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Relative tangent cone of analytic curves

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Abstract. The purpose of this paper is to give a characterization of the relative tangent cone of two analytic curves in $\mathbb{C}^m$ with an isolated intersection.

1. Introduction. We consider analytic curves X and Y in a neighbourhood Ω of 0 in $\mathbb{C}^m$ (by an analytic curve we mean an analytic set of pure dimension 1) such that $X \cap Y = \{0\}$ and study the relative tangent cone to these curves at 0 (see Section 2 of [ATW]). We restrict our attention to analytic curves with irreducible germs at 0. This involves no loss of generality as our considerations are local and the relative tangent cone and the intersection multiplicity of analytic curves are additive.

The main result of this paper, that is, the equality $C_0(X,Y) + C_0(X) = C_0(X,Y)$, is proved in Section 3 after preliminary results for analytic curves. This theorem gives a strong geometric characterization of the relative tangent cone of analytic curves.

This research was inspired by the paper [ChKT]. In the last section we present a method which reduces the calculations of the intersection multiplicity to the calculations of the multiplicity of a holomorphic mapping at a point. We use ideas from [ChKT] and our characterization of the relative tangent cone. The method we apply can simplify the proof of the formula for the intersection multiplicity of analytic curves presented in [ChKT].

2. Preliminary results. We start with the following lemma which will be used in the proof of the main theorem of this paper.

Lemma 2.1. *Let d be a positive integer. Suppose that $\{t_n\}$ is a sequence of complex numbers convergent to* 0 *and such that $\{nt_n\}$ is convergent in $\widehat{\mathbb{C}}$. Then for each $c \in \mathbb{C}$ there exists a sequence $\{h_n\}$ such that*:

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(1) $h_n \to 0$,

(2) $n^d((t_n + h_n)^d - t_n^d) \to c$,

(3) *for any holomorphic function* $\varphi : \Omega \to \mathbb{C}$ *defined in an open neighbourhood* Ω *of* $0 \in \mathbb{C}$ *with* $\operatorname{ord}_0 \varphi > d$ *we have* $n^d(\varphi(t_n + h_n) - \varphi(t_n)) \to 0$.

Proof. We have to consider three cases depending on the behaviour of the sequence $\{nt_n\}$. First, in each case we prove assertions (1) and (2).

(i) If $nt_n \to 0$, then by a simple calculation the sequence $h_n = c^{1/d} n^{-1}$ satisfies the assertion of our lemma.

(ii) If $nt_n \to a \in \mathbb{C} \setminus \{0\}$, then take an $\alpha \in \mathbb{C}$ such that $(1+\alpha)^d - 1 = ca^{-d}$ and define $h_n = \alpha t_n$. We have $h_n \to 0$ and

$$n^d((t_n + h_n)^d - t_n^d) = (nt_n)^d((1+\alpha)^d - 1) = (nt_n)^d c a^{-d} \to c.$$

(iii) If $nt_n \to \infty$, then we take $h_n = c(nd)^{-1}(nt_n)^{1-d}$. We get $h_n \to 0$ and

$$n^d((t_n + h_n)^d - t_n^d) = \sum_{i=1}^{d} \binom{d}{i} n^d h_n^i t_n^{d-i} = \sum_{i=1}^{d} \binom{d}{i} (c/d)^i (nt_n)^{d(1-i)} \to c.$$

Now, we can prove (3) starting from a common observation. For each h_n, there is a θ_n $(0 \le \theta_n \le 1)$ such that

$$|\varphi(t_n + h_n) - \varphi(t_n)| \le |h_n||\varphi'(t_n + \theta_n h_n)|.$$

Since $\operatorname{ord}_0 \varphi > d$, by the property of the order of φ' at 0 we have

$$|\varphi'(t_n + \theta_n h_n)| \le M|t_n + \theta_n h_n|^d$$

for some $M > 0$. The task is now to show that $n^d h_n (t_n + \theta_n h_n)^d$ converges to zero as $n \to \infty$. In the cases (i) and (ii) this is obvious. For the sequence $h_n = (c/nd)(nt_n)^{1-d}$ from case (iii) we have

$$n^d h_n (t_n + \theta_n h_n)^d = \frac{c}{d} t_n \left(1 + \theta_n \frac{c}{d} (nt_n)^{-d}\right)^d \to 0, \qquad n \to \infty,$$

due to $nt_n \to \infty$ as $n \to \infty$. This finishes the proof.

3. Main result. Let Ω be a neighbourhood of $0 \in \mathbb{C}^m$ $(m \ge 2)$ and let X, Y be analytic curves in Ω with irreducible germs at the origin such that $X \cap Y = \{0\}$. The *relative tangent cone* $C_0(X,Y)$ to the analytic sets X, Y at the point 0 (see [ATW]) is the set of $v \in \mathbb{C}^m$ with the following property: there exist sequences $\{x_\nu\}$ of points of X, $\{y_\nu\}$ of points of Y and $\{\lambda_\nu\}$ of complex numbers such that $x_\nu \to 0$, $y_\nu \to 0$ and $\lambda_\nu(x_\nu - y_\nu) \to v$ as $\nu \to \infty$.

The main goal of this paper is to prove the following theorem.

THEOREM 3.1. *If* $X \cap Y = \{0\}$ *then* $C_0(X,Y) + C_0(X) = C_0(X,Y)$.

Proof. We consider two cases.

First, if $C_0(X) \cap C_0(Y) = \{0\}$ then by Property 2.9 in [ATW] we obtain the equality $C_0(X,Y) = C_0(X) + C_0(Y)$, which completes the proof in this case.

Second, if X and Y have common tangent cone $C_0(X) = C_0(Y)$ then after a suitable biholomorphic change of coordinates we may assume that this cone is the line $\mathbb{C}_1 := \{x \in \mathbb{C}^m : x_2 = \ldots = x_m = 0\}$.

By the second version of the Puiseux Theorem ([Ł], II.6.2) we can parametrize X, Y (shrinking Ω if necessary) in the following way:

$$U \ni t \mapsto (t^p, \varphi(t)) \in X, \qquad \operatorname{ord}_0 \varphi > p,$$
$$U \ni \tau \mapsto (\tau^q, \psi(\tau)) \in Y, \qquad \operatorname{ord}_0 \psi > q,$$

where $\operatorname{ord}_0 \varphi$ denotes the smallest number among the orders $\operatorname{ord}_0 \varphi_i =: k_i$ at 0 ($i = 2, \ldots, m$). Put $d := p \cdot q$ and consider new parametrizations

$$\widetilde{U} \ni t \mapsto (t^d, \widetilde{\varphi}(t)) \in X, \qquad \operatorname{ord}_0 \widetilde{\varphi} > d,$$
$$\widetilde{U} \ni \tau \mapsto (\tau^d, \widetilde{\psi}(\tau)) \in Y, \qquad \operatorname{ord}_0 \widetilde{\psi} > d,$$

where $\widetilde{\varphi}(t) = \varphi(t^q)$ and $\widetilde{\psi}(\tau) = \psi(\tau^p)$.

Define $\widetilde{\varphi}(t) = (\widetilde{\varphi}_2(t), \ldots, \widetilde{\varphi}_m(t))$ and $\widetilde{\psi}(\tau) = (\widetilde{\psi}_2(\tau), \ldots, \widetilde{\psi}_m(\tau))$. Fix $v = (v_1, \ldots, v_m) \in C_0(X,Y)$ and $(c, 0) \in \mathbb{C}_1$. By the definition of $C_0(X,Y)$ and Theorem 3.11 from [W] we have sequences $\{t_n\}$ and $\{\tau_n\}$ of complex numbers converging to zero and such that

$$n^d(t_n^d - \tau_n^d, \widetilde{\varphi}_2(t_n) - \widetilde{\psi}_2(\tau_n), \ldots, \widetilde{\varphi}_m(t_n) - \widetilde{\psi}_m(\tau_n)) \to v.$$

Applying Lemma 2.1 to the sequence $\{t_n\}$ yields a sequence $\{h_n\}$ which has properties (1)–(3). By substituting $t_n + h_n$ for t_n, we move the points on the curve X a little and for the first coordinate we obtain the following limit (as $n \to \infty$):

$$n^d((t_n + h_n)^d - \tau_n^d) = n^d((t_n + h_n)^d - t_n^d) + n^d(t_n^d - \tau_n^d) \to c + v_1$$

while the equality

$$n^d(\widetilde{\varphi}_i(t_n + h_n) - \widetilde{\psi}_i(\tau_n)) = n^d(\widetilde{\varphi}_i(t_n + h_n) - \widetilde{\varphi}_i(t_n)) + n^d(\widetilde{\varphi}_i(t_n) - \widetilde{\psi}_i(\tau_n))$$

shows that for $i \ge 2$ each coordinate converges to the respective v_i. Consequently, $v + (c, 0) \in C_0(X,Y)$. Since $C_0(X) = \mathbb{C}_1$ we conclude that $C_0(X,Y) + C_0(X) = C_0(X,Y)$ and the theorem follows.

Corollary 3.2. *The relative tangent cone to analytic curves X and Y is a finite union of planes, all of which contain both $C_0(X)$ and $C_0(Y)$.*

Proof. According to Property 2.9 in [ATW], the relative tangent cone to X and Y is an algebraic cone of pure dimension 2. If $C_0(X) \cap C_0(Y) = \{0\}$, then by Property 2.10 in [ATW] we have $C_0(Y) + C_0(X) = C_0(X,Y)$ and our corollary follows. If $C_0(X) = C_0(Y)$ and a hypersurface H intersects tangent lines isolatedly, then we have $C_0(X,Y) = C_0(X,Y) \cap H + C_0(X)$.

This implies that the cone $C_0(X,Y)\cap H$ has pure dimension 1, therefore it is a finite system of lines in H. This ends the proof.

4. Multiplicity of the intersection of analytic curves. Theorem 3.1 can be applied to counting the multiplicity of intersection of analytic curves in a simpler way. We begin with a theorem which reduces the calculations of the intersection multiplicity $i(X\cdot Y;0)$ to the calculations of the multiplicity of a special holomorphic mapping.

Let curves X and Y have parametrizations

$$U\ni t\mapsto \Phi(t):=\ (t^p,\varphi(t))\in X,\qquad \operatorname{ord}_0\varphi>p,$$
$$U\ni \tau\mapsto \Psi(\tau):=(\tau^q,\psi(\tau))\in Y,\qquad \operatorname{ord}_0\psi>q,$$

as in the second version of the Puiseux Theorem ([Ł], II.6.2). By Corollary 3.2 there exists a finite system of vector lines $\mathbb{S}\subset\mathbb{C}^{m-1}$ such that

$$C_0(X,Y)=\mathbb{C}\times\mathbb{S}=\mathbb{C}_1+(\{0\}\times\mathbb{S})\subset\mathbb{C}\times\mathbb{C}^{m-1}=\mathbb{C}^m.$$

For a linear form $l:\mathbb{C}^{m-1}\to\mathbb{C}$ we define

$$f_l:\mathbb{C}^m\times\mathbb{C}^m\ni(x,y)=((x_1,x'),(y_1,y'))\mapsto(x_1-y_1,l(x'-y'))\in\mathbb{C}^2.$$

If the mapping $f_l\circ(\Phi\times\Psi)$ has an isolated zero at $0\in\mathbb{C}^2$ then we denote by $\mu_0(f_l\circ(\Phi\times\Psi))$ the multiplicity of the mapping at 0. We use the above notation in the following theorem.

THEOREM 4.1. *If* $\operatorname{Ker} l\cap\mathbb{S}=\{0\}$ *then* $i(X\cdot Y;0)=\mu_0(f_l\circ(\Phi\times\Psi))$.

Proof. Let $T:=X\times Y$, $\pi:\mathbb{C}^m\times\mathbb{C}^m\ni(x,y)\mapsto x-y\in\mathbb{C}^m$ and $\Delta:=\operatorname{Ker}\pi$. Using the theory developed in [ATW] to finding the multiplicity of the isolated intersection of X and Y we calculate ([ATW], Def. 5.1) the multiplicity of the isolated intersection of T with the subspace Δ at the point $0\in\mathbb{C}^m\times\mathbb{C}^m$.

Denote by L the set $\operatorname{Ker} f_l$; L is a linear subspace of $\mathbb{C}^m\times\mathbb{C}^m$ of codimension 2. Moreover, we have (see [ATW], Lemma 2.4)

$$C_0(T,\Delta)=\pi^{-1}(\mathbb{C}\times\mathbb{S})=\{(x,y)\in\mathbb{C}^m\times\mathbb{C}^m : x'-y'\in\mathbb{S}\}.$$

Since $L=\{(x,y)\in\mathbb{C}^m\times\mathbb{C}^m : x_1=y_1,\ x'-y'\in\operatorname{Ker} l\}$ our assumption implies $L\cap C_0(T,\Delta)=\Delta$. From Theorem 4.4 in [ATW] we get

$$i(X\cdot Y;0)=i(T\cdot L;0).$$

According to Definition 4.1 in [TW], the mapping $\Phi\times\Psi:U^2\to T=X\times Y$ is a 1-parametrization of T and $L=Z_{f_l}$ is a cycle of zeros of the mapping f_l. By Theorem 4.2 in [TW],

$$\deg(T\cdot Z_{f_l})=\deg(Z_{f_l\circ(\Phi\times\Psi)}).$$

We have $\deg(T\cdot Z_{f_l})=\deg(T\cdot L)=i(X\cdot Y;0)$ and $\deg(Z_{f_l\circ(\Phi\times\Psi)})=\mu_0(f_l\circ(\Phi\times\Psi))$, which completes the proof.

EXAMPLE 4.2. Take the algebraic curves

$$X = \{(t^2, t^3, 0) \in \mathbb{C}^3 : t \in \mathbb{C}\}, \quad Y = \{(\tau^2, 0, \tau^3) \in \mathbb{C}^3 : \tau \in \mathbb{C}\}.$$

We have $X \cap Y = \{0\}$ and $C_0(X) = C_0(Y) = \mathbb{C}_1$. A simple calculation shows that $C_0(X, Y) = \{(x, y, z) \in \mathbb{C}^3 : y^2 = z^2\}$ is a union of two planes. By Theorem 4.1, $i(X \cdot Y; 0)$ is equal to the multiplicity at $0 \in \mathbb{C}^2$ of the mapping

$$F : \mathbb{C}^2 \ni (t, \tau) \mapsto (t^2 - \tau^2, t^3) \in \mathbb{C}^2$$

produced (as above) by the linear form $l : \mathbb{C}^2 \ni (y, z) \mapsto y \in \mathbb{C}$. Since $\mu_0(F) = 6$ we conclude that the intersection multiplicity of X and Y at 0 is equal to 6.

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