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Abstract separation theorems of Rodé type and their applications

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Abstract. Sufficient and necessary conditions are presented under which two given functions can be separated by a function Π-affine in Rodé sense (resp. Π-convex, Π-concave). As special cases several old and new separation theorems are obtained.

1. Introduction. The starting point of our investigations is one of the most general versions of the Hahn–Banach theorem due to Rodé [13]. This abstract theorem states that if a function f is Π-concave, g is Π-convex and $f \le g$, then there exists a Π-affine function h such that $f \le h \le g$ (cf. the definitions below). A simpler proof of his result is given by König [7]. A geometric version of this separation theorem can be found in Páles [12]. The work [14] of Volkmann and Weigel offers an essential generalization of this result by showing that the linear combinations (in the definitions of the Π-convexity and concavity) can also be replaced by more abstract operations.

The above assumptions on f and g are sufficient but not necessary for f and g to admit a separation by a Π-affine function. In the present paper, we give a full characterization of functions which can be separated by Π-convex (resp. Π-concave, Π-affine) functions. As special cases of these results, we obtain several new and old separation (or sandwich) theorems, among them the theorems due to Kaufman, Kranz and Mazur–Orlicz.

2. Notations. The notations we use are similar to those of [7] (but our definition of a saturated family Π is different from that of [7]).

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Let X be a non-empty set. For every $m \in \mathbb{N}$, denote by $\mathcal{P}^m(X)$ the family of all pairs (σ, s) such that $\sigma : X^m \to X$ is an arbitrary function and there exist $s_0 \in \mathbb{R}$ and $s_1, \ldots, s_m \in [0, \infty)$ such that $s = [s_0, s_1, \ldots, s_m] : \mathbb{R}^m \to \mathbb{R}$ is an affine function defined by the formula

$$s(y_1, \ldots, y_m) := s_0 + s_1 y_1 + \ldots + s_m y_m.$$

(In the sequel the functions of the type $\sigma : X^m \to X$ are called *operations*).

Put $\mathcal{P}(X) = \bigcup_{m\in\mathbb{N}} \mathcal{P}^m(X)$.

Let Π be a fixed subset of $\mathcal{P}(X)$ and $\Pi^m = \Pi \cap \mathcal{P}^m(X)$, $m \in \mathbb{N}$. The set Π is said to be *commutative* if for any $m, n \in \mathbb{N}$, $(\sigma, s) \in \Pi^m$, $(\tau, t) \in \Pi^n$, the operations σ, τ and s, t commute, i.e.

$$\sigma(\tau(x_1^1, \ldots, x_n^1), \ldots, \tau(x_1^m, \ldots, x_n^m)) = \tau(\sigma(x_1^1, \ldots, x_1^m), \ldots, \sigma(x_n^1, \ldots, x_n^m))$$

for all $x_j^i \in X$ $(i = 1, \ldots, m,\ j = 1, \ldots, n)$ and

$$s(t(y_1^1, \ldots, y_n^1), \ldots, t(y_1^m, \ldots, y_n^m)) = t(s(y_1^1, \ldots, y_1^m), \ldots, s(y_n^1, \ldots, y_n^m))$$

for all $y_j^i \in \mathbb{R}$ $(i = 1, \ldots, m,\ j = 1, \ldots, n)$.

It is easy to verify that s and t commute if and only if

$$s_0 + t_0(s_1 + \ldots + s_m) = t_0 + s_0(t_1 + \ldots + t_n).$$

This condition holds automatically in two important cases: (1) if $s_0 = 0$ for all $(\sigma, s) \in \Pi$; (2) if $s_1 + \ldots + s_m = 1$ for all $m \in \mathbb{N}$ and $(\sigma, s) \in \Pi^m$.

Given $(\sigma, s) \in \Pi^m$ and $(\tau_1, t^1) \in \Pi^{n_1}, \ldots, (\tau_m, t^m) \in \Pi^{n_m}$, we define an operation $\sigma \circ (\tau_1, \ldots, \tau_m) : X^{n_1+\ldots+n_m} \to X$ by

$$\begin{aligned}\sigma \circ (\tau_1, \ldots, \tau_m)&(x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^m, \ldots, x_{n_m}^m)\\ &= \sigma(\tau_1(x_1^1, \ldots, x_{n_1}^1), \ldots, \tau_m(x_1^m, \ldots, x_{n_m}^m))\end{aligned}$$

and set

$$s\circ(t^1, \ldots, t^m) = [s_0 + s_1 t_0^1 + \ldots + s_m t_0^m, s_1 t_1^1, \ldots, s_1 t_{n_1}^1, \ldots, s_m t_1^m, \ldots, s_m t_{n_m}^m].$$

We say that Π is *saturated* if $(\mathrm{id}, [0, 1]) \in \Pi^1$ and, for every $(\sigma, s) \in \Pi^m$ and $(\tau_1, t^1) \in \Pi^{n_1}, \ldots, (\tau_m, t^m) \in \Pi^{n_m}$, we have

$$(\sigma \circ (\tau_1, \ldots, \tau_m), s \circ (t^1, \ldots, t^m)) \in \Pi^{n_1+\ldots+n_m}.$$

A function $f : X \to [-\infty, \infty)$ is called *Π-convex* if

$$f(\sigma(x_1, \ldots, x_m)) \le s_0 + s_1 f(x_1) + \ldots + s_m f(x_m)$$

for all $m \in \mathbb{N}$, $(\sigma, s) \in \Pi^m$ and $x_1, \ldots, x_m \in X$; f is *Π-concave* if it satisfies the reverse inequality, and f is *Π-affine* if it is Π-convex and Π-concave. Here, as usual, we adopt the following conventions:

$$0 \cdot (-\infty) = 0, \quad c \cdot (-\infty) = -\infty\ (\forall c > 0), \quad c + (-\infty) = -\infty\ (\forall c \in \mathbb{R}).$$

It is easy to check that if a function is Π-convex (resp. Π-concave, Π-affine), then it is also $\overline{\Pi}$-convex (resp. $\overline{\Pi}$-concave, $\overline{\Pi}$-affine), where $\overline{\Pi}$ is the smallest saturated subset of $\mathcal{P}(X)$ containing Π. If Π is commutative, then it can also be proved that the smallest saturated subset $\overline{\Pi}$ of $\mathcal{P}(X)$ is commutative as well. Therefore, we may restrict our attention to saturated classes in the rest of the paper.

3. The main results

THEOREM 1. *Let $\Pi \subset \mathcal{P}(X)$ be saturated and $f, g : X \to [-\infty, \infty)$. The following two conditions are equivalent*:

(i) *there exists a Π-convex function $h : X \to [-\infty, \infty)$ such that $f \le h \le g$*;

(ii) $f(\tau(x_1, \ldots, x_n)) \le t_0 + \sum_{j=1}^{n} t_j g(x_j)$ *for all $n \in \mathbb{N}$, $(\tau, t) \in \Pi^n$ and $x_1, \ldots, x_n \in X$.*

P r o o f. Since the implication (i)⇒(ii) is obvious, it is enough to show its converse. For all $x, x_1, \ldots, x_n \in X$ and $(\tau, t) \in \Pi^n$ with $x = \tau(x_1, \ldots, x_n)$, we have

$$f(x) \le t_0 + \sum_{j=1}^{n} t_j g(x_j). \tag{1}$$

In particular, taking $(\mathrm{id}, (0, 1)) \in \Pi^1$ and $x = x_1$, we get $f(x) \le g(x)$. Define

$$A(x) = \Big\{ t_0 + \sum_{j=1}^{n} t_j g(x_j) : n \in \mathbb{N},\ x_1, \ldots, x_n \in X \\ \text{and } (\tau, t) \in \Pi^n \text{ with } \tau(x_1, \ldots, x_n) = x \Big\}$$

and put

$$h(x) = \begin{cases} \inf A(x) & \text{if } A(x) \text{ is bounded below,} \\ -\infty & \text{otherwise.} \end{cases} \tag{2}$$

It follows from (1) that $f(x) \le h(x)$. We also have $h(x) \le g(x)$, because $1 \cdot g(x) \in A(x)$.

To show that h is Π-convex, fix arbitrarily $x_1, \ldots, x_m \in X$ and $(\sigma, s) \in \Pi^m$. If $h(\sigma(x_1, \ldots, x_m)) = -\infty$, then trivially

$$h(\sigma(x_1, \ldots, x_m)) \le s_0 + \sum_{i=1}^{m} s_i h(x_i). \tag{3}$$

So, assume that $h(\sigma(x_1, \ldots, x_m))$ is finite and take arbitrary representations $x_i = \tau_i(y_1^i, \ldots, y_{n_i}^i)$, $i = 1, \ldots, m$, where $(\tau_i, t^i) \in \Pi^{n_i}$. Since Π is saturated, $(\sigma \circ (\tau_1, \ldots, \tau_m), s \circ (t^1, \ldots, t^m)) \in \Pi$ and

$$\sigma(x_1, \ldots, x_m) = \sigma \circ (\tau_1, \ldots, \tau_m)(y_1^1, \ldots, y_{n_1}^1, \ldots, y_1^m, \ldots, y_{n_m}^m).$$

Hence, by condition (ii) of the theorem,

$$h(\sigma(x_1,\ldots,x_m)) \le s_0 + \sum_{i=1}^{m} s_i\Big(t_0^i + \sum_{j=1}^{n_i} t_j^i g(y_j^i)\Big). \tag{4}$$

The sums $t_0^i + \sum_{j=1}^{n_i} t_j^i g(y_j^i)$ are arbitrary elements of the sets $A(x_i)$, $i = 1,\ldots,m$. Taking the infimum of $A(x_i)$, $i = 1,\ldots,m$, we get (3). ∎

The next theorem gives a condition under which two functions $f, g : X \to [-\infty,\infty)$ can be separated by a Π-concave function. In the case when $f, g : X \to \mathbb{R}$, this result is an immediate consequence of the above theorem. The general case requires a separate (although similar) proof.

THEOREM 2. *Let $\Pi \subset \mathcal{P}(X)$ be saturated and $f, g : X \to [-\infty,\infty)$. The following conditions are equivalent*:

(i) *there exists a Π-concave function $h : X \to [-\infty,\infty)$ such that $f \le h \le g$*;

(ii) *$g(\tau(x_1,\ldots,x_n)) \ge t_0 + \sum_{j=1}^{n} t_j f(x_j)$ for all $n \in \mathbb{N}$, $(\tau,t) \in \Pi^n$ and $x_1,\ldots,x_n \in X$.*

Proof. The necessity is clear. To prove (ii)⇒(i), fix an $x \in X$, define

$$B(x) = \Big\{t_0 + \sum_{j=1}^{n} t_j f(x_j) : n \in \mathbb{N},\ x_1,\ldots,x_n \in X \text{ and } (\tau,t) \in \Pi^n \text{ with } \tau(x_1,\ldots,x_n) = x\Big\}$$

and put

$$h(x) = \begin{cases} \sup B(x) & \text{if } B(x) \ne \{-\infty\},\\ -\infty & \text{otherwise.}\end{cases} \tag{5}$$

Since $(\mathrm{id},1) \in \Pi^1$, we have $B(x) \ne \emptyset$ and $f(x) \le h(x)$. By (ii) and the definition of $B(x)$, we also get $h(x) \le g(x)$. As in the proof of Theorem 1, it can be shown that h is Π-concave. ∎

THEOREM 3. *Let $\Pi \subset \mathcal{P}(X)$ be saturated and commutative*, *and $f, g : X \to [-\infty,\infty)$. The following conditions are equivalent*:

(i) *there exists a Π-affine function $h : X \to [-\infty,\infty)$ such that $f \le h \le g$*;

(ii) *$s_0 + \sum_{i=1}^{m} s_i f(x_i) \le t_0 + \sum_{j=1}^{n} t_j g(y_j)$ for all $m, n \in \mathbb{N}$, $(\sigma,s) \in \Pi^m$, $(\tau,t) \in \Pi^n$ and $x_1,\ldots,x_m,y_1,\ldots,y_n \in X$ such that $\sigma(x_1,\ldots,x_m) = \tau(y_1,\ldots,y_n)$.*

Proof. The implication (i)⇒(ii) is clear. We prove its converse. Using (ii) and the fact that $(\mathrm{id},(0,1)) \in \Pi^1$, we obtain the inequalities appearing in Theorems 1 and 2. Hence we get a Π-convex function $\overline{h} : X \to [-\infty,\infty)$

and a Π-concave function $\underline{h} : X \to [-\infty, \infty)$ defined by (2) and (5) which separate f and g. Using (ii) once more, we infer that $\underline{h} \le \overline{h}$. By the Theorem of Rodé (cf. [13]) and also by its extension due to Volkmann and Weigel [14], there exists a Π-affine function $h : X \to [-\infty, \infty)$ separating $\underline{h}$ and $\overline{h}$ (and also f and g). ■

We say that a function $h : X \to [-\infty, \infty)$ *supports* $g : X \to [-\infty, \infty)$ *at a point* $x_0 \in X$ if $h(x_0) = g(x_0)$ and $h(x) \le g(x)$ for all $x \in X$. As an immediate consequence of Theorem 3, we get the following

PROPOSITION 1. *Let* $\Pi \subset \mathcal{P}(X)$ *be saturated and commutative such that, for all* $(\sigma, s) \in \Pi^m$, *we have* $s_i > 0$ *for* $i = 1, \ldots, m$. *Assume that* $g : X \to [-\infty, \infty)$ *is* Π*-convex*, $x_0 \in X$ *and for all* $m \in \mathbb{N}$ *and* $(\sigma, s) \in \Pi^m$,

$$g(\sigma(x_0, \ldots, x_0)) = s_0 + \sum_{i=1}^{m} s_i g(x_0). \tag{6}$$

Then there exists a Π*-affine function* $h : X \to [-\infty, \infty)$ *supporting* g *at* x_0.

Proof. Take the function $f : X \to [-\infty, \infty)$ defined by

$$f(x) = \begin{cases} g(x_0) & \text{if } x = x_0, \\ -\infty & \text{if } x \ne x_0. \end{cases}$$

We show that f and g satisfy the condition (ii) of Theorem 3. Let $(\sigma, s) \in \Pi^m$, $(\tau, t) \in \Pi^n$, $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$ and $\sigma(x_1, \ldots, x_m) = \tau(y_1, \ldots, y_n)$. If $x_i \ne x_0$ for some $i \in \{1, \ldots, n\}$ then $f(x_i) = -\infty$ and (ii) holds. If $x_i = x_0$ for all $i \in \{1, \ldots, n\}$, then by (6) and by the Π-convexity of g, we get

$$\begin{aligned} s_0 + \sum_{i=1}^{m} s_i f(x_i) &= s_0 + \sum_{i=1}^{m} s_i g(x_0) = g(\sigma(x_0, \ldots, x_0)) = g(\tau(y_1, \ldots, y_n)) \\ &\le t_0 + \sum_{j=1}^{n} t_j g(y_j). \end{aligned}$$

Then, by Theorem 3, there is a Π-affine function $h : X \to [-\infty, \infty)$ separating f and g. Since $h(x_0) = g(x_0)$, this h supports g at x_0. ■

REMARK 1. If a function $h : X \to [-\infty, \infty)$ is Π-affine and $h(x_0) \ne -\infty$ for some $x_0 \in X$ such that for every $x \in X$,

$$x_0 \in \{\sigma(x, x_2, \ldots, x_m) : (\sigma, s) \in \Pi^m,\ s_1 \ne 0,\ x_2, \ldots, x_m \in X,\ m \in \mathbb{N}\},$$

then h has finite values only. Indeed, fix an $x \in X$ and take $(\sigma, s) \in \Pi^m$ with $s_1 \ne 0$ and $x_2, \ldots, x_m \in X$ such that $x_0 = \sigma(x, x_2, \ldots, x_m)$. Then

$$h(x_0) = h(\sigma(x, x_2, \ldots, x_m)) = s_0 + s_1 h(x) + \sum_{i=2}^{m} s_i h(x_i),$$

which implies that $h(x) \ne -\infty$.

4. Applications. Many known as well as new results can be obtained as corollaries of Theorems 1–3 by an appropriate specification of X and Π. In this section we present several such results (for Π-convex and Π-affine case; the Π-concave versions are similar).

4.1. *Separation by subadditive and additive functions.* Let S be an abelian semigroup and $\Pi = \{(\sigma_m, [0,1,\dots,1]) : m \in \mathbb{N}\}$, where $\sigma_m : S^m \to S$ are defined by $\sigma_m(x_1,\dots,x_m) = x_1 + \dots + x_m$. Clearly, Π is commutative and saturated, and a function $f : S \to [-\infty,\infty)$ is Π-convex (resp. Π-affine) iff it is subadditive (resp. additive). Therefore, as a consequence of Theorems 1 and 3, we obtain the following corollaries. A direct proof of the first of them (for $f, g : S \to \mathbb{R}$) can be found in [10].

Corollary 1. *Let $f, g : S \to [-\infty,\infty)$. There exists a subadditive function $h : S \to [-\infty,\infty)$ such that $f \le h \le g$ iff*

$$f\Big(\sum_{i=1}^{m} x_i\Big) \le \sum_{i=1}^{m} g(x_i)$$

for all $x_1,\dots,x_m \in S$, $m \in \mathbb{N}$.

Corollary 2. *Let $f, g : S \to [-\infty,\infty)$. There exists an additive function $h : S \to [-\infty,\infty)$ such that $f \le h \le g$ iff*

$$\sum_{i=1}^{m} f(x_i) \le \sum_{j=1}^{n} g(y_j) \tag{7}$$

for all $m, n \in \mathbb{N}$ and $x_1,\dots,x_m, y_1,\dots,y_n \in S$ such that $\sum_{i=1}^m x_i = \sum_{j=1}^n y_j$.

Condition (7) is satisfied if, in particular, g is subadditive, f is superadditive and $f \le g$, or if g is subadditive and

$$\sum_{i=1}^{m} f(x_i) \le g\Big(\sum_{i=1}^{m} x_i\Big) \tag{8}$$

for all $x_1,\dots,x_m \in S$ and $m \in \mathbb{N}$. Therefore Corollary 2 generalizes the following two results.

Corollary 3 (Kranz [8]). *If $f : S \to [-\infty,\infty)$ is superadditive, $g : S \to [-\infty,\infty)$ is subadditive and $f \le g$, then there exists an additive function $h : S \to [-\infty,\infty)$ separating f and g.*

Corollary 4 (Kaufman [6]). *If $f, g : S \to [-\infty,\infty)$ satisfy* (8) *and g is subadditive, then there exists an additive function $h : S \to [-\infty,\infty)$ separating f and g.*

4.2. *Separation by midconvex and Jensen functions.* Let D be a convex subset of a real vector space. A function $f : D \to [-\infty,\infty)$ is said to be *midconvex* if

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}, \qquad x, y \in D; \tag{9}$$

it is called a *Jensen function* if (9) holds with equality.

Let

$$\Pi = \Big\{(\sigma_{k_1,\ldots,k_m}, [0, 2^{-k_1}, \ldots, 2^{-k_m}]) : m \in \mathbb{N},\\ k_1, \ldots, k_m \in \mathbb{N} \cup \{0\},\ \sum_{i=1}^{m} 2^{-k_i} = 1\Big\},$$

where $\sigma_{k_1,\ldots,k_m} : D^m \to D$ are defined by $\sigma_{k_1,\ldots,k_m}(x_1, \ldots, x_m) = \sum_{i=1}^{m} 2^{-k_i} x_i$. It is easy to check that Π is commutative and saturated. Moreover, f is Π-convex (resp. Π-affine) iff it is a midconvex (resp. Jensen) function. Hence, using Theorems 1 and 3, we get the following results.

COROLLARY 5. *Let $f, g : D \to [-\infty, \infty)$. There exists a midconvex function $h : D \to [-\infty, \infty)$ such that $f \le h \le g$ iff*

$$f\left(\frac{1}{2^n}\sum_{i=1}^{2^n} x_i\right) \le \frac{1}{2^n}\sum_{i=1}^{2^n} g(x_i) \tag{10}$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_{2^n} \in D$.

Proof. The necessity is obvious. To prove the sufficiency, we show that (10) yields

$$f\Big(\sum_{i=1}^{m} 2^{-k_i} x_i\Big) \le \sum_{i=1}^{m} 2^{-k_i} g(x_i) \tag{11}$$

for all $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in D$ and $k_1, \ldots, k_m \in \mathbb{N} \cup \{0\}$ with $\sum_{i=1}^{m} 2^{-k_i} = 1$.

Indeed, take $x_1, \ldots, x_m \in D$ and $k_1, \ldots, k_m$ as above and define

$$(\overline{x}_1, \ldots, \overline{x}_{2^n}) = (\underbrace{x_1, \ldots, x_1}_{2^{n-k_1}\text{ times}}, \ldots, \underbrace{x_m, \ldots, x_m}_{2^{n-k_m}\text{ times}}),$$

where n is defined to be the maximum of the integers $k_1, \ldots, k_m$. If we apply (10) to the elements $\overline{x}_1, \ldots, \overline{x}_{2^n}$, the resulting inequality reduces to (11). Now, applying Theorem 1, we get the existence of a separating midconvex function. ■

The proof of the following result is completely analogous to that of the previous corollary.

COROLLARY 6. *Let $f, g : D \to [-\infty, \infty)$. There exists a Jensen function $h : D \to [-\infty, \infty)$ such that $f \le h \le g$ iff*

$$\sum_{i=1}^{2^n} f(x_i) \le \sum_{i=1}^{2^n} g(y_i) \tag{12}$$

for all $n \in \mathbb{N}$ *and* $x_1, \ldots, x_{2^n}, y_1 \ldots, y_{2^n} \in D$ *such that* $\sum_{i=1}^{2^n} x_i = \sum_{i=1}^{2^n} y_i$.

Proof. It suffices to prove that (12) is equivalent to the following condition:

$$\sum_{i=1}^{m} 2^{-k_i} f(x_i) \le \sum_{j=1}^{n} 2^{-l_j} g(y_j)$$

for all $m, n \in \mathbb{N}$, $k_1, \ldots, k_m, l_1, \ldots, l_n \in \mathbb{N} \cup \{0\}$ and $x_1, \ldots, x_m, y_1, \ldots, y_n \in D$ such that $\sum_{i=1}^{m} 2^{-k_i} = \sum_{j=1}^{n} 2^{-l_j} = 1$ and $\sum_{i=1}^{m} 2^{-k_i} x_i = \sum_{j=1}^{n} 2^{-l_j} y_j$.

Then, by Theorem 3, the result follows. ∎

4.3. *Separation by convex and affine functions.* Let D be a convex subset of a real vector space and

$$\Pi = \Big\{ (\sigma_{s_1,\ldots,s_m}, [0, s_1, \ldots, s_m]) : m \in \mathbb{N},\ s_1, \ldots, s_m \ge 0,\ \sum_{i=1}^{m} s_i = 1 \Big\},$$

where $\sigma_{s_1,\ldots,s_m} : D^m \to D$, $\sigma_{s_1,\ldots,s_m}(x_1, \ldots, x_m) = \sum_{i=1}^{m} s_i x_i$. Then Π is commutative and saturated and a function $f : D \to [-\infty, \infty)$ is Π-convex (resp. Π-affine) iff it is convex (resp. affine) in the usual sense. By Theorem 1, we get the following result due to Baron, Matkowski and Nikodem (in the case $f, g : D \to \mathbb{R}$) [1, Theorem 1b].

Corollary 7. *Let* $f, g : D \to [-\infty, \infty)$. *There exists a convex function* $h : D \to [-\infty, \infty)$ *such that* $f \le h \le g$ *iff*

$$f\Big(\sum_{i=1}^{m} s_i x_i\Big) \le \sum_{i=1}^{m} s_i g(x_i) \tag{13}$$

for all $m \in \mathbb{N}$, $x_1, \ldots, x_m \in D$ *and* $s_1, \ldots, s_m \ge 0$ *summing up to* 1.

Remark 2. It is proved in [1] that if D is a convex subset of $\mathbb{R}^k$, then it is enough to take in (13) the convex combinations of $k+1$ points. In particular, two real functions f, g defined on an interval $I \subset \mathbb{R}$ can be separated by a convex function iff

$$f(sx + (1-s)y) \le sg(x) + (1-s)g(y), \qquad x, y \in I,\ s \in [0,1].$$

The next result is a direct consequence of Theorem 3. It can also be found in [4, p. 35].

Corollary 8. *Let* $f, g : D \to [-\infty, \infty)$. *There exists an affine function* $h : D \to [-\infty, \infty)$ *such that* $f \le h \le g$ *iff*

$$\sum_{i=1}^{m} s_i f(x_i) \le \sum_{j=1}^{n} t_j g(y_j) \tag{14}$$

for all $m, n \in \mathbb{N}$, $s_1, \ldots, s_m, t_1, \ldots, t_n \geq 0$ *and* $x_1, \ldots, x_m, y_1, \ldots, y_n \in D$ *such that* $\sum_{i=1}^m s_i = \sum_{j=1}^n t_j = 1$ *and* $\sum_{i=1}^m s_i x_i = \sum_{j=1}^n t_j y_j$.

REMARK 3. A similar result for real functions defined on a convex subset of $\mathbb{R}^k$ is obtained in [2]. In that case, it is enough to take in (14) $m, n \in \mathbb{N}$ such that $m + n = k + 2$. For $k = 1$, we get the result obtained earlier by Nikodem and Wąsowicz [11] stating that two real functions f, g defined on an interval $I \subset \mathbb{R}$ can be separated by an affine function iff

$$\begin{cases} f(sx + (1-s)y) \leq sg(x) + (1-s)g(y), \\ g(sx + (1-s)y) \geq sf(x) + (1-s)f(y) \end{cases}$$

for all $x, y \in I$ and $s \in [0, 1]$.

4.4. *Separation by sublinear and linear functions.* Let E be a real vector space and

$$\Pi = \{(\sigma_{s_1,\ldots,s_m}, [0, s_1, \ldots, s_m]) : m \in \mathbb{N},\ s_1, \ldots, s_m \geq 0\},$$

where $\sigma_{s_1,\ldots,s_m} : E^m \to E$ is given by $\sigma_{s_1,\ldots,s_m}(x_1, \ldots, x_m) = \sum_{i=1}^m s_i x_i$. Obviously, Π is commutative and saturated, and a function $f : E \to [-\infty, \infty)$ is Π-convex (resp. Π-affine) iff it is sublinear (resp. linear). In this case, Theorem 1 reduces to the following result (for the real case cf. [10]).

COROLLARY 9. *Let* $f, g : E \to [-\infty, \infty)$. *There exists a sublinear function* $h : E \to [-\infty, \infty)$ *such that* $f \leq h \leq g$ *iff*

$$f\Big(\sum_{i=1}^m s_i x_i\Big) \leq \sum_{i=1}^m s_i g(x_i)$$

for all $m \in \mathbb{N}$, $x_1, \ldots, x_m \in E$ *and* $s_1, \ldots, s_m \geq 0$.

The next result is a consequence of Theorem 3.

COROLLARY 10. *Let* $f, g : E \to [-\infty, \infty)$. *There exists a linear function* $h : E \to [-\infty, \infty)$ *such that* $f \leq h \leq g$ *iff*

$$\sum_{i=1}^m s_i f(x_i) \leq \sum_{j=1}^n t_j g(y_j) \tag{15}$$

for all $m, n \in \mathbb{N}$, $s_1, \ldots, s_m, t_1, \ldots, t_n \geq 0$ *and* $x_1, \ldots, x_m, y_1, \ldots, y_n \in E$ *such that* $\sum_{i=1}^m s_i x_i = \sum_{j=1}^n t_j y_j$.

Condition (15) is satisfied if, in particular, g is sublinear and

$$\sum_{i=1}^m s_i f(x_i) \leq g\Big(\sum_{i=1}^m s_i x_i\Big) \tag{16}$$

for all $m \in \mathbb{N}$, $x_1, \ldots, x_m \in E$ and $s_1, \ldots, s_m \geq 0$. Hence we get a result which is an analogue of Kaufman's theorem (cf. Corollary 4 above). It is also a special version of the well known Mazur–Orlicz Theorem [9].

Corollary 11. *If* $f, g : E \to [-\infty, \infty)$ *satisfy* (16) *and* g *is sublinear, then there exists a linear function* $h : E \to [-\infty, \infty)$ *separating* f *and* g.

By the above result (using a method similar to [6]), we can also obtain the full version of the Mazur–Orlicz Theorem.

Corollary 12 ([9, Theorem 2.41]; cf. also [3, Theorem 37]). *Let* T *be a non-empty set and* $\varphi : T \to E$, $\alpha : T \to \mathbb{R}$. *Assume that* $g : E \to \mathbb{R}$ *is a sublinear function. Then the following conditions are equivalent*:

(i) *there exists a linear function* $h : E \to \mathbb{R}$ *such that* $h \le g$ *on* E *and* $\alpha(u) \le h(\varphi(u))$, $u \in T$;
(ii) $\sum_{i=1}^m s_i\alpha(u_i) \le g(\sum_{i=1}^m s_i\varphi(u_i))$ *for all* $m \in \mathbb{N}$, $s_1, \ldots, s_m \ge 0$ *and* $u_1, \ldots, u_m \in T$.

Proof. The implication (i)⇒(ii) is obvious. To prove the converse take

$$f(x) = \begin{cases} \sup\{\alpha(u) : u \in T \text{ such that } \varphi(u) = x\} & \text{if } x \in \varphi(T), \\ -\infty & \text{if } x \notin \varphi(T). \end{cases}$$

By (ii), $f : E \to [-\infty, \infty)$ is well defined and $f \le g$. We show that f and g satisfy (16). Fix $s_1, \ldots, s_m \ge 0$ and $x_1, \ldots, x_m \in E$. If $x_i \notin \varphi(T)$ for some $i \in \{1, \ldots, m\}$, then (16) is obvious. If $x_i \in \varphi(T)$ for all $i \in \{1, \ldots, m\}$, then for arbitrary $u_i \in T$ such that $\varphi(u_i) = x_i$, $i = 1, \ldots, m$, we have

$$\sum_{i=1}^m s_i\alpha(u_i) \le g\Big(\sum_{i=1}^m s_i x_i\Big).$$

Taking the suprema over all u_i we obtain (16). By Corollary 11, there exists a linear function $h : E \to [-\infty, \infty)$ separating f and g. Notice that h is finite. Indeed, fix an $x_0 \in \varphi(T)$ and take any $x \in E$. Then $-\infty < h(x_0) = h(x_0 - x) + h(x)$, whence $h(x) > -\infty$. It is easy to see that h satisfies (i). ∎

Note that the condition (16) is satisfied if, in particular, f is concave and g is sublinear. Therefore, the following theorem of Hirano, Komiya and Takahashi is a consequence of Corollary 11.

Corollary 13 ([5, Theorem 1]). *Let* $g : E \to \mathbb{R}$ *be sublinear*, $C \subset E$ *be a non–empty convex set and let* $f : C \to \mathbb{R}$ *be a concave function such that* $f(x) \le g(x)$ *for all* $x \in C$. *Then there exists a linear function* $h : E \to \mathbb{R}$ *such that* $f(x) \le h(x)$, $x \in C$, *and* $h \le g$ *on* E.

Proof. Let E_0 be the subspace of E generated by C. Put

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in C, \\ -\infty & \text{if } x \in E_0 \setminus C. \end{cases}$$

It is easy to check that $\bar{f} : E_0 \to [-\infty, \infty)$ is a concave function such that $\bar{f} \le g$ on E_0. Moreover, $\bar{f}$ and g satisfy (16) on E_0. By Corollary 11 there exists a linear function $\bar{h} : E_0 \to [-\infty, \infty)$ such that $\bar{f} \le \bar{h} \le g$ on E_0. Since

$\bar{f}$ is finite on C, also $\overline{h}$ must be finite on E_0. Now, by the Hahn–Banach Theorem we get a linear extension $h : E \to \mathbb{R}$ of $\overline{h}$ such that $h \le g$ on E. Clearly, $f \le h$ on C. ■

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