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On the multivariate transfinite diameter

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Abstract. We prove several new results on the multivariate transfinite diameter and its connection with pluripotential theory: a formula for the transfinite diameter of a general product set, a comparison theorem and a new expression involving Robin's functions. We also study the transfinite diameter of the pre-image under certain proper polynomial mappings.

1. Introduction. We present several new results on the multivariate transfinite diameter which, we believe, should clarify its rather close connection with well-known objects of pluripotential theory. We shall first recall the definition, fix the notation and provide the necessary background. An outline of the paper appears at the end of this introductory section.

The space of all polynomials in $\mathbb{C}^n$ is denoted by $\mathcal{P}(\mathbb{C}^n)$ and the subspace of polynomials of degree at most d by $\mathcal{P}_d(\mathbb{C}^n)$. The dimension of the latter is $N := N_d(n) := \binom{d+n}{d}$, which is also the number of multi-indices whose length does not exceed d. We arrange the multi-indices in a sequence (α_i), $i = 1, 2, \ldots$, such that $\alpha_i \prec \alpha_{i+1}$ for every i where $\prec$ is the usual graded lexicographic order. Recall that this order is defined by $\alpha \prec \beta$ if either $|\alpha| \le |\beta|$ or $|\alpha| = |\beta|$ but the first (starting from the left) non-zero entry of $\alpha-\beta$ is negative. Thus, for example, $\alpha_1 = (0, \ldots, 0, 0)$ and $\alpha_N = (d, 0, \ldots, 0)$ and the z^{α_i}, $i = 1, \ldots, N_d(n)$, form the usual monomial basis of $\mathcal{P}_d(\mathbb{C}^n)$.

The *Vandermonde determinant* of a collection of $N_d(n)$ points $z_i = (z_{i1}, \ldots, z_{in}) \in \mathbb{C}^n$ is the $N \times N$ determinant defined by

$$\mathrm{VDM}(z_1, \ldots, z_N) = \det(z_i^{\alpha_j})_{i,j=1}^N. \tag{1.1}$$

As a function of the $N \times n$ complex variables z_{ij}, VDM is a polynomial of

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degree

$$l_d = \sum_{j=1}^{d} j h_j(n) = n\binom{n+d}{n+1} \quad \text{where} \quad h_j(n) := \binom{n+j-1}{j} \tag{1.2}$$

is the dimension of the space of homogeneous polynomials of degree j.

Now, given a compact set $E \subset \mathbb{C}^n$, we define the dth *diameter* of E by

$$D_d(E) = \sqrt[l_d]{\sup_{z_i \in E} |\mathrm{VDM}(z_1, \ldots, z_N)|}. \tag{1.3}$$

When $n = 1$, we have $N_d = d + 1$, $|\mathrm{VDM}(z_1, \ldots, z_N)| = \prod_{1 \le i < j \le N} |z_i - z_j|$ and $l_d = \binom{d+1}{2}$ so that $D_d(E)$ converges to the classical transfinite diameter of Fekete that coincides—this is a basic result of potential theory in the complex plane—with the logarithmic capacity. The question whether or not the sequence $D_d(E)$ converges as well for every compact set $E \subset \mathbb{C}^n$ was posed by Leja in 1959 and answered affirmatively by Zakharyuta [Za] in 1975. This limit

$$D(E) = \lim_{d \to \infty} D_d(E) \tag{1.4}$$

is naturally called the (*multivariate*) *transfinite diameter*. The proof of this result provides an interesting link with approximation theory.

Given a compact set $E \subset \mathbb{C}^n$ and a multi-index α we define

$$\mathcal{T}(\alpha, E) = \inf\Big\{\Big\| z^\alpha + \sum_{\beta \prec \alpha} a_\beta z^\beta \Big\|_E\Big\} \tag{1.5}$$

where the infimum runs over all the possible choices of the coefficients a_β. A polynomial $t_{\alpha,E}(z) = z^\alpha + \sum_{\beta \prec \alpha} a_\beta z^\beta$ for which the infimum above is attained, i.e.

$$\|t_{\alpha,E}\|_E = \mathcal{T}(\alpha, E), \tag{1.6}$$

will be termed a $\mathcal{P}(\alpha)$-*minimal polynomial*. Such a polynomial is, in general, not unique.

Now, for every $\theta \in \Sigma_0$ where

$$\Sigma_0 := \Big\{\theta \in \mathbb{R}^n : \sum_{i=1}^{n} \theta_i = 1,\ \theta_i > 0\ (i = 1, \ldots, n)\Big\} \tag{1.7}$$

is the interior of the standard simplex in $\mathbb{R}^n$, Zakharyuta [Za] proved that the following limit exists:

$$\tau(E, \theta) = \lim_{\alpha/|\alpha| \to \theta} \mathcal{T}(\alpha, E)^{1/|\alpha|}, \qquad |\alpha| \to \infty, \tag{1.8}$$

and established the existence of $D(E)$ via the following remarkable formula:

$$D(E) = \exp\left(\frac{1}{m(\Sigma_0)} \int_{\Sigma_0} \log(\tau(E,\theta))\, dm(\theta)\right). \tag{1.9}$$

The numbers $\tau(E,\theta)$ are called the *directional Chebyshev constants* of E. In the course of the proof, Zakharyuta [Za, Lem. 6, p. 356] also showed that

$$D(E) = \lim_{d\to\infty} \Big(\prod_{|\alpha|=d} \mathcal{T}(\alpha, E)\Big)^{1/(dh_d)}, \tag{1.10}$$

which will be used in the sequel.

This multivariate transfinite diameter is involved in several questions of pluripotential or approximation theory but few relevant results can be found in the literature. We list here some known results.

(i) (Transfinite diameter of a product of planar compact sets) If $E_i \subset \mathbb{C}$, $i = 1, \ldots, n$, and $E = E_1 \times \ldots \times E_n$ then

$$D(E) = \sqrt[n]{D(E_1)\ldots D(E_n)}.$$

This is a result of Schiffer and Siciak [SS].

(ii) (Sheĭnov's formula) If $A \in \mathrm{GL}_n(\mathbb{C})$ and $E \subset \mathbb{C}^n$ then

$$D(A(E)) = \sqrt[n]{|\det A|} \cdot D(E). \tag{1.11}$$

The original (elementary) proof of [Sh] is somewhat cumbersome. In fact the result is not difficult to establish directly from the definition when A is a diagonal matrix while when A is unitary, Levenberg and Taylor have given a fairly simple proof in [LT]. Since unitary and diagonal automorphisms generate $\mathrm{GL}_n(\mathbb{C})$, formula (1.11) follows.

(iii) (Continuity under decreasing sequences of compact sets) If $E_i \subset \mathbb{C}^n$ is a decreasing sequence ($E_i \supset E_{i+1}$, $i = 1, 2, \ldots$) of compact sets such that $E = \bigcap_{i=1}^{\infty} E_i$ then $D(E_i) \searrow D(E)$ as i tends to ∞. This is a result of Znamienskiĭ, subsequently (independently) proved by Levenberg [Le].

(iv) $D(E)$ has been computed for balls of the form $E = \{\sum_{i=1}^{n} |z_i|^{p_i} \le M\}$ by Jędrzejowski [Je] and for $E = \{x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2 \subset \mathbb{C}^2$ by Bos who was motivated by a problem on multivariate Lagrange interpolation (see [Bo] and [BBCL]).

(v) Finally we mention that there exist some comparison theorems between the transfinite diameter and the logarithmic capacity (see [LT]). In particular $D(E) = 0$ if and only if E is pluripolar.

In the next section we shall extend the result (i) above to the case where the factor sets are not necessarily plane. The third section will exhibit a close relationship between the transfinite diameter and the Robin function of a regular (see below) compact set. Several applications will be given. The final section will study the transfinite diameter of the pre-image of a compact set under suitable polynomial mappings, thus giving a partial generalization of the corresponding classical one variable theorem of Fekete.

We assume that the reader is familiar with the basic notions of pluripotential theory—the standard reference is the book of Klimek [Kl]—but we shall provide the necessary background on Robin's functions. Let us just recall here that a compact set $E \subset \mathbb{C}^n$ is said to be *regular* if its pluricomplex Green function V_E is continuous on $\mathbb{C}^n$.

2. Transfinite diameter of a product set. The purpose of this section is to prove the following

THEOREM 1. *Let* $E \subset \mathbb{C}^n$ *and* $F \subset \mathbb{C}^m$ *be compact sets. Then*

$$D(E \times F) = \sqrt[n+m]{D^n(E) \cdot D^m(F)}. \tag{2.1}$$

The proof will use a connection of independent interest between the transfinite diameter and orthogonal polynomials with respect to suitable probability measures.

Let μ be a probability measure on a compact set $E \subset \mathbb{C}^n$. We say that μ satisfies the *Bernstein–Markov inequality* if for every $\varepsilon > 0$, there exists a finite positive constant $M = M(\varepsilon)$ such that for every (analytic) polynomial p we have

$$\|p\|_E \le M(1+\varepsilon)^{\deg p} \|p\|_\mu, \tag{2.2}$$

where $\|p\|_\mu^2 := \int |p|^2 \, d\mu$. Roughly, (2.2) means that the $L^2(\mu)$ norm and the supremum norm of polynomials are asymptotically comparable. For such measures and for E unisolvent the monomials z^α are linearly independent and, using the standard Gram–Schmidt procedure, we can therefore form the sequence of monic orthogonal polynomials p_α, that is, polynomials p_α of the form

$$p_\alpha(z) = z^\alpha + \sum_{\beta \prec \alpha} c_\beta z^\beta \tag{2.3}$$

such that

$$\beta \prec \alpha \Rightarrow (p_\alpha, z^\beta) = 0$$

where $(\cdot,\cdot)$ is the hermitian product of $L^2(\mu)$.

LEMMA 1. *Let E be a compact unisolvent set in $\mathbb{C}^n$ and μ a probability measure on E satisfying* (2.2). *Then*

$$D(E) = \lim_{d\to\infty} \Big(\prod_{|\alpha|=d} \|p_\alpha\|_\mu \Big)^{1/(dh_d)}. \tag{2.4}$$

NOTE. The assumption of this lemma is satisfied on every regular compact set for example by the *equilibrium measure* μ_E of E (see [Kl, Cor. 5.6.7]) and [NZ]).

Proof. Due to the L^2 minimality property of monic orthogonal polynomials we have, for every multi-index α,

$$\|p_\alpha\|_\mu \le \|t_{\alpha,E}\|_\mu \le \|t_{\alpha,E}\|_E = \mathcal{T}(\alpha, E) \tag{2.5}$$

while, given $\varepsilon > 0$, the Bernstein–Markov inequality (2.2) gives

$$\mathcal{T}(\alpha, E) = \|t_{\alpha,E}\|_E \le \|p_\alpha\|_E \le M(1+\varepsilon)^{|\alpha|}\|p_\alpha\|_\mu. \tag{2.6}$$

Inequalities (2.5) and (2.6) give upper and lower bounds for $\prod_{|\alpha|=d}\|p_\alpha\|_\mu$ in terms of $\prod_{|\alpha|=d}\mathcal{T}(\alpha, E)$ and the lemma follows from (1.10) since ε can be made arbitrarily small. ■

NOTE. Under the same hypothesis but using (1.8) rather than (1.10) it follows (see [Le]) that

$$\tau(E, \theta) = \lim_{\alpha/|\alpha|\to\theta} \|p_\alpha\|_\mu^{1/|\alpha|}, \qquad |\alpha| \to \infty. \tag{2.7}$$

(This will be used in Section 3.)

Lemma 1 shows that, as in the one variable case, the transfinite diameter reflects the asymptotic behavior of orthogonal polynomials with respect to a measure satisfying (2.2). It would be interesting to know if the equality (2.4) is a sufficient condition for μ to satisfy (2.2) on E.

The idea of applying orthogonal polynomials to the study of the multivariate transfinite diameter is not new. Bos [Bo] employed it in a slightly different formulation. Roughly, he proved that the Vandermonde determinant (1.1) can be replaced by the Gram determinant of the z^{α_i}, $i = 1, \ldots, N_d$, in the definition of the transfinite diameter.

The following remark should explain the interest in considering orthogonal polynomials for computing the transfinite diameter of a product set.

Let μ_1 (resp. μ_2) be a measure satisfying (2.2) on $E \subset \mathbb{C}^n$ (resp. $F \subset \mathbb{C}^m$) for which the orthogonal polynomials can be constructed. Then we can also construct the orthogonal polynomials with respect to the product measure $\mu_1 \otimes \mu_2$ and these polynomials can be easily expressed in terms of the former. Here is a precise statement.

If α is an $n+m$-multi-index, we write $\alpha = (\alpha_1, \alpha_2)$ where α_1 (resp. α_2) is an n-multi-index (resp. an m-multi-index). We denote by p, q and r (properly indexed) the monic orthogonal polynomials for the measures $\mu := \mu_1 \otimes \mu_2$, μ_1, μ_2 respectively. Recall that the three families are constructed using the graded lexicographic order in the corresponding space. It is easily seen that if $\alpha = (\alpha_1, \alpha_2)$ then

$$p_\alpha(z) = q_{\alpha_1}(z_1) r_{\alpha_2}(z_2), \qquad z = (z_1, z_2) \in \mathbb{C}^n \times \mathbb{C}^m, \tag{2.8}$$

and consequently

$$\|p_\alpha\|_\mu = \|q_{\alpha_1}\|_{\mu_1}\|r_{\alpha_2}\|_{\mu_2}. \tag{2.9}$$

Indeed, this follows from the fact that if $\alpha = (\alpha_1, \alpha_2) \prec \beta = (\beta_1, \beta_2)$ then $\alpha_1 \prec \beta_1$ or $\alpha_2 \prec \beta_2$.

We shall need the following

LEMMA 2. *Let μ_1 (resp. μ_2) be a measure satisfying* (2.2) *on $E \subset \mathbb{C}^n$ (resp. $F \subset \mathbb{C}^m$) with E, F unisolvent. Then $\mu_1 \otimes \mu_2$ satisfies* (2.2) *on $E \times F \subset \mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m$.*

Proof. We use the previous discussion. Fix $\varepsilon > 0$. We want to find a positive constant C such that

$$\|p\|_{E\times F} \le C(1+\varepsilon)^{\deg p}\|p\|_\mu \quad (p \in \mathcal{P}(\mathbb{C}^{n+m})). \tag{2.10}$$

Since both μ_1 and μ_2 are regular, we can find positive constants C_1 and C_2 such that for every $P \in \mathcal{P}(\mathbb{C}^n)$ and $Q \in \mathcal{P}(\mathbb{C}^m)$ we have

$$\|P\|_E \le C_1(1+\varepsilon/2)^{\deg P}\|P\|_{\mu_1}, \quad \|Q\|_F \le C_2(1+\varepsilon/2)^{\deg Q}\|Q\|_{\mu_2}.$$

Now, take $p \in \mathcal{P}(\mathbb{C}^{n+m})$ and set $d := \deg p$. Using the orthogonal polynomial basis of (2.8), we can write

$$p = \sum_{|\alpha|\le d} a_\alpha p_\alpha \tag{2.11}$$

where $\|p_\alpha\|_\mu^2 a_\alpha = (p, p_\alpha)$. Using the Cauchy–Schwarz inequality and (2.8) we deduce

$$\begin{aligned}
\|p\|_{E\times F} &\le \|p\|_\mu \sum_{|\alpha|\le d} \frac{\|p_\alpha\|_{E\times F}}{\|p_\alpha\|_\mu} \\
&\le \|p\|_\mu \sum_{|\alpha|\le d} \frac{\|q_{\alpha_1}\|_E\|r_{\alpha_2}\|_F}{\|q_{\alpha_1}\|_{\mu_1}\|r_{\alpha_2}\|_{\mu_2}} \\
&\le \|p\|_\mu C_1C_2 \sum_{|\alpha|\le d} (1+\varepsilon/2)^{|\alpha_1|}(1+\varepsilon/2)^{|\alpha_2|} \\
&\le C_1C_2\|p\|_\mu N_d(m+n)(1+\varepsilon/2)^d \le C_3\|p\|_\mu(1+\varepsilon)^d.
\end{aligned}$$

(The last inequality holds true because, as $d \to \infty$, $N_d(n+m)$ grows slower than any δ^d with $\delta > 1$.) This shows that $\mu_1 \otimes \mu_2$ satisfies (2.2) and the lemma is proved. ■

REMARK. Z. Błocki [Blo] has shown that $\mu_{E\times F} = \mu_E \otimes \mu_F$ (answering a question of J. Szczepański).

Proof of Theorem 1. It suffices to prove formula (2.1) in the case where both E and F are *regular*. The general case can be deduced by using Property (iii) (in the introduction) and approximating E and F by sequences of decreasing regular compact sets. Such sequences always exist (see [Kl, Cor. 5.1.5]).

Now take a probability measure μ_1 (resp. μ_2) satisfying (2.2) on E (resp. F). This is possible since these compact sets are regular. Then, by Lemmas 1 and 2, using again the orthogonal polynomials p_α with respect to $\mu = \mu_1 \otimes \mu_2$ and the notation $\alpha = (\alpha_1, \alpha_2)$ we have

$$D(E \times F) = \lim_{d\to\infty} \Big(\prod_{|\alpha|=d} \|p_\alpha\|_\mu\Big)^{1/(dh_d(n+m))},$$

hence

$$D(E \times F) = \lim_{d\to\infty} \Big(\prod_{|\alpha|=d} \|q_{\alpha_1}\|_{\mu_1} \|r_{\alpha_2}\|_{\mu_2}\Big)^{1/(dh_d(n+m))}. \tag{2.12}$$

Define sequences (u_k) and (v_k) by

$$u_k := \prod_{|\alpha_1|=k} \|q_{\alpha_1}\|_{\mu_1}, \quad v_k := \prod_{|\alpha_2|=k} \|r_{\alpha_2}\|_{\mu_2},$$

and take $0 < \lambda < 1 < \Lambda < \infty$. According to Lemma 1, there exist four positive constants $c_1 < C_1$ and $c_2 < C_2$ such that

$$c_1(\lambda D(E))^{kh_k(n)} \le u_k \le C_1(\Lambda D(E))^{kh_k(n)} \quad (k \in \mathbb{N}), \tag{2.13}$$

$$c_2(\lambda D(F))^{kh_k(m)} \le v_k \le C_2(\Lambda D(F))^{kh_k(m)} \quad (k \in \mathbb{N}). \tag{2.14}$$

Now, returning to (2.12), we have

$$\prod_{|\alpha|=d} \|q_{\alpha_1}\|_{\mu_1} \|r_{\alpha_2}\|_{\mu_2} = \prod_{|\alpha|=d} \|q_{\alpha_1}\|_{\mu_1} \prod_{|\alpha|=d} \|r_{\alpha_2}\|_{\mu_2} = \prod_{k=0}^{d} u_k^{h_{d-k}(m)} \prod_{k=0}^{d} v_k^{h_{d-k}(n)}.$$

Using (2.13) and (2.14) we can find lower and upper bounds of the quantity above that involve $D(E)$ to the power $\sum_{k=0}^{d} kh_k(n)h_{d-k}(m)$ and $D(F)$ to the power $\sum_{k=0}^{d} kh_k(m)h_{d-k}(n)$. Assume for a while that for every integer m and n,

$$\sum_{k=0}^{d} kh_k(n)h_{d-k}(m) = \frac{n}{n+m} dh_d(n+m);$$

then the lower bound of $\prod_{|\alpha|=d} \|q_{\alpha_1}\|_{\mu_1} \|r_{\alpha_2}\|_{\mu_2}$ is precisely

$$c_1^{N_d(m)}(\lambda D(E))^{\frac{n}{n+m} dh_d(n+m)} \cdot c_2^{N_d(n)}(\lambda D(F))^{\frac{m}{n+m} dh_d(n+m)}$$

while the upper bound is given by

$$C_1^{N_d(m)}(\Lambda D(E))^{\frac{n}{n+m} dh_d(n+m)} \cdot C_2^{N_d(n)}(\Lambda D(F))^{\frac{m}{n+m} dh_d(n+m)}.$$

Since $\lim_{d\to\infty} N_d(n)/(dh_d(n+m)) = 0$, we deduce, by taking roots, that

$$(\lambda D(E))^{n/(n+m)}(\lambda D(F))^{m/(n+m)} \le \varliminf_{d\to\infty} \Big(\prod_{|\alpha|=d} \|p_\alpha\|_\mu\Big)^{1/(dh_d(n+m))}$$

and

$$\varlimsup_{d\to\infty} \Big(\prod_{|\alpha|=d} \|p_\alpha\|_\mu\Big)^{1/(dh_d(n+m))} \le (\Lambda D(E))^{n/(n+m)}(\Lambda D(F))^{m/(n+m)}.$$

Since λ (< 1) and Λ (> 1) can be taken arbitrarily close to 1 the desired formula follows. This proof will therefore be completed as soon as the simple combinatorial lemma below is proved. ■

REMARK. As pointed out by Prof. Siciak, it is sufficient to prove the result for the δ-neighborhoods of a compact set E and on these sets the Lebesgue measure itself is known to satisfy the Bernstein–Markov inequality. Also the spirit of our proof can be compared with that of Siciak's [Si1] proof of the product formula for the extremal function.

LEMMA 3. *Let m and n be positive integers. Then*

$$\sum_{k=0}^{d} kh_k(n)h_{d-k}(m) = \frac{n}{n+m} dh_d(n+m).$$

Proof. Set $c_d := \sum_{k=0}^{d} kh_k(n)h_{d-k}(m)$ and consider the formal power series F_1 and F_2 defined by

$$\begin{aligned} F_1(X) &= \sum_{d=0}^{\infty} dh_d(n)X^d = X\sum_{d=1}^{\infty} dh_d(n)X^{d-1} \\ &= X\frac{d}{dX}\frac{1}{(1-X)^n} = \frac{nX}{(1-X)^{n+1}}, \\ F_2(X) &= \sum_{d=0}^{\infty} h_d(m)X^d = \frac{1}{(1-X)^m}. \end{aligned}$$

Now c_d is the coefficient of X^d in

$$F_1(X)F_2(X) = \frac{nX}{(1-X)^{m+n+1}}$$

that is,

$$c_d = n\binom{m+n+d-1}{d-1}.$$

Hence we get

$$\frac{1}{dh_d(m+n)}\sum_{k=0}^{d} kh_k(n)h_{d-k}(m) = \frac{n\binom{m+n+d-1}{d-1}}{d\binom{m+n+d-1}{d}} = \frac{n}{m+n}. \ ■$$

3. Transfinite diameter and Robin's functions. Let E be a regular compact set in $\mathbb{C}^n$ with pluricomplex Green function $V_E(z)$. The *Robin function* of E is defined in complex projective $(n-1)$-space $\mathbb{P}^{n-1}$ (see [BT]) via

$$\varrho_E([z]) = \overline{\lim_{|\lambda|\to\infty}}(V_E(\lambda z) - \log|\lambda z|) \tag{3.1}$$

where $z \in \mathbb{C}^n - \{0\}$, $|\cdot|$ is the euclidean norm on $\mathbb{C}^n$ and $[z]$ is the point that z determines in $\mathbb{P}^{n-1}$. Of course, the pluricomplex Green function V_E and the Robin function are defined whether or not E is regular but we shall only use these notions in the regular case. In this case, $\varrho_E([z])$ is continuous on $\mathbb{P}^{n-1}$ ([Le] or [Si2]). The *logarithmic capacity* of E, denoted by $C(E)$, is defined (see e.g. [BT]) by

$$C(E) = \exp(-\sup_{\mathbb{P}^{n-1}} \varrho_E([z])). \tag{3.2}$$

We remark that the Robin function (and the logarithmic capacity) depends on the norm we work with. Replacing the euclidean norm $|\cdot|$ by any complex norm ν in the definition above, we may similarly define the Robin function ϱ_E^ν and the capacity $C_\nu(E)$. The two functions ϱ_E and ϱ_E^ν behave essentially in the same way (for the two norms are equivalent) but the values of $C(E)$ and $C_\nu(E)$ are different and this will play some role in our study. (Note that the computation of the transfinite diameter does not require the use of a norm on $\mathbb{C}^n$.) Another function is often called the Robin function in the literature. It is the function

$$\overline{\varrho}_E(z) = \overline{\lim_{|\lambda|\to\infty}}(V_E(\lambda z) - \log|\lambda|) \quad (z \neq 0), \tag{3.3}$$

which has the advantage of being independent of the norm but the inconvenience of not being homogeneous and therefore not defined on the projective space. Of course for every complex norm we have

$$\overline{\varrho}_E(z) = \varrho_E^\nu(z) + \log\nu(z). \tag{3.4}$$

Another description of this function is the following. Under the above regularity assumption on E there exists (see e.g. [Si2]) a unique homogeneous continuous plurisubharmonic function $\overline{V}_E(z)$ on $\mathbb{C}^n \times \mathbb{C}$ such that for $t \neq 0$,

$$\overline{V}_E(z,t) = \log|t| + V_E(z/t) \tag{3.5}$$

for every $z \in \mathbb{C}^n$. (We refer to [Si2] for the meaning of "homogeneous" in this context.) Then we have

$$\overline{\varrho}_E(z) = \overline{V}_E(z,0). \tag{3.6}$$

Much of the content of this section will be based on a polynomial approximation theorem of Bloom [Bl, Th. 3.2]. We first need to consider another polynomial approximation problem.

Let h be an *homogeneous* polynomial of degree d. Then an *h-minimal polynomial* for a compact set E is a polynomial of the form $h + p$ where $p \in \mathcal{P}_{d-1}(\mathbb{C}^n)$ which is of least deviation from 0 on E, that is,

$$\|h + p\|_E = \inf\{\|h + r\|_E : r \in \mathcal{P}_{d-1}(\mathbb{C}^n)\}. \tag{3.7}$$

Such a polynomial is generally not unique. The notation $\mathbf{Che}_E(h)$ will be used for any of these polynomials.

Of course the uniform norm $\|\mathbf{Che}_E(h)\|_E$ depends only on h and E and we have

$$\|\mathbf{Che}_E(\lambda h)\|_E = |\lambda| \cdot \|\mathbf{Che}_E(h)\|_E \quad (\lambda \in \mathbb{C}^*). \tag{3.8}$$

Theorem 2. *Let E be a regular polynomially convex compact subset of $\mathbb{C}^n$ and let $\{Q_d\}$ $(d = 1, 2, 3, \ldots)$ be a sequence of homogeneous polynomials such that* $\deg Q_d$ *strictly increases. If*

$$\overline{\lim_{d\to\infty}} \frac{1}{\deg Q_d} \log |Q_d(z)| - \log |z| \le \varrho_E([z]) \quad (z \ne 0) \tag{3.9}$$

then

$$\overline{\lim_{d\to\infty}} \|\mathbf{Che}_E(Q_d)\|_E^{1/\deg Q_d} \le 1. \tag{3.10}$$

This theorem is stated in [Bl] with $\deg Q_d = d$ but the proof works just as well under the assumption that $\deg Q_d$ increases. Bloom [Bl] based it on a Cauchy–Weil integral formula in order to construct explicitly a competitor to $\mathbf{Che}(Q_d)$ that satisfies (3.10), and Siciak [Si2] gave an alternate (somewhat more abstract) proof.

We are now able to relate these notions to the multivariate transfinite diameter.

Theorem 3. *Let E and F be regular compact subsets of $\mathbb{C}^n$. Then*

$$\inf_{\mathbb{P}^{n-1}} \frac{e^{-\varrho_E}}{e^{-\varrho_F}} \le \frac{D(E)}{D(F)} \le \sup_{\mathbb{P}^{n-1}} \frac{e^{-\varrho_E}}{e^{-\varrho_F}}. \tag{3.11}$$

In particular,

$$\varrho_E = \varrho_F \Rightarrow D(E) = D(F). \tag{3.12}$$

Note. Since E and F are regular, both ϱ_E and ϱ_F are continuous on the compact space $\mathbb{P}^{n-1}$ so that the extrema above are attained and finite. In (3.11) we may replace simultaneously ϱ_E by $\overline{\varrho}_E$ and ϱ_F by $\overline{\varrho}_F$, as follows from (3.4).

Proof (of Theorem 3). First we note we may suppose without loss of generality that E and F are *polynomially convex*. Indeed, if K is a regular compact subset of $\mathbb{C}^n$ and if $\widehat{K}$ is its polynomial hull then $D(K) = D(\widehat{K})$ (see [Za]) and $\varrho_K = \varrho_{\widehat{K}}$ for $V_K = V_{\widehat{K}}$ (see [Kl]).

Let α be a non-zero multi-index that we keep fixed for a moment.

Take a $\mathcal{P}(\alpha)$-minimal polynomial $t_{\alpha,F}$ for F (see (1.6)) and consider the function

$$u(z) = \frac{1}{|\alpha|}(\log|t_{\alpha,F}(z)| - \log\|t_{\alpha,F}\|_F). \tag{3.13}$$

This is a plurisubharmonic function on $\mathbb{C}^n$ of logarithmic growth that is smaller than or equal to zero on F, therefore

$$u(z) \le V_F(z) \quad (z \in \mathbb{C}^n), \tag{3.14}$$

as V_F is the supremum of such functions. Consequently, we have

$$\varlimsup_{|\lambda|\to\infty}(u(\lambda z) - \log|\lambda z|) \le \varrho_F([z]) \quad (z \ne 0). \tag{3.15}$$

But an easy calculation shows that

$$\varlimsup_{|\lambda|\to\infty}(u(\lambda z) - \log|\lambda z|) = \frac{1}{|\alpha|}(\log|\widehat{t}_{\alpha,F}(z)| - \log\|t_{\alpha,F}\|_F) - \log|z| \tag{3.16}$$

where $\widehat{t}_{\alpha,F}$ is the homogeneous part of degree $|\alpha|$ in $t_{\alpha,F}$. Hence we deduce

$$\begin{aligned}\frac{1}{|\alpha|}&(\log|\widehat{t}_{\alpha,F}(z)| - \log\|t_{\alpha,F}\|_F) - \log|z| \\ &\le \varrho_F([z]) \le \varrho_E([z]) + [\varrho_F([z]) - \varrho_E([z])] \\ &\le \varrho_E([z]) + \sup_{\mathbb{P}^{n-1}}[\varrho_F([z]) - \varrho_E([z])].\end{aligned}$$

We define the finite number γ by the relation

$$-\log\gamma = \sup_{\mathbb{P}^{n-1}}[\varrho_F([z]) - \varrho_E([z])]. \tag{3.17}$$

Let Y be a sequence of multi-indices for which $|\alpha|$ increases and $\alpha/|\alpha| \to \theta \in \Sigma_0$. Then since (see (1.8))

$$\varlimsup_{|\alpha|\to\infty,\,\alpha\in Y}\|t_{\alpha,F}\|_F^{1/|\alpha|} = \tau(F,\theta), \tag{3.18}$$

from the inequality above we get

$$\varlimsup_{|\alpha|\to\infty,\,\alpha\in Y}\frac{1}{|\alpha|}\log|\widehat{t}_{\alpha,F}(z)| - \log|z| \le \varrho_E([z]) + \log\frac{\tau(F,\theta)}{\gamma}. \tag{3.19}$$

Set finally

$$\nu := \tau(F,\theta)/\gamma. \tag{3.20}$$

The relation (3.19) gives

$$\varlimsup_{|\alpha|\to\infty,\,\alpha\in Y}\frac{1}{|\alpha|}\log\left|\frac{\widehat{t}_{\alpha,F}}{\nu^{|\alpha|}}(z)\right| - \log|z| \le \varrho_E([z]) \quad (z \ne 0).$$

Therefore Theorem 2 implies

$$\varlimsup_{|\alpha|\to\infty,\,\alpha\in Y}\|\mathbf{Che}_E(\widehat{t}_{\alpha,F}/\nu^{|\alpha|})\|_E^{1/|\alpha|} \le 1; \tag{3.21}$$

and, in view of (3.8), this also gives

$$\varlimsup_{|\alpha|\to\infty,\,\alpha\in Y} \|\mathbf{Che}_E(\widehat{t}_{\alpha,F})\|_E^{1/|\alpha|} \le \nu. \tag{3.22}$$

So, since

$$\|t_{\alpha,E}\| \le \|\mathbf{Che}_E(\widehat{t}_{\alpha,F})\|_E, \tag{3.23}$$

taking roots, letting $|\alpha| \to \infty, \alpha \in Y$ and using (3.22) together with (1.8), we obtain

$$\tau(E,\theta) \le \nu = \tau(F,\theta)/\gamma, \tag{3.24}$$

that is,

$$\log\tau(E,\theta) \le \log\tau(F,\theta) + \sup_{\mathbb{P}^{n-1}}[\varrho_F([z]) - \varrho_E([z])].$$

Substituting this inequality in Zakharyuta's formula (1.9) we obtain

$$\log D(E) \le \log D(F) + \sup_{\mathbb{P}^{n-1}}[\varrho_F([z]) - \varrho_E([z])].$$

Taking exponentials, we have

$$\frac{D(E)}{D(F)} \le \sup_{\mathbb{P}^{n-1}} \frac{e^{-\varrho_E}}{e^{-\varrho_F}}. \tag{3.25}$$

This is the first of the required inequalities. As for the second, since E and F play a symmetric role, we also obtain

$$\frac{D(F)}{D(E)} \le \sup_{\mathbb{P}^{n-1}} \frac{e^{-\varrho_F}}{e^{-\varrho_E}}, \quad \text{and so} \quad \frac{D(E)}{D(F)} \ge \inf_{\mathbb{P}^{n-1}} \frac{e^{-\varrho_E}}{e^{-\varrho_F}}. \ \blacksquare \tag{3.26}$$

We now apply our result to the computation of the transfinite diameter of the level sets E_R of E. They are defined for $R > 1$ by

$$E_R = \{z \in \mathbb{C}^n : V_E(z) \le \log R\}.$$

Each E_R is compact and $V_{E_R}(z) = \max\{0, V_E(z) - \log R\}$. It follows that $\varrho_{E_R}([z]) = \varrho_E([z]) - \log R$ and $C(E_R) = RC(E)$. Taking $F = E_R$ in Theorem 3 we immediately obtain the following corollary that answers (affirmatively) a question posed by Zakharyuta [Za].

Corollary 1. *Let E be compact, polynomially convex and regular in $\mathbb{C}^n$, and $R > 1$. Then*

$$D(E_R) = RD(E).$$

Remark. Examination of the proof shows that, in fact, we have $\tau(E_R,\theta) = R\tau(E,\theta)$ for every $\theta \in \Sigma_0$.

Corollary 2. *For every compact set $E \subset \mathbb{C}^n$, $n > 1$, we have*

$$C(E) \le \exp\biggl(+\frac{1}{2}\sum_{k=2}^{n}\frac{1}{k}\biggr)D(E). \tag{3.27}$$

Proof. We may assume that E is regular; otherwise we can use an approximation argument since the transfinite diameter and the logarithmic capacity are continuous under decreasing sequences of compact sets. Then the result follows immediately from Theorem 3 on taking F equal to $B(0,1)$, the unit (closed) euclidean ball, and using $D(B(0,1)) = \exp\big(-\frac{1}{2}\sum_{k=2}^{n}\frac{1}{k}\big)$, which is a result of Jędrzejowski [Je]. ■

Inequality (3.27) is sharp—a less precise bound appeared in [LT]—for it reduces to an equality if $E = B(0,R)$, actually if E is a regular compact set such that $\lim_{|z|\to\infty}(V_E(z) - \log|z|)$ exists.

More generally, if E is regular compact set such that there exists a complex norm ν (depending on E) for which the limit

$$\lim_{|z|\to\infty}(V_E(z) - \log\nu(z)) \text{ exists} \tag{3.28}$$

then it must be $-\log C_\nu(E)$, and

$$(3.28) \Rightarrow C_\nu(E) = \frac{D(E)}{D(B_\nu(0,1))} \tag{3.29}$$

where $B_\nu(0,1)$ is the closed unit ball for the norm ν. More specifically, if $\nu = \|\cdot\|$, the polydisc norm, then $D(E) = C_{\|\cdot\|}(E)$.

Of course the hypothesis (3.28) is very strong, but several compact sets for which one can explicitly find the pluricomplex Green function do have this property. Let us give two examples that can be derived from the corresponding formulas for V_E due to Baran, as in [Kl]. If $E = \Sigma$, the standard simplex in $\mathbb{R}^n \subset \mathbb{C}^n$, then the norm $\nu(z) = \sum|z_i| + |\sum z_i|$ works and the limit in (3.28) is $\log 2$. If $E = B_{\mathbb{R}}(0,1)$, the real euclidean ball in $\mathbb{R}^n \subset \mathbb{C}^n$, then we can take $\nu(z) = ((|z|^2 + |\sum z_i^2|)/2)^{1/2}$ and the limit is again $\log 2$. Recent results of Baran [Ba] imply existence of such a norm for every convex symmetric subset of $\mathbb{C}^n$. It would be interesting to study whether this holds true for other classes of regular compact sets.

Theorem 3 reveals that $D(E)$ is in fact a function of ϱ_E. We now show how the transfinite diameter of a regular compact set E can be (theoretically) computed with the sole knowledge of the Robin function of E. In view of Zakharyuta's formula, it suffices in fact to express the directional Chebyshev constant in terms of the Robin function.

We first need some new definitions.

A polynomial p is said to have *leading monomial* z^α if it is of the form $p(z) = a_\alpha z^\alpha + \sum_{\beta\prec\alpha} a_\beta z^\beta$ with $a_\alpha \neq 0$. The coefficient a_α is then termed the *leading coefficient* of p and we write $a_\alpha = \mathrm{Lead}(p)$. (The order $\prec$ is defined in the introduction.)

Now let $H(\alpha)$ denote the space of all the homogeneous polynomials of degree $|\alpha|$ whose leading monomial is z^α.

DEFINITION. Let $\theta \in \Sigma_0$. A sequence of homogeneous polynomials Q_d, $d = 0, 1, 2, \ldots$, is said to be a *θ-extremal* (*polynomial*) *sequence* for F if the following two conditions are satisfied.

(1) For every d, $Q_d \in H(\alpha^d)$ with $|\alpha^d|$ increasing as d increases and $\alpha^d/|\alpha^d| \to \theta$ as $d \to \infty$.

(2) For every $z \neq 0$, $|\alpha^d|^{-1} \log |Q_d(z)| - \log |z| \le \varrho_F([z])$.

COROLLARY 3 (to the proof of Theorem 3). *Let F be a regular polynomially convex compact subset of $\mathbb{C}^n$ and $\theta \in \Sigma_0$. Then*

$$\frac{1}{\tau(F,\theta)} = \sup\{\overline{\lim_{d\to\infty}} \operatorname{Lead}(Q_d)^{1/|\alpha^d|} : (Q_d) \text{ is a } \theta\text{-extremal sequence for } F\}. \tag{3.30}$$

Proof. First we note that $\tau(F,\theta)$ cannot be zero for otherwise, by [Za, Cor. 1, p. 353], $\tau(F,\cdot)$ would vanish on the whole of Σ_0. This would imply $D(F) = 0$ and thus $C(F) = 0$, which would contradict the regularity of F.

Now, take a sequence Y of multi-indices of increasing length such that $\alpha/|\alpha| \to \theta$ for $\alpha \in Y$. Then the sequence $\widehat{t}_{\alpha,F}/\|t_{\alpha,F}\|$ ($\alpha \in Y$) is a θ-extremal sequence for F (see the proof of Theorem 3). Since

$$\left(\operatorname{Lead} \frac{\widehat{t}_{\alpha,F}}{\|t_{\alpha,F}\|}\right)^{1/|\alpha|} = \left(\frac{1}{\|t_{\alpha,F}\|}\right)^{1/|\alpha|} \to \frac{1}{\tau(F,\theta)} \quad (|\alpha| \to \infty,\ \alpha \in Y),$$

we deduce that $1/\tau(F,\theta)$ is not greater than the right hand side in (3.30).

For the reverse inequality, we can proceed as follows. Take a θ-extremal polynomial sequence Q_d for F. In view of condition (2) in the definition, we may apply Theorem 2 to get

$$\overline{\lim_{d\to\infty}} \|\mathbf{Che}_F(Q_d)\|_F^{1/\deg Q_d} \le 1. \tag{3.31}$$

On the other hand the polynomial $\mathbf{Che}_F(Q_d)$ is of the form

$$\mathbf{Che}_F(Q_d) = Q_d + (\text{lower degree terms}) = \operatorname{Lead}(Q_d) z^{\alpha^d} + \sum_{\beta \prec \alpha^d} a_\beta z^\beta,$$

which implies that $\mathbf{Che}_F(Q_d)/\operatorname{Lead}(Q_d)$ is a competitor to be a $\mathcal{P}(\alpha^d)$-minimal polynomial for F and consequently

$$\|t_{\alpha^d,F}\|_F \le \left\| \frac{\mathbf{Che}_F(Q_d)}{\operatorname{Lead}(Q_d)} \right\|_F$$

or equivalently

$$\operatorname{Lead}(Q_d) \cdot \|t_{\alpha^d,F}\|_F \le \|\mathbf{Che}_F(Q_d)\|_F.$$

Taking $|\alpha^d|$th roots and letting $d \to \infty$ we obtain, using (3.31) and the fact that $\alpha^d/|\alpha^d| \to \theta$,

$$\tau(F,\theta)\cdot \varlimsup_{d\to\infty} \operatorname{Lead}(Q_d)^{1/|\alpha^d|} \le 1,$$

from which the required inequality immediately follows. ■

We conclude this section by another comparison theorem involving the second Robin function.

THEOREM 4. *Let E and F be regular compact subsets of $\mathbb{C}^n$. Then*

$$\frac{D(F)}{D(E)} \le \sup_{S(F)} e^{\overline{\varrho}_E} \tag{3.32}$$

where $S(F)$ is the Shilov boundary of F.

Proof. We return to the orthogonal polynomial method used in Section 2. Let μ_F be the equilibrium measure of the regular compact set F. We denote by p_α the sequence of monic orthogonal polynomials in $L^2(\mu_F)$. Recall that $(\cdot,\cdot)$ denotes the corresponding hermitian product. Now, for a nonzero multi-index α, we consider once again the polynomial $t_{\alpha,E}/\|t_{\alpha,E}\|_E$. We have repeatedly used the inequality

$$\frac{1}{|\alpha|}\log\left(\frac{|\widehat{t}_{\alpha,E}(z)|}{\|t_{\alpha,E}\|_E}\right) \le \log|z| + \varrho_E([z]) = \overline{\varrho}_E(z) \qquad (z\ne 0),$$

that is,

$$\frac{|\widehat{t}_{\alpha,E}(z)|}{\|t_{\alpha,E}\|_E} \le \exp(|\alpha|\overline{\varrho}_E(z)) \qquad (z\in\mathbb{C}^n).$$

(In fact this holds true even when $z=0$ for in this case the left hand side reduces to 0.) Here this inequality is exploited as follows:

$$\begin{aligned}\frac{1}{\|t_{\alpha,E}\|_E}\|p_\alpha\|_{\mu_F}^2 &= \left(\frac{\widehat{t}_{\alpha,E}}{\|t_{\alpha,E}\|_E}, p_\alpha\right) \le \left\|\frac{\widehat{t}_{\alpha,E}}{\|t_{\alpha,E}\|_E}\right\|_{\mu_F}\|p_\alpha\|_{\mu_F}\\ &\le \|(\exp\overline{\varrho}_E)^{|\alpha|}\|_{\mu_F}\|p_\alpha\|_{\mu_F}.\end{aligned}$$

Now, taking roots and letting $|\alpha|\to\infty$ with $\alpha/|\alpha|\to\theta\in\Sigma_0$, we derive, with the help of (2.7),

$$\frac{1}{\tau(E,\theta)} \le \varlimsup\Big(\int(\exp\overline{\varrho}_E)^{2|\alpha|}\,d\mu_F\Big)^{1/(2|\alpha|)}\cdot\frac{1}{\tau(F,\theta)}.$$

(The use of (2.7) is valid here for μ_F is regular.) Now the limsup above is equal to the supremum of the integrated function on the support of μ_F. Since this support is known to be the Shilov boundary of F (see [BT2]), we are done. ■

The bound (3.32) can be improved by using the invariance of the transfinite diameter under translations ($z\mapsto z+a$) and under the group $\mathrm{SL}_n(\mathbb{C}) :=$

$\{A \in \mathrm{GL}_n(\mathbb{C}) : |\det A| = 1\}$, i.e., $b \in \mathbb{C}^n$ and $A \in \mathrm{SL}_n$ imply $D(A(F)+b) = D(F)$. The translation invariance follows via the Zakharyuta formula from the invariance of the directional Chebyshev constants or, via Theorem 3, from the simple fact that $\varrho_{F+b} = \varrho_F$. The SL_n invariance follows from Sheĭnov's formula (1.11).

COROLLARY. *Under the same assumptions as in Theorem* 4 *we have*

$$\frac{D(F)}{D(E)} \le \inf_{b\in\mathbb{C}^n} \inf_{A\in \mathrm{SL}_n} \sup_{z\in S(F)} \exp(\overline{\varrho}_E(A(z)+b)). \tag{3.33}$$

Proof. Take $A \in \mathrm{SL}_n$ and $b \in \mathbb{C}^n$. Applying Theorem 4 to $A(F)+b$ we get

$$\frac{D(F)}{D(E)} = \frac{D(A(F)+b)}{D(E)} \le \sup_{S(A(F)+b)} e^{\overline{\varrho}_E(z)} = \sup_{u\in S(F)} e^{\overline{\varrho}_E(A(u)+b)}.$$

Taking the infimum over A and b we obtain the desired inequality. ∎

This leads in turn to a purely geometric bound for the transfinite diameter. For brevity we work only with the euclidean norm $|\cdot|$. Given a compact set K in $\mathbb{C}^n$ we define its *radius* $\mathrm{rad}(K)$ by

$$\mathrm{rad}(K) := \inf_{b\in\mathbb{C}^n} \sup_{z\in K} |z-b|.$$

We have $\mathrm{rad}(K) \le \mathrm{diam}(K) \le 2\,\mathrm{rad}(K)$. Now, taking $n \ge 2$ and $E = B(0,1)$ in the corollary above we get

$$D(F) \le \exp\biggl(-\frac{1}{2}\sum_{k=2}^{n} \frac{1}{k}\biggr) \cdot \mathrm{rad}(S(A(F)))$$

where A is any element of SL_n.

Note finally that the use of the SL_n invariance could also provide some improvement in the bound of Theorem 3.

4. Transfinite diameter of a pre-image. Let $q(z) = a_d z^d + a_{d-1}z^{d-1} + \ldots + a_1 z + a_0$ be a univariate polynomial of exact degree d ($a_d \ne 0$). Given a planar compact set E, a classical theorem of Fekete states that the transfinite diameter of the compact set $F = q^{-1}(E)$ is

$$D(F) = \biggl(\frac{D(E)}{|a_d|}\biggr)^{1/d}. \tag{4.1}$$

The classical (mostly algebraic) proof of Fekete can be found in [Go] and a direct potential-theoretic proof is available e.g. in [Ra]. It is the purpose of this section to study the generalization of this result to the multivariate transfinite diameter.

We consider polynomial mappings $q = (q_1, \ldots, q_n)$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ such that

$$(4.2)\quad \deg q_i = m \geq 1\ (1 \leq i \leq n), \quad \widehat{q}^{\,-1}(0) := \widehat{q}_1^{\,-1}(0) \cap \ldots \cap \widehat{q}_n^{\,-1}(0) = \{0\},$$

where $\widehat{q} = (\widehat{q}_1, \ldots, \widehat{q}_n)$ and $\widehat{q}_i$ is as usual the leading homogeneous term of q_i. Under these assumptions, q is a proper polynomial mapping [Kl, Th. 5.3.1] so that $F := q^{-1}(E)$ is compact if E is. We will study the relation between $D(F)$ and $D(E)$. Our first result shows that, as in the one-variable formula above, the relation between $D(E)$ and $D(F)$ depends only on the highest degree components of q.

THEOREM 5. *Let q be a polynomial mapping satisfying conditions* (4.2) *and E a regular compact subset of $\mathbb{C}^n$. If $F = q^{-1}(E)$ and $\widehat{F} = \widehat{q}^{\,-1}(E)$ then $D(F) = D(\widehat{F})$.*

Proof. This follows from Theorem 3 since, as shown in the lemma below, both F and $\widehat{F}$ have the same Robin function. ■

LEMMA 4. *Under the above hypothesis, we have $\varrho_F = \varrho_{\widehat{F}}$.*

Proof. We prove that for every $z \in \mathbb{C}^n$,

$$(4.3)\qquad \overline{\varrho}_F(z) = \frac{1}{m}\overline{\varrho}_E(\widehat{q}(z)).$$

Applying this to $\widehat{q}$ instead of q we get $\overline{\varrho}_F(z) = \overline{\varrho}_{\widehat{F}}(z)$ and this implies equality of the Robin functions.

By a theorem of Klimek [Kl, Th. 5.3.1], the pluricomplex Green function of F is given by $V_F(z) = m^{-1}V_E(q(z))$. (Note that the regularity of E implies that of F.) We use the corresponding homogeneous plurisubharmonic functions $\overline{V}_F(z,t)$ (see (3.5)). For every $z \in \mathbb{C}^n$ and every non-zero t we have

$$(4.4)\qquad \overline{V}_F(z,t) = \log|t| + V_F(z/t) = \log|t| + \frac{1}{m}V_E(q(z/t)).$$

One can write $q(z/t) = t^{-m}(\widehat{q}(z) + R(z,t))$ where $\lim_{t\to 0} R(z,t) = 0$. Fix $z \in \mathbb{C}^n$ and define $\xi(\lambda) := \widehat{q}(z) + R(z,\lambda)$ for $\lambda \in \mathbb{C}^*$. Note that $(\xi(\lambda), \lambda^m) \to (\widehat{q}(z), 0)$ as $\lambda \to 0$. Now, (4.4) gives

$$\overline{V}_F(z,\lambda) = \frac{1}{m}\overline{V}_E(\xi(\lambda), \lambda^m).$$

Letting $\lambda \to 0$ and using the continuity of $\overline{V}_F$ at $(z,0)$ and of $\overline{V}_E$ at $(\widehat{q}(z), 0)$ [Si2, Prop. 2.3], in view of (3.6) we get

$$(4.5)\qquad \overline{\varrho}_F(z) = \frac{1}{m}\overline{\varrho}_E(\widehat{q}(z)).\ ■$$

We are able to give a precise multivariate version of (4.1) only in very particular cases. A polynomial map q is said to be *simple* (of degree m)

if $\widehat{q}_i(z) = z_i^m$ for $i = 1, \ldots, d$. Simple mappings obviously satisfy conditions (4.2).

THEOREM 6. *Let E be a compact set in $\mathbb{C}^n$, q a simple polynomial mapping of degree m and $F = q^{-1}(E)$. Then*

$$D(F) = \sqrt[m]{D(E)}. \tag{4.6}$$

This is a consequence of the following much stronger statement proved in [BC]. (We use the notation explained in the introduction.)

THEOREM 7. *Let E be a compact set in $\mathbb{C}^n$ and q a simple polynomial mapping of degree d. If p is a $\mathcal{P}(\alpha)$-minimal polynomial for E then $p \circ q$ is a $\mathcal{P}(m\alpha)$-minimal polynomial for $F = q^{-1}(E)$.*

Proof of Theorem 6. The previous theorem says that $\mathcal{T}(m\alpha, F) = \mathcal{T}(\alpha, E)$ so that

$$\mathcal{T}(m\alpha, F)^{1/(m|\alpha|)} = \sqrt[m]{\mathcal{T}(\alpha, E)^{1/|\alpha|}}. \tag{4.7}$$

Now since the condition $\alpha/|\alpha| \to \theta$ is equivalent to $m\alpha/|m\alpha| \to \theta$, passing to the limit in (4.7), we obtain $\tau(F, \theta) = \sqrt[m]{\tau(E, \theta)}$, and the theorem then follows immediately from Zakharyuta's formula. ∎

Bos [Bo] has proved that the transfinite diameter $D(\mathbf{D})$ of the unit disc $\mathbf{D} := \{x_1^2 + x_2^2 \le 1\} \subset \mathbb{R}^2 \subset \mathbb{C}^2$ is $1/\sqrt{2e}$. Thus, using the mapping $q(z) = (z_1^2, z_2^2)$, we deduce from Theorem 6 that the transfinite diameter of the triangle $\mathbf{T}$ of vertices $(0,0), (0,1), (1,0)$ is $D(\mathbf{T}) = 1/(2e)$.

Formula (4.6) can be extended to some more general polynomial mappings by using Sheĭnov's formula.

COROLLARY 1. *Let q be a polynomial mapping of degree m satisfying (4.2). Suppose that the highest homogeneous parts of its components q_i are of the form*

$$\widehat{q}_i(z) = a_{i1} l_1(z)^m + a_{i2} l_2(z)^m + \ldots + a_{in} l_n(z)^m \quad (1 \le i \le n) \tag{4.8}$$

where $A = (a_{ij})$ is a non-singular matrix and the l_i's are linear forms. Then for every compact set E and $F = q^{-1}(E)$ we have

$$D(F) = |\det l|^{-1/n} |\det A|^{-1/(nm)} \sqrt[m]{D(E)} \tag{4.9}$$

where l is the linear automorphism of $\mathbb{C}^n$ defined by $l := (l_1, \ldots, l_n)$.

Proof. Note that since q is proper, l must be invertible and $\det l$ does not vanish. Now, thanks to (4.8), one can find a simple map Q of degree m such that $q = A \circ Q \circ l$. Hence $F = (l^{-1} \circ Q^{-1} \circ A^{-1})(E)$. Using twice Sheĭnov's formula (1.11) and once Theorem 6 we get

$$D(F) = |\det l|^{-1/n} D(Q^{-1}(A^{-1}(E))) = |\det l|^{-1/n} \sqrt[m]{D(A^{-1}(E))}$$
$$= |\det l|^{-1/n} \sqrt[m]{|\det A|^{-1/n} D(E)}. \ \blacksquare$$

In fact, every proper map of degree 2 on $\mathbb{C}^2$ that satisfies (4.2) also satisfies conditions (4.8). Up to a linear change of variable, such a map has a highest homogeneous part of the form

$$\widehat{q}(z) = (z_1^2, a^2 z_1^2 + 2bz_1z_2 + c^2 z_2^2) \quad \text{or} \quad \widehat{q}(z) = (z_1z_2, a^2 z_1^2 + 2bz_1z_2 + c^2 z_2^2),$$

where, of course, $\widehat{q} = (\widehat{q}_1, \widehat{q}_2)$. We can make our result completely explicit in both cases.

EXAMPLE 1. *Let q be a proper polynomial map of degree 2 on $\mathbb{C}^2$, E a compact set and $F = q^{-1}(E)$.*

(i) *If $\widehat{q}$ is of the form $\widehat{q}(z) = (z_1^2, a^2 z_1^2 + 2bz_1z_2 + c^2 z_2^2)$ then $D^2(F) = |c|^{-1} D(E)$.*

(ii) *If $\widehat{q}$ is of the form $\widehat{q}(z) = (z_1z_2, a^2 z_1^2 + 2bz_1z_2 + c^2 z_2^2)$ then $D^2(F) = |ac|^{-1/2} D(E)$.*

Proof. Note that in case (i), since q is proper one must have $c \neq 0$. Now, setting $l_1(z) = z_1$ and $l_2(z) = (b/c)z_1 + cz_2$ we have

$$\widehat{q}_1(z) = l_1(z)^2 + 0 \cdot l_2(z)^2, \qquad \widehat{q}_2(z) = (a^2 - b^2/c^2) l_1(z)^2 + l_2(z)^2,$$

and the claim follows from Corollary 1.

In case (ii), q being proper implies that both a and c are different from zero. The result is again a consequence of Corollary 1 with the help of

$$(2ac - 2b)\widehat{q}_1(z) + \widehat{q}_2(z) = (az_1 + cz_2)^2,$$
$$(-2ac - 2b)\widehat{q}_1(z) + \widehat{q}_2(z) = (az_1 - cz_2)^2. \ \blacksquare$$

We can also compute the relation in terms of the roots of the $\widehat{q}_i$. This is easy for proper maps of type (i).

EXAMPLE 2. *If q is a proper polynomial map of degree 2 on $\mathbb{C}^2$ such that*

$$\widehat{q}(z) = ((az_1 + bz_2)^2, (a'z_1 + b'z_2)(a''z_1 + b''z_2))$$

then

$$D^2(F) = (|b'a - ba'| \cdot |b''a - ba''|)^{-1/2} D(E).$$

Proof. This is again a simple computation. We omit the details. The change of variable $Z_1 = az_1 + bz_2$ and $Z_2 = a'z_1 + b'z_2$ which is non-singular since q is proper brings us back to case (i) of Example 1. $\blacksquare$

A little can be said on the computation of the transfinite diameter of the (filled-in) Julia sets of certain polynomial maps. Recall that, given a polynomial map q, the *(filled-in) Julia set* of q is

$$J(q) := \{z \in \mathbb{C}^n : (q^{(k)}(z))_k \text{ is a bounded sequence}\}.$$

If q is proper then $J(q)$ is a compact set obviously invariant under q, that is, $J(q)=q^{-1}(J(q))$. A classical and simple theorem of Brolin (see [Ra]) states that if q is a one-variable polynomial of the form $q(z)=a_m z^m+\ldots+a_0$ with $m\ge 2$ then the (one-variable) transfinite diameter $D(J(q))$ of $J(q)$ is $|a_m|^{-1/(m-1)}$.

EXAMPLE 3. *Let q be a proper polynomial mapping as in Corollary* 1, *with $m\ge 2$. Then the transfinite diameter of the corresponding Julia set is given by*

$$D^{m-1}(J(q))=|\det l|^{-m/n}\cdot|\det A|^{-1/n}.$$

Proof. It suffices to apply formula (4.9) with $E=J(q)=F$. ∎

A precise generalization of (4.1) to all polynomial mappings satisfying (4.2) remains to be found. (We think that such a formula does exist.)

We mention that there is no such formula for the logarithmic capacity in $\mathbb{C}^n$ $(n>1)$. Klimek [Kl2, p. 2769] showed that for a polynomial mapping of degree m one has

$$C(F)\Big(\underline{\lim}_{|z|\to\infty}\frac{|q(z)|}{|z|^m}\Big)^{1/m}\le\sqrt[m]{C(E)}\le C(F)\Big(\overline{\lim}_{|z|\to\infty}\frac{|q(z)|}{|z|^m}\Big)^{1/m},\tag{4.10}$$

which follows from (4.5). The example below shows that one may have equality in either the right or left inequalities in (4.10).

Let $E=\Delta(0,r)\times\Delta(0,s)\subset\mathbb{C}^2$ where $\Delta(0,\varrho)$ is the closed disc in the plane with center 0 and radius ϱ. Then $V_E(z)=\max(0,\log|z_1/r|,\log|z_2/s|)$ and $C(E)=\min(r,s)$.

Let $q(z_1,z_2)=(z_1^2,2z_2^2)$. Then $F=\Delta(0,\sqrt{r})\times\Delta(0,\sqrt{s/2})$ and if $\sqrt{r}<\sqrt{s/2}$ then $\sqrt{C(E)}=C(F)$ and there is equality in the left part of (4.10), whereas if $\sqrt{r}>\sqrt{s}$ then $\sqrt{C(E)}=\sqrt{2}\,C(F)$ and there is equality in the right part of (4.10).

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