}{\nu!}\Big) = l(e^{(t-t_0)\overline{A}}). \end{aligned}$$

In view of the arbitrariness of l,

$$\prod_{t_0}^{t} e^{A(\tau)d\tau} = e^{(t-t_0)\overline{A}}. \tag{49}$$

The formula (49) can be proved directly. If, additionally, R is an invertible element of the algebra $\mathcal{A}$, then the solution of (43) with the initial condition

$$a(t_0) = R, \tag{50}$$

after multiplying by R^{-1}, becomes a solution of $\{(43), (44)\}$; thus,

(51) $\mathcal{J}_\star \ni t \mapsto Re^{(t-t_0)\overline{A}} \in \mathcal{A}$ is a solution of problem $\{(43), (50)\}$.

Next, assume that A is a simple curve, i.e.

$$\forall t \in \mathcal{J}_\star : \quad A(t) = \sum_{j=1}^{N} \chi_{[t_{j-1},t_j[}(t)\overline{A}_j$$

for some $\overline{A}_1, \dots, \overline{A}_N \in \mathcal{A}$ and $t_0 < t_1 < \dots < t_N = t_\star$ (in the case when $t_0 < t_\star$). Using (51) successively in the intervals $[t_0, t_1[, [t_1, t_2[, \dots$ we get

$$\prod_{t_0}^{t} e^{A(\tau)d\tau} = e^{(t_1-t_0)\overline{A}_1} \dots e^{(t_{j-1}-t_{j-2})\overline{A}_{j-1}} e^{(t-t_{j-1})\overline{A}_j}$$

for $t \in [t_{j-1}, t_j[$, $1 \le j \le N$. In other words,

$$\prod_{t_0}^{t} e^{A(\tau)d\tau} = \exp\Big(\int_{t_0}^{t} A(\tau)\,d\tau\Big) \quad \text{for all } t \in \mathcal{J}_\star. \tag{52}$$

The analogous reasoning leads to (52) when $t_0 > t_\star$.

In the general case the sequence of simple curves

$$A_\nu(t) := \sum_{j=1}^{\nu} \chi_{[s_{j-1}^{(\nu)},s_j^{(\nu)}[}(t)\frac{1}{s_j^{(\nu)} - s_{j-1}^{(\nu)}} \int_{s_{j-1}^{(\nu)}}^{s_j^{(\nu)}} A(\tau)\,d\tau,$$

where $s_j^{(\nu)} := t_0 + \frac{j}{\nu}(t_\star - t_0)$, converges to A in $L^1(\mathcal{J}_\star, \mathcal{A})$ as $\nu \to \infty$ (which can be easily shown through density of the subspace of continuous functions in $L^1(\mathcal{J}_\star, \mathcal{A})$). Each curve A_ν satisfies (52), so, thanks to (48), also A satisfies (52).

Finally, the formula (47) takes the following concrete form:

$$y(t) = \Big(\exp\int_{t_0}^{t} A(s)\,ds\Big)\Big(y_0 + \int_{t_0}^{t}\Big(\exp\int_{\tau}^{t_0} A(s)\,ds\Big)C(\tau)\,d\tau\Big). \tag{53}$$

Taking as $\mathcal{A}$ the closure of the subalgebra (39) in $\mathrm{End}X$ and putting

$$M = X, \qquad A(t) = \Big(\sum_\alpha a_\alpha(t)(i\xi)^\alpha\Big)^{\sim}, \qquad C(t) = \psi\widehat{f}(t), \qquad y_0 = \mathsf{x}_0 = \psi\widehat{u}_0,$$

we get

$$x(t) = e^{\mathop{\mathrm{T}}\limits_{t_0}^{t}(\sum_\alpha a_\alpha(s)(i\xi)^\alpha)^{\sim} ds}\Big(\psi\widehat{u}_0 + \int_{t_0}^{t} e^{\mathop{\mathrm{T}}\limits_{\tau}^{t_0}(\sum_\alpha a_\alpha(s)(i\xi)^\alpha)^{\sim} ds}\psi\widehat{f}(\tau)\,d\tau\Big). \tag{54}$$

Assume that for all $t_1 \in \mathcal{J}$,

$$\sup\Big\{\mathrm{Re}\Big(\sum_\alpha (i\xi)^\alpha \int_s^t a_\alpha(\tau)\,d\tau\Big) : \xi \in \mathbb{R}^n,\ t_0 \le s \le t \le t_1\Big\} < \infty. \tag{55}$$

Then, in particular, for any $t, s \in \mathcal{J}$ satisfying $s \in [t_0, t]$ the function

$$\eta_s^t : \mathbb{R}^n \ni \xi \overset{\mathrm{def}}{\mapsto} \exp\Big(\sum_\alpha (i\xi)^\alpha \int_s^t a_\alpha(\tau)\,d\tau\Big) \in \mathbb{C}$$

is polynomially bounded together with all its derivatives. Since at the same time $e^{\widetilde{P}} = (e^P)^{\sim}$ for each $P \in \mathcal{C}^\infty(\mathbb{R}^n)$, as shown in Holly [7] (Chap. II, Sec. 2), we can get rid of the product integrals in (54):

$$x(t) = \psi\eta_{t_0}^t\widehat{u}_0 + \underbrace{\int_{t_0}^t \psi\eta_\tau^t\widehat{f}(\tau)\,d\tau}_{\text{integral in } X} = \psi\Big(\eta_{t_0}^t\widehat{u}_0 + \underbrace{\int_{t_0}^t \eta_\tau^t\widehat{f}(\tau)\,d\tau}_{\text{integral in } \mathcal{D}'_{\mathrm{temp}}}\Big). \tag{56}$$

Above, we have additionally factored out the function ψ; it was possible in view of (35) applied to the operator $L : \mathcal{S} \ni \varphi \mapsto \varphi\psi \in \mathcal{S}$. The equality (56) holds for all $t \in \mathcal{J}_\star$. In particular,

$$\psi\widehat{u}(t_\star) = x(t_\star) = \psi\Big(\eta_{t_0}^{t_\star}\widehat{u}_0 + \int_{t_0}^{t_\star} \eta_\tau^{t_\star}\widehat{f}(\tau)\,d\tau\Big).$$

In view of the arbitrariness of ψ,

$$\widehat{u}(t_\star) = \eta_{t_0}^{t_\star}\,\widehat{u}_0 + \int_{t_0}^{t_\star} \eta_\tau^{t_\star}\widehat{f}(\tau)\,d\tau.$$

In view of the arbitrariness of $t_\star$,

$$u(t) = \Im^{-1}\Big(\eta_{t_0}^t\widehat{u}_0 + \int_{t_0}^t \eta_\tau^t\widehat{f}(\tau)\,d\tau\Big) \qquad \text{for all } t \in \mathcal{J}. \tag{57}$$

The reasoning above is the proof of uniqueness for the Cauchy problem $\{(36), (37)\}$ and, simultaneously, it can lead to a construction of solution. In [7] it was proved that, indeed, the curve given by (57) *is* the required solution. Moreover, in virtue of (35), we have $\Im^{-1}\int_{t_0}^{t} = \int_{t_0}^{t}\Im^{-1}$. So, finally, the curve $u : \mathcal{J} \to \mathcal{D}'_{\mathrm{temp}}$ given by

$$(58)\quad u(t) = \Im^{-1}\big(e^{\sum_\alpha (i\xi)^\alpha \int_{t_0}^{t} a_\alpha(\tau)\,d\tau}\,\widehat{u}_0\big) + \int_{t_0}^{t} \Im^{-1}\big(e^{\sum_\alpha (i\xi)^\alpha \int_{s}^{t} a_\alpha(\tau)\,d\tau}\cdot \widehat{f(s)}\big)\,ds$$

is the unique solution of the problem $\{(36), (37)\}$ whenever (55) holds.

The condition (55) is satisfied e.g. in the case of the *heat equation* ($t_0 = \inf\mathcal{J}$, $a_{2e_k} \equiv \sigma > 0$ for $1 \le k \le n$ and $a_\alpha \equiv 0$ for $\alpha \in \mathbb{N}^n \setminus \{2e_1,\dots,2e_n\}$) and in the case of the *Schrödinger equation* ($t_0 \in \mathcal{J}$ arbitrary, $a_{2e_k} \equiv i\sigma$ for $1 \le k \le n$ and $a_\alpha \equiv 0$ for $\alpha \in \mathbb{N}^n \setminus \{2e_1,\dots,2e_n\}$).

5. Appendix. In general, if our Banach space X is not reflexive, then a curve $x : [a,b] \to X$ which is absolutely continuous in the traditional sense mentioned at the beginning of Section 2 need not be a function of the upper limit of integration of some summable curve (see Yosida [19]). However, it is so if x is differentiable almost everywhere. This follows from

THEOREM 5.1. *Let $x : I \to X$ be an absolutely continuous* (*in the usual sense*) *curve mapping a compact interval $I \subset \mathbb{R}$ into a Banach space X. Then*

(a) *the set*

$$\operatorname{dom}\dot{x} := \{t \in I : \textit{the function } h \mapsto (x(t+h)-x(t))/h \textit{ satisfies the Cauchy condition as } h \to 0\}$$

is a Borel set (*in* $\mathbb{R}$), *and the curve*

$$\dot{x} : \operatorname{dom} x \ni t \overset{\mathrm{def}}{\mapsto} \lim_{h\to 0}\frac{x(t+h)-x(t)}{h} \in X$$

is Borel measurable while its range is separable;

(b) $\int_{\operatorname{dom}\dot{x}} |\dot{x}(t)|\,dt < \infty$ (*so $\dot{x}$ is summable in the Bochner sense with respect to the one-dimensional Lebesgue measure*);

(c) *provided $I \setminus \operatorname{dom}\dot{x}$ is a set of measure zero*, *we have*

$$\forall t_1, t_2 \in \operatorname{dom} x,\ t_1 \le t_2 : \quad x(t_2) - x(t_1) = \int_{[t_1,t_2]\cap\operatorname{dom}\dot{x}} \dot{x}(t)\,dt.$$

The last integral is, for short, usually written as

$$\int_{t_1}^{t_2} \dot{x}(t)\,dt.$$

This abbreviation is often used in the paper, also when talking about oriented integrals (with a minus sign if $t_1 \ge t_2$). The symbol $|\dot{x}(t)|$ in claim (b) stands for the norm of the vector $\dot{x}(t)$. To be completely precise, let us finally mention that the difference quotient in (a) is defined in a both-sided or one-sided neighbourhood of zero, depending on whether $t \in \operatorname{int} I$ or $t \in \partial I$. (It is also possible, without losing generality, to regard $\dot{x}(t)$ only for $t \in \operatorname{int} I$.)

We suspect that the lemma below is known, but, for the reader's convenience, we give its proof. Our original proof was based on the Hahn–Banach theorem. The present elementary reasoning is due to J. Mateja [12].

LEMMA 5.2. *Consider the retraction R of a normed space X onto its unit ball, given by*

$$R(\mathsf{x}) = \begin{cases} \mathsf{x} & \text{if } |\mathsf{x}| \le 1, \\ \mathsf{x}/|\mathsf{x}| & \text{if } |\mathsf{x}| \ge 1, \end{cases}$$

where $|\cdot|$ is the norm of X. Then R satisfies the Lipschitz condition with constant 2.

Proof. Of course, R satisfies the Lipschitz condition with constant 1 in the unit ball. Let x_1 and x_2 be not in the unit ball. Then

$$\begin{aligned} |R(\mathsf{x}_1) - R(\mathsf{x}_2)| &= \left| \frac{|\mathsf{x}_2|\mathsf{x}_1 - |\mathsf{x}_1|\mathsf{x}_2}{|\mathsf{x}_1|\,|\mathsf{x}_2|} \right| = \frac{\big|\,|\mathsf{x}_2|(\mathsf{x}_1 - \mathsf{x}_2) + (|\mathsf{x}_2| - |\mathsf{x}_1|)\mathsf{x}_2\big|}{|\mathsf{x}_1|\,|\mathsf{x}_2|} \\ &\le \frac{|\mathsf{x}_1 - \mathsf{x}_2| + \big|\,|\mathsf{x}_2| - |\mathsf{x}_1|\,\big|}{|\mathsf{x}_1|} \le 2|\mathsf{x}_1 - \mathsf{x}_2|. \end{aligned}$$

So, it remains to prove $|R(a) - R(b)| \le 2|a-b|$ for $|a| \le 1$ and $|b| > 1$. Let $|a| \le 1$ and $|b| > 1$. Then

$$\begin{aligned} |R(a) - R(b)| &= \left| a - \frac{b}{|b|} \right| \le \left| a - \frac{a}{|b|} \right| + \left| \frac{a}{|b|} - \frac{b}{|b|} \right| \\ &\le |a| \cdot \left| 1 - \frac{1}{|b|} \right| + \frac{1}{|b|} \cdot |a-b| = \frac{1}{|b|}(|a|(|b|-1) + |a-b|) \\ &\le |a| \cdot |b| - |a| + |a-b| \le |b| - |a| + |a-b| \le 2|a-b|, \end{aligned}$$

which completes the proof. ■

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