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On some generalization of box splines

by Zygmunt Wronicz (Kraków)

Abstract. We give a generalization of box splines. We prove some of their properties and we give applications to interpolation and approximation of functions.

1. Introduction. An extended complete Chebyshev system $U_n=\{u_i\}_{i=0}^n$ of functions of class C^n in the interval $I=[a,b]$ is a generalization of the system $\{t^i\}_{i=0}^n$ of power functions. We can write such a system in the canonical form (see [9, 12])

$$u_0(t) = w_0(t),$$
$$u_i(t) = w_0(t)\int_a^t w_1(\tau_1)\int_a^{\tau_1} w_2(\tau_2)\ldots\int_a^{\tau_{i-1}} w_i(\tau_i)\,d\tau_i\ldots d\tau_1, \tag{1}$$

$i=1,\ldots,n$, where $w_j\in C^{n-j}(I)$, $w_j(t)>0$ for $t\in I$, $j=0,\ldots,n$.

For $u_j(t)=(t-a)^j$, $j=0,\ldots,n$, we have $w_0=1$, $w_i=i$, $i=1,\ldots,n$.

To generalize box splines we shall use a similar method to that used to obtain (1). Box splines are used in approximation theory, in interpolation of functions, in the finite element variational method, in the theory of wavelets and in other branches of mathematics (see [3, 5, 7, 8, 10]). Not only algebraic splines but also Chebyshevian splines play an important role in approximation theory. We give a generalization of box splines and a few of their main properties; we call the new splines Chebyshevian box splines. Then we define a fundamental function and give some applications of it to interpolation and approximation of functions.

2. Chebyshevian box splines. We need some notation and definitions. We say that an integer $n\times s$ matrix $V_s=\{v_1,\ldots,v_s\}$, $v_i\in\mathbb{Z}^n\setminus\{0\}$,

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$i = 1, \ldots, s$, is *admissible* if $\operatorname{rank} V_n = n$ (the first n columns of V_s are linearly independent), $s \ge n$. Below we assume that the matrix V_s is admissible.

Define

$$\langle V_s \rangle = \Big\{ \sum_{j=1}^{s} t_j v_j : 0 \le t_j \le 1,\ j = 1, \ldots, s \Big\}.$$

We see that

$$\langle V_s \rangle = V_s([0,1]^s).$$

Let $W = \{w_1, \ldots, w_{s-n}\}$ be a sequence of continuous functions on $\mathbb{R}^n$ such that

(i) each w_j is periodic, i.e. $w_j(x+\alpha) = w_j(x)$ for $\alpha \in \mathbb{Z}^n$,

(ii) $0 < a_j \le w_j(x) \le b_j < \infty$ for $x \in \mathbb{R}^n$, where a_j and b_j are some constants.

Definition 1. The box spline $B(x \mid V_s, W)$ with respect to V_s and W is defined as follows:

$$B(x \mid V_n, W) = \frac{\chi_{\langle V_n \rangle}(x)}{|\det V_n|}, \tag{1}$$

where $\chi_{\langle V_n \rangle}$ is the characteristic function of the set $\langle V_n \rangle$ and

$$\begin{aligned} &B(x \mid V_{n+k}, W) \\ &\quad = \int_0^1 w_k(x - t v_{n+k}) B(x - t v_{n+k} \mid V_{n+k-1}, W)\, dt, \qquad k = 1, \ldots, s-n. \end{aligned} \tag{2}$$

For $w_j = 1$, $j = 1, \ldots, s-n$, we obtain algebraic box splines, denoted by $B(x \mid V_s)$ (see [1, 2, 4, 6]).

Theorem 1. *For any function $f \in C(\mathbb{R}^n)$ we have*

$$\int_{\mathbb{R}^n} f(x) B(x \mid V_n, W)\, dx = \int_{[0,1]^n} f(V_n u)\, du,$$

$$\begin{aligned} &\int_{\mathbb{R}^n} f(x) B(x \mid V_s, W)\, dx \\ &\qquad = \int_{[0,1]^s} f(V_s u) w_{s-n}(\widetilde{V}_s u) w_{s-n-1}(\widetilde{V}_{s-1} u) \ldots w_1(\widetilde{V}_{n+1} u)\, du, \end{aligned} \tag{3}$$

where $\widetilde{V}_{n+k} = \widetilde{V}_{s,n+k} = [v_1, \ldots, v_{n+k-1}, 0, \ldots, 0]$ with $s-n-k+1$ zeros, $u = [u_1, \ldots, u_s]$, $du = du_1 \ldots du_s$.

Proof (by induction on s). For $s = n$ we change variables in the integral (see [6])

$$\int_{\mathbb{R}^n} f(x) B(x \mid V_n, W)\, dx = \int_{\langle V_n \rangle} f(x) \frac{\chi_{\langle V_n \rangle}}{|\det V_n|}\, dx = \int_{[0,1]^n} f(V_n u)\, du$$

because $V_n([0,1]^n) = \langle V_n \rangle$. Assume that (3) holds true for $s = n + k$. Then

$$\int_{\mathbb{R}^n} f(x)B(x \mid V_{n+k+1}, W)\,dx$$
$$= \int_0^1 \int_{\mathbb{R}^n} f(x) w_{k+1}(x - tv_{n+k+1}) B(x - tv_{n+k+1} \mid V_{n+k}, W)\,dx\,dt$$
$$= \int_0^1 \int_{\mathbb{R}^n} f(x + tv_{n+k+1}) w_{k+1}(x) B(x \mid V_{n+k}, W)\,dx\,dt$$
$$= \int_0^1 \int_{[0,1]^{n+k}} f(V_{n+k}u + tv_{n+k+1}) w_{k+1}(V_{n+k}u) w_k(\widetilde{V}_{n+k}u) \ldots w_1(\widetilde{V}_{n+1}u)\,du\,dt$$
$$= \int_{[0,1]^{n+k+1}} f(V_{n+k+1}y) w_{k+1}(\widetilde{V}_{n+k+1}y) w_k(\widetilde{V}_{n+k}y) \ldots w_1(\widetilde{V}_{n+1}y)\,dy$$

and we have proved (3) for $s = n + k + 1$.

Assume that $\int_{[0,1]^s} w_{s-n}(\widetilde{V}_{s-n}u) \ldots w_1(\widetilde{V}_{n+1}u)\,du = 1$. Putting $f = 1$ we obtain

$$\int_{\mathbb{R}^n} B(x \mid V_s, W)\,dx = 1.$$

LEMMA 1. *Let $v \in \mathbb{R}^n \setminus \{0\}$ and the function $G : \mathbb{R}^n \to \mathbb{R}$ be continuous on its convex support D. Then the function*

$$F(x) = \int_0^1 G(x - tv)\,dt$$

is continuous on its support

$$\operatorname{supp} F = \{x : [x, x - v] \cap D \neq \emptyset\}.$$

P r o o f. Since D is convex, the supports of the functions $g_1(t) = G(y - tv)$ and $g(t) = G(x - tv)$ are segments whose lengths depend continuously on y and x from G. Hence by continuity of G we obtain $\underline{\lim}_{y \to x} F(y) = F(x)$.

COROLLARY. *The function $B(x \mid V_s, W)$ is continuous on its support $\langle V_s \rangle$.*

THEOREM 2 (cf. [6]). *Let C be a nonsingular integer $n \times n$ matrix. Then*

$$B(x \mid CV_s, W) = \frac{1}{|\det C|} B(C^{-1}x \mid V_s, \widetilde{W}), \tag{4}$$

where $\widetilde{W} = \{\widetilde{w}_1, \ldots, \widetilde{w}_{s-n}\}$, $\widetilde{w}_j(x) = w_j(Cx)$, $j = 1, \ldots, s - n$.

Proof. Let $f \in C(\mathbb{R}^n)$. Applying Theorem 1 we have

$$\int_{\mathbb{R}^n} f(x)B(x\,|\,CV_s,W)\,dx$$
$$= \int_{[0,1]^s} f(CV_su)w_{s-n}(\widetilde{CV_s}u)w_{s-n-1}(\widetilde{CV}_{s-1}u)\ldots w_1(\widetilde{CV}_{n+1}u)\,du$$
$$= \int_{[0,1]^s} f(CV_su)w_{s-n}(C\widetilde{V}_su)w_{s-n-1}(C\widetilde{V}_{s-1}u)\ldots w_1(C\widetilde{V}_{n+1}u)\,du.$$

Putting $g(x) = f(Cx)$ and applying Theorem 1 again we obtain

$$\int_{\mathbb{R}^n} f(x)B(x\,|\,CV_s,W)\,dx$$
$$= \int_{[0,1]^s} g(V_su)\widetilde{w}_{s-n}(\widetilde{V}_su)\widetilde{w}_{s-n-1}(\widetilde{V}_{s-1}u)\ldots\widetilde{w}_1(\widetilde{V}_{n+1}u)\,du$$
$$= \int_{\mathbb{R}^n} f(Cx)B(x\,|\,V_s,\widetilde{W})\,dx = \int_{\mathbb{R}^n} f(x)\frac{B(C^{-1}x\,|\,V_s,\widetilde{W})}{|\det C|}\,dx.$$

Since the function f was chosen arbitrarily and $B(x\,|\,V_s,W)$ is continuous (Corollary of Theorem 1) we have proved (4).

THEOREM 3. *Let*

$$D_{v_s,w_{s-n}}f(x) = \frac{1}{w_{s-n}(x)}\lim_{t\to0+}\frac{1}{t}[f(x+tv_s)-f(x)].$$

Then

$$D_{v_s,w_{s-n}}B(x\,|\,V_s,W) = B(x\,|\,V_{s-1},W) - B(x-v_s\,|\,V_{s-1},W) \tag{5}$$

at every point of continuity of $B(x\,|\,V_{s-1},W)$.

Proof (cf. [6]). Let $g(x) = w_{s-n}(x)B(x\,|\,V_{s-1},W)$. Then

$$w_{s-n}(x)D_{v_s.w_{s-n}}B(x\,|\,V_s,W)$$
$$= \lim_{t\to0+}\frac{1}{t}\Big\{\int_0^1 g[x-(r-t)v_s]\,dr - \int_0^1 g(x-rv_s)\,dr\Big\}$$
$$= \lim_{t\to0+}\frac{1}{t}\Big[\int_{-t}^{1-t} g(x-rv_s)\,dr - \int_0^1 g(x-rv_s)\,dr\Big]$$
$$= \lim_{t\to0+}\frac{1}{t}\Big[\int_{-t}^{0} g(x-rv_s)\,dr - \int_{1-t}^1 g(x-rv_s)\,dr\Big]$$

$$= \lim_{t\to 0+} \frac{1}{t}\Big[\int_0^t g(x+rv_s)\,dr - \int_0^t g(x+rv_s-v_s)\,dr\Big]$$
$$= w_{s-n}(x)B(x\,|\,V_{s-1},W) - w_{s-n}(x-v_s)B(x-v_s\,|\,V_{s-1},W)$$
$$= w_{s-n}(x)[B(x\,|\,V_{s-1},W) - B(x-v_s\,|\,V_{s-1},W)]$$

and we have got (5).

3. A fundamental function. Let

$$G_{V_s} = \bigcup_{i_1<\ldots<i_{n-1}} \langle v_{i_1,\ldots,}v_{i_{n-1}}\rangle + \Big\{\sum_{j\neq i_1,\ldots,i_{n-1}}^{s} \varepsilon_j v_j : \varepsilon_j = 0,1\Big\} \qquad \text{for } n>1$$

and

$$G_{V_s} = \Big\{\sum_{j=1}^{s} \varepsilon_j v_j : \varepsilon_j = 0,1\Big\} \qquad \text{for } n=1.$$

As in the algebraic case we can prove that in the set $\mathbb{R}^n \setminus G_{V_s}$,

$$D_{v_{n,1}} D_{v_{n+1},w_1} \ldots D_{v_s,w_{s-n}} B(x\,|\,V_s,W) = 0.$$

Now we may consider the space $S(V_s,W)$ spanned by the integer translates of the box spline $B(x\,|\,V_s,W)$:

$$S(V_s,W) = \operatorname{span}\{B(x-\alpha\,|\,V_s,W) : \alpha\in\mathbb{Z}^n\}.$$

For $n=1$ the space $S(V_s,W)$ is included in the space of cardinal Chebyshevian splines. In the algebraic case for $n=1$ we have cardinal B-splines

$$N_1(x) = \chi_{[0,1)}(x),$$
$$N_m(x) = (N_{m-1} * N_1)(x) = \int_0^1 N_{m-1}(x-t)\,dt, \qquad m\geq 2.$$

Now we need some definitions, lemmas and theorems.

DEFINITION 2. The family of columns of the matrix V_s is called *unimodular* if the first n columns are linearly independent and $\forall_{Y\subset V_s,\sharp Y=n} |\det Y|\leq 1$.

Let $\widehat{f}$ denote the Fourier transform of f, i.e.

$$\widehat{f}(x) = \int_{\mathbb{R}^n} f(t)e^{-2\pi i t\cdot x}\,dt.$$

In the algebraic case we have the following

LEMMA 2 (see [4, 6]). *The family $V=V_s$ is unimodular if and only if*

$$(6) \qquad \{x\in\mathbb{R}^n : \forall_{\alpha\in\mathbb{Z}^n} \widehat{B}(x-\alpha\,|\,V) = 0\} = \emptyset.$$

For Chebyshevian box splines the condition (6) may not hold for a family which is not unimodular:

EXAMPLE.

$$V_4 = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad w_1 = 1,$$
$$w_2(x_1, x_2) = e^{-|2x_1-1|} \qquad \text{for } 0 \le x_1 \le 1,$$
$$w_2(x_1+\alpha, x_2) = w_2(x_1, x_2) \qquad \text{for } \alpha \in \mathbb{Z}.$$

Applying (3) we obtain

$$\begin{aligned}\widehat{B}&(x \,|\, V_4, W)\\ &= \int\limits_{[0,1]^4} e^{-2\pi i[(u_1+2u_3)x_1+(u_2+u_4)x_2]}\, w_2(u_1+2u_3, u_2)\, du_1\, du_2\, du_3\, du_4\\ &= \left(\frac{1-e^{-2\pi i x_2}}{2\pi i x_2}\right)^2 \sum_{k=1}^{8} \iint\limits_{D_k} e^{-2\pi i(u_1+2u_3)x_1} g_k(u_1, u_3)\, du_1\, du_3,\end{aligned}$$

where

$$\begin{aligned}
D_1 &= \{(u_1,u_3) : 0 \le u_1 \le 1/2,\ 0 \le u_3 \le 1/4 - u_1/2\},\\
D_2 &= \{(u_1,u_3) : 0 \le u_1 \le 1/2,\ 1/4 - u_1/2 \le u_3 \le 1/2 - u_1/2\},\\
D_3 &= \{(u_1,u_3) : 1/2 \le u_1 \le 1,\ 0 \le u_3 \le 1/2 - u_1/2\},\\
D_4 &= \{(u_1,u_3) : 0 \le u_1 \le 1,\ 1/2 - u_1/2 \le u_3 \le 3/4 - u_1/2\},\\
D_5 &= \{(u_1,u_3) : 0 \le u_1 \le 1,\ 3/4 - u_1/2 \le u_3 \le 1 - u_1/2\},\\
D_6 &= \{(u_1,u_3) : 0 \le u_1 \le 1/2,\ 1 - u_1/2 \le u_3 \le 1\},\\
D_7 &= \{(u_1,u_3) : 1/2 \le u_1 \le 1,\ 1 - u_1/2 \le u_3 \le 5/4 - u_1/2\},\\
D_8 &= \{(u_1,u_3) : 1/2 \le u_1 \le 1,\ 5/4 - u_1/2 \le u_3 \le 1\},\\
g_1 &= e^{-1+2u_1+4u_3}, \qquad g_2 = g_3 = e^{1-2u_1-4u_3}, \qquad g_4 = e^{-3+2u_1+4u_3},\\
g_5 &= e^{3-2u_1-4u_3}, \qquad g_6 = g_7 = e^{-5+2u_1+4u_3}, \qquad g_8 = e^{5-2u_1-4u_3}.
\end{aligned}$$

Hence

$$\widehat{B}(x \,|\, V_4, W) = \left(\frac{1-e^{-2\pi i x_2}}{2\pi i x_2}\right)^2 \frac{g(x_1)}{8e(1+\pi^2 x_1^2)^2},$$

where

$$\begin{aligned}
g(x_1) &= (1 + 2e \cdot e^{-\pi i x_1} - 5e^{-2\pi i x_1} + 4e \cdot e^{-3\pi i x_1} - 5e^{-4\pi i x_1}\\
&\quad + 2e \cdot e^{-5\pi i x_1} + e^{-6\pi i x_1})\\
&\quad + 2\pi i(1 - 2e \cdot e^{-\pi i x_1} + e^{-2\pi i x_1} - e^{-4\pi i x_1}\\
&\quad + 2e \cdot e^{-5\pi i x_1} - e^{-6\pi i x_1})x_1\\
&\quad + \pi^2(-1 + 2e \cdot e^{-\pi i x_1} - 3e^{-2\pi i x_1} + 4e \cdot e^{-3\pi i x_1} - 3e^{-4\pi i x_1}\\
&\quad + 2e \cdot e^{-5\pi i x_1} - e^{-6\pi i x_1})x_1^2\\
&= \sum_{k=0}^{2} \alpha_k(e^{\pi i x_1}) x_1^k
\end{aligned}$$

and the coefficients α_k depend only on $e^{\pi i x_1}$. If $g(\widetilde{x}_1)=0$, then $g(\widetilde{x}_1+2j)=\sum_{k=0}^{2}\alpha_k(e^{\pi i\widetilde{x}_1})(\widetilde{x}_1+2j)^k$ and since it is a polynomial of degree 2 with respect to j, it is different from zero for some j except for the case when all the coefficients of g are zero, and this happens for $x_1=1/2$ and $x_1=3/2$. Hence the condition (6) is not satisfied.

Let $X=V\cup -V=\{v_1,\dots,v_s,-v_1,\dots,-v_s\}$. Then we define

$$\widetilde{B}(x\,|\,X,W)=(B(\cdot\,|\,V,W)*B(\cdot\,|\,-V,W^-))(x)=\int_{\mathbb{R}^n}B(x-t\,|\,V,W)B(t\,|\,-V,W^-)\,dt.$$

LEMMA 3. *We have*

$$B(-x\,|\,V,W)=B(x\,|\,-V,W^-),\qquad \widetilde{B}(x\,|\,X,W)=\widetilde{B}(-x\,|\,X,W)$$

and

$$\widehat{B}(x\,|\,-V,W^-)=\overline{\widehat{B}(x\,|\,V,W)},$$

where $W^-=\{f:f(-x)\in W$ *and* $-V=\{x\in V:-x\in V\}$.

The proof follows directly from the definition of box splines.

LEMMA 4. *Let* $D_{V_s,W}f=D_{v_{n+1},w_1}\dots D_{v_s,w_{s-n}}f$ *and let* $B(x\,|\,V_s)$ *and* $B(x\,|\,X)$ *be the algebraic box splines with respect to* V_s *and* $X=V_s\cup -V_s$ *respectively*; *moreover*, *in the cases* $W=1$ *let all the functions* w_j *be equal to one. Then*

$$D_{-V_s,W^-}D_{V_s,W}\widetilde{B}(x\,|\,X,W)=D_{-V_s,1}D_{V_s,1}B(x\,|\,X).$$

Proof. Using the fact that convolution is commutative and Theorem 3 we obtain the equality

$$\begin{aligned}D_{-V_s,W^-}&D_{V_s,W}\widetilde{B}(x\,|\,X,W)\\&=\int_{\mathbb{R}^n}D_{V_s,W}B(x-t\,|\,V_s,W)D_{-V_s,W^-}B(t\,|\,-V_s,W^-)\,dt\\&=\int_{\mathbb{R}^n}D_{V_s,1}B(x-t\,|\,V_s)D_{-V_s,1}B(t\,|\,-V_s)\,dt\\&=D_{-V_s,1}D_{V_s1}\int_{\mathbb{R}^n}B(x-t\,|\,V_s)B(t\,|\,-V_s)\,dt=D_{-V_s,1}D_{V_s,1}B(x\,|\,X)\end{aligned}$$

since $B(x\,|\,V_s)*B(x\,|\,-V_s)=B(x,X)$ (see [4, 6]) and we have proved the lemma.

COROLLARY. $\widetilde{B}(x\,|\,X,W)=B(x\,|\,\widetilde{V},\widetilde{W})$, *where*

$$\widetilde{V}=\{v_1,\dots,v_n,-v_1,\dots,-v_n,-v_{n+1},\dots,-v_s,-v_{n+1},\dots,-v_s\},$$
$$\widetilde{W}=\{\underbrace{1,\dots,1}_{n},w_1(x),\dots,w_{s-n}(x),w_1(-x),\dots,w_{s-n}(-x)\}.$$

Lemma 5. *Let* $f, \widehat{f} \in C(\mathbb{R})$ *and for some* $\delta > 0$,

$$|f(x)| \le C(1+\|x\|)^{-n-\delta}, \qquad x \in \mathbb{R}^n,$$
$$|\widehat{f}(x)| \le C(1+\|x\|)^{-n-\delta}, \qquad x \in \mathbb{R}^n.$$

Then for every $x \in \mathbb{R}^n$,

$$\sum_{\alpha\in\mathbb{Z}^n} f(x-\alpha) = \sum_{\alpha\in\mathbb{Z}^n} \widehat{f}(\alpha)e^{2\pi i\alpha\cdot x}.$$

For the proof we refer to [11] (see also [6]).

Lemma 6. *Let functions* φ *and* ψ *be integrable, bounded and with compact support. Then for every* $x \in \mathbb{R}^n$,

$$\sum_{\alpha\in\mathbb{Z}^n} \widehat{\varphi}(x-\alpha)\widehat{\psi}(x-\alpha) = \sum_{\alpha\in\mathbb{Z}^n} (\varphi * \psi)(-\alpha)e^{2\pi i\alpha\cdot x}.$$

For the proof we refer to [6].

Theorem 4. *Let the family* V *be admissible and satisfy* (6). *Then for every* $x \in \mathbb{R}^n$,

$$P_{X,W}(x) = \sum_{\alpha\in\mathbb{Z}^n} \widetilde{B}(\alpha \mid X, W)e^{2\pi i\alpha\cdot x} \neq 0.$$

Proof (cf. [6]). We apply Lemma 6 to the functions $\varphi(x) = B(x \mid V, W)$ and $\psi(x) = B(x \mid -V, W^-)$. Hence

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^n} &[B(\cdot \mid V, W)]^\wedge(x-\alpha)[B(\cdot \mid -V, W^-)]^\wedge(x-\alpha)\\ &= \sum_{\alpha\in\mathbb{Z}^n} (B(\cdot \mid V, W) * B(\cdot \mid -V, W^-))(-\alpha)e^{2\pi i\alpha\cdot x}\\ &= \sum_{\alpha\in\mathbb{Z}^n} \widetilde{B}(-\alpha|X, W)e^{2\pi i\alpha\cdot x} = \sum_{\alpha\in\mathbb{Z}^n} \widetilde{B}(\alpha|X, W)e^{2\pi i\alpha\cdot x}.\end{aligned}$$

Using Lemma 3 we obtain

$$[B(\cdot \mid V, W)]^\wedge(x)[B(\cdot \mid -V, W^-)]^\wedge(x) = |[B(\cdot \mid V, W)]^\wedge(x)|^2.$$

Hence

$$\sum_{\alpha\in\mathbb{Z}^n} |[B(\cdot \mid V, W)]^\wedge(x-\alpha)|^2 = \sum_{\alpha\in\mathbb{Z}^n} \widetilde{B}(\alpha \mid X, W)e^{2\pi i\alpha\cdot x}$$

and by (6) we obtain the theorem.

Since the trigonometric polynomial $P_{X,W}$ is different from zero for every $x \in \mathbb{R}^n$, the function $1/P_{X,W}$ is periodic and of class C^∞. Hence we may expand it in a Fourier series:

$$\frac{1}{P_{X,W}(x)} = \sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha)e^{2\pi i\alpha\cdot x}, \tag{7}$$

where

$$b_{X,W}(\alpha) = \int\limits_{[0,1]^n} \frac{1}{P_{X,W}(x)} e^{-2\pi i\alpha\cdot x}\,dx.$$

LEMMA 7. *Let the family V be admissible and satisfy* (6). *Then the sequence $b_{X,W} = \{b_{X,W}(\alpha)\}$ of coefficients of the expansion of $1/P_{X,W}$ is exponentially decaying*, *i.e. there exist constants $C > 0$ and $0 < q < 1$ such that*

$$|b_{X,W}(\alpha)| \le Cq^{\|\alpha\|}, \qquad \alpha \in \mathbb{Z}^n, \tag{8}$$

where $\|\alpha\| = |\alpha_1| + \ldots + |\alpha_n|$.

The proof is the same as in the algebraic case (see [6]).

Let V satisfy the condition (6).

DEFINITION 3. The *fundamental function* associated with the family $X = V \cup -V$ and the family W is the function

$$\Phi_{X,W}(x) = \sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha)\widetilde{B}(x-\alpha \mid X, W).$$

LEMMA 8. *For every* $\alpha \in \mathbb{Z}^n$,

$$\Phi_{X,W}(\alpha) = \delta_{0,\alpha}. \tag{9}$$

Proof (see [6]). Taking the Cauchy product of the Fourier series for $P_{X,W}$ and $1/P_{X,W}$ we obtain (7).

As in the algebraic case we obtain (see [6])

LEMMA 9. *There exist constants $C > 0$ and $0 < q < 1$ such that*

$$|\Phi_{X,W}(x)| \le Cq^{\|x\|}, \qquad x \in \mathbb{R}^n.$$

Now we may define interpolating operators I and I_h as follows: for every function g defined on $\mathbb{Z}^n$ we put

$$Ig(x) = \sum_{\alpha\in\mathbb{Z}^n} g(\alpha)\Phi_{X,W}(x-\alpha),$$

$$I_h g(x) = \sum_{\alpha\in\mathbb{Z}^n} g(\alpha)\Phi_{X,W}(x/h-\alpha).$$

The problem of the convergence of the operators I_h will be considered in another paper.

Let

$$B^*_{V,W}(x) = \sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha) B(x-\alpha \mid V, W), \qquad x \in \mathbb{R}^n.$$

We have the following (see [6])

LEMMA 10. *For every* $\beta \in \mathbb{Z}^n$,

$$(B^*_{V,W}, B(\cdot - \beta \,|\, V, W))_{\mathbb{R}^n} = \delta_{0,\beta}.$$

Moreover, there exist constants $C > 0$ *and* $0 < q < 1$ *such that*

$$|B^*_{V,W}(x)| \le C q^{\|x\|}, \qquad x \in \mathbb{R}^n.$$

Proof. Applying Lemma 3 and changing variables we obtain

$$\begin{aligned}
(B^*_{X,W}, B(\cdot - \beta \,|\, V, W))_{\mathbb{R}^n}
&= \sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha) \int_{\mathbb{R}^n} B(x - \alpha \,|\, V, W) B(x - \beta \,|\, V, W)\, dx \\
&= \sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha) \int_{\mathbb{R}^n} B(x - (\alpha - \beta) \,|\, V, W) B(x \,|\, V, W)\, dx \\
&= \sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha) \int_{\mathbb{R}^n} B(x \,|\, -V, W^-) B((\beta - \alpha) - x \,|\, V, W)\, dx \\
&= \sum_{\alpha\in\mathbb{Z}^n} b_{X,W}(\alpha) \widetilde{B}(\beta - \alpha \,|\, X, W) = \Phi_{X,W}(\beta) = \delta_{0,\beta}.
\end{aligned}$$

The inequality is proved as in the algebraic case.

Because of this lemma we call the function $B^*_{V,W}$ the *biorthogonal function.*

Using the Jensen inequality and Lemma 9 we may prove (see [6]) the following

THEOREM 5. *Let the family* V *be admissible and satisfy the condition* (6). *Then for every* $1 \le p \le \infty$ *there exist constants* $C_1 > 0$ *and* $C_2 > 0$ *such that for every sequence* $a = \{a_\alpha\} \subset l^p$,

$$C_1 \|a\|_{l^p} \le \Big\| \sum_{\alpha\in\mathbb{Z}^n} a_\alpha B(x - \alpha \,|\, V, W) \Big\|_{L^p} \le C_2 \|a\|_{l^p}.$$

COROLLARY. *Let the family* V *be admissible and satisfy the condition* (6). *Then the system of box splines* $\{B(\cdot - \alpha \,|\, V, W)\}_{\alpha\in\mathbb{Z}^n}$ *is linearly independent.*

The results of this paper were announced by the author in [13] and [14].

PROBLEM. Prove that if V_s is unimodular then V_s satisfies (6).

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Faculty of Applied Mathematics
Academy of Mining and Metallurgy
Al. Mickiewicza 30
30-059 Kraków, Poland
E-mail: wronicz@uci.agh.edu.pl

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