ANNALES
POLONICI MATHEMATICI
LXXII.2 (1999)

On a theorem of Cauchy–Kovalevskaya type for a class of nonlinear PDE's of higher order with deviating arguments

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Abstract. We prove an existence theorem of Cauchy–Kovalevskaya type for the equation

$$D_t u(t,z) = f(t,z,u(\alpha^{(0)}(t,z)), D_z u(\alpha^{(1)}(t,z)), \ldots, D_z^k u(\alpha^{(k)}(t,z)))$$

where f is a polynomial with respect to the last k variables.

1. Introduction. We study the existence and uniqueness of solutions to the following Cauchy problem:

$$(1)\qquad \begin{aligned} &D_t u(t,z) = f(t,z,u(\alpha^{(0)}(t,z)), D_z u(\alpha^{(1)}(t,z)), \ldots, D_z^k u(\alpha^{(k)}(t,z))),\\ &u(0,z) = 0. \end{aligned}$$

The presence of deviating arguments $\alpha^{(1)}, \ldots, \alpha^{(k)}$ makes problem (1) difficult. The classical methods, such as the theory of characteristics, difference schemes for $k = 1$, transformations to a differential-integral equation (when $k \geq 2$ and f is linear with respect to the last variable), fail to work if $\alpha^{(k)}(t,z) \neq (t,z)$.

In the case of $k = 1$ and real variables, applying the Banach contraction principle, the Neumann series and the Fourier series methods resulted in getting certain existence theorems for limited classes of deviating arguments (see [1]), and for some linear equations ([9], [5]). There are more effective methods concerning analytic solutions to (1). These methods are based on power series expansions ([2]–[4]), properties of the Bernstein classes of analytic functions ([11]) and on the Nagumo lemma ([6, 7, 10], [12]–[15]). The last method is used in the present paper.

1991 *Mathematics Subject Classification*: Primary 35A10; Secondary 35A07, 35G25, 35R10.

Key words and phrases: Cauchy–Kovalevskaya theorem, deviating argument, nonlinear equation, analytic solution.

The classical Kovalevskaya counterexample $D_t u = D_z^2 u$, $u(0,z) = (1-z)^{-1}$ ([8, 12]) shows that if $k > 1$ then problem (1) may have no analytic solutions, even for elementary right-hand side. In [6, 7], existence results were obtained under the assumption that the deviating arguments are separated from the lateral boundary of the Haar pyramid. We relax this condition when the right-hand side in (1) is a polynomial with respect to the last k variables.

2. Banach spaces E_p. Nagumo lemma. Let Ω be an open bounded subset of the complex plane $\mathbb{C}$ and

$$d(z) = \operatorname{dist}(z, \partial\Omega), \qquad d(t,z) = d(z) - |t|/\eta,$$
$$G_\eta = \{(t,z) \in \mathbb{C}^2 : z \in \Omega,\ d(t,z) > 0,\ |t| < t_0\},$$

where $\eta, t_0 > 0$ are fixed. The set G_η is the *Haar pyramid* with slope η, and $d(t,z)$ is the distance between (t,z) and the boundary of t-intersection of G_η.

Let $H(G)$ denote the space of all analytic functions on G. For $p \geq 0$ and $u \in H(G_\eta)$ we define

$$\|u\|_p = \sup_{(t,z)\in G_\eta} |u(t,z)| d(t,z)^p, \qquad E_p = \{u \in H(G_\eta) : \|u\|_p < +\infty\}.$$

The set E_p is a Banach space with the natural linear structure and the norm $\|\cdot\|_p$.

Our investigations are based on the following

Lemma 1. *If* $a, u \in H(G_\eta)$, *then*

(1) $\|D_z u\|_{p+1} \leq C_p \|u\|_p$, *where* $C_p = (p+1)(1+1/p)^p$, $C_0 = 1$,
(2) $\|a(\cdot)u(\cdot)\|_{p+q} \leq \|a\|_q \|u\|_p$,
(3) $\|u(\alpha(\cdot))\|_p \leq \lambda_\alpha^p \|u\|_p$ *if* $\alpha(G_\eta) \subset G_\eta$, *where*

$$\lambda_\alpha = \sup_{(t,z)\in G_\eta} \frac{d(t,z)}{d(\alpha(t,z))},$$

(4) $\|Iu\|_p \leq (\eta/p)\|u\|_{p+1}$, *where* $(Iu)(t,z) = \int_0^t u(s,z)\, ds$.

The assertion (1) is the Nagumo lemma (cf. [10]). Conditions (2)–(3), (4) are proved in [6], [13], respectively.

3. Existence and uniqueness results. In order to present the main idea, we consider a simple case of equation (1):

$$(2) \qquad \begin{aligned} &D_t u(t,z) = a(t,z,u(\alpha(t,z)))(D_z^k u(\beta(t,z)))^n + b(t,z,u(\gamma(t,z))), \\ &u(0,z) = 0. \end{aligned}$$

THEOREM 1. *Suppose that for some* $r, h > 0$ *and* $\kappa \in (0,1)$, *there exist* $\omega \in [0,\kappa)$, $\lambda, \eta > 0$, *and* $A, B \geq 0$ *such that* a, b *are analytic on* $G_\eta \times \overline{K}(0,r)$ (*where* $\overline{K}(0,r)$ *is the closed ball in* $\mathbb{C}$ *centered at the origin and with radius* r), *and* $\alpha, \beta, \gamma : G_\eta \to G_\eta$ *are analytic. Assume that for* $(t,z) \in G_\eta$, $|u| \leq r$, *we have*

$$\begin{gathered} \|a(\cdot,u)\|_\omega \leq A, \quad \|b(\cdot,u)\|_\kappa \leq B, \\ d(t,z)^{\kappa-\omega} \leq \lambda d(\beta(t,z))^{n(\kappa+k-1)}, \\ \frac{\eta}{1-\kappa}\widehat{d}^{\,1-\kappa}[A\lambda(C_{\kappa,k-1}h)^n + B] \leq r, \\ \frac{\eta}{\kappa}[A\lambda(C_{\kappa,k-1}h)^n(C_\omega + C_{\kappa-\omega}) + C_\kappa B] \leq h, \end{gathered} \tag{3}$$

where

$$\widehat{d} = \sup_{(t,z)\in G_\eta} d(t,z), \quad C_{p,j} = C_pC_{p+1}\ldots C_{p+j-1}, \quad C_{p,0} = 1.$$

Then problem (2) *has an analytic solution defined on* G_η. *Moreover*, *if there exist constants* $\lambda_1, \lambda_3, p > 0$ *and* $A', B', \omega', \kappa' \geq 0$ *such that*

$$\begin{gathered} |a(t,z,u) - a(t,z,v)| \leq A'd(t,z)^{-\omega'}|u-v|, \\ |b(t,z,u) - b(t,z,v)| \leq B'd(t,z)^{-\kappa'}|u-v|, \\ d(t,z)^{p+1+\omega-\kappa-\omega'} \leq \lambda_1 d(\alpha(t,z))^p, \quad d(t,z)^{p+1-\kappa'} \leq \lambda_3 d(\gamma(t,z))^p, \\ L = \frac{\eta}{p}[A'(C_{\kappa,k-1}h)^n\lambda\lambda_1 + An(C_{\kappa,k-1}h)^{n-1}C_{p,k}\lambda\lambda_2^{p+1-\kappa} + B'\lambda_3] < 1 \end{gathered} \tag{4}$$

for $(t,z) \in G_\eta$, $|u|, |v| \leq r$, *where* $\lambda_2 = \sup\{d(t,z)d(\beta(t,z))^{-1} : (t,z) \in G_\eta\}$, *then the solution is unique in the set*

$$D = \{u \in E_0 : \|u\|_0 \leq r,\ \|D_z u\|_\kappa \leq h\}.$$

REMARK 1. If $\delta : G_\eta \to G_\eta$ and $d(t,z) \leq \tau d(\delta(t,z))$, then $d(t,z)^q \leq \tau' d(\delta(t,z))^{q'}$ for $q \geq q'$ and some $\tau' > 0$. This shows that the existence of constants λ_1 and λ_3 follows from the natural assumption

$$d(t,z) \leq \tau_1 d(\alpha(t,z)), \quad d(t,z) \leq \tau_2 d(\gamma(t,z)) \quad \text{if } \kappa + \omega' \leq 1 + \omega \text{ and } \kappa' \leq 1.$$

Since $\kappa - \omega \leq n(\kappa + k - 1)$, from (3) we have $\lambda_2 < +\infty$. Observe also that $L < 1$ and the last two inequalities in (3) are satisfied, provided η is sufficiently small.

Proof (of Theorem 1). Define

$$(Fu)(t,z) = \int_0^t [a(s,z,u(\alpha(s,z)))(D_z^k u(\beta(s,z)))^n + b(s,z,u(\gamma(s,z)))]\,ds.$$

We now prove that $F(D)\subset D$. If $u\in D$ then

$$\begin{aligned}|(D_z^k u(\beta(t,z)))^n| &\le (\|D_z^k u\|_{\kappa+k-1} d(\beta(t,z))^{-\kappa-k+1})^n\\ &\le (C_{\kappa,k-1}\|D_z u\|_\kappa)^n d(\beta(t,z))^{-n(\kappa+k-1)}\\ &\le (C_{\kappa,k-1}h)^n \lambda d(t,z)^{-\kappa+\omega},\end{aligned}$$

so we obtain

$$\begin{aligned}|D_t(Fu)(t,z)| &\le A\lambda d(t,z)^{-\omega} d(t,z)^{-\kappa+\omega}(C_{\kappa,k-1}h)^n + Bd(t,z)^{-\kappa}\\ &= (A\lambda(C_{\kappa,k-1}h)^n + B)d(t,z)^{-\kappa},\end{aligned}$$

hence

$$|(Fu)(t,z)| \le \frac{\eta}{1-\kappa}\widehat{d}^{1-\kappa}[A\lambda(C_{\kappa,k-1}h)^n + B] \le r.$$

Moreover, we get

$$\begin{aligned}|D_tD_z(Fu)(t,z)| &\le AC_\omega d(t,z)^{-\omega-1}(C_{\kappa,k-1}h\, d(\beta(t,z))^{-\kappa-k+1})^n\\ &\quad + Ad(t,z)^{-\omega}\left|\frac{\partial}{\partial z}(D_z^k u(\beta(t,z)))^n\right| + C_\kappa B d(t,z)^{-\kappa-1}\\ &\le AC_\omega(C_{\kappa,k-1}h)^n\lambda d(t,z)^{-\kappa-1}\\ &\quad + A\lambda C_{\kappa-\omega}(C_{\kappa,k-1}h)^n d(t,z)^{-\kappa-1} + C_\kappa B d(t,z)^{-\kappa-1}\end{aligned}$$

hence

$$|D_z(Fu)(t,z)|d(t,z)^\kappa \le \frac{\eta}{\kappa}[A\lambda(C_{\kappa,k-1}h)^n(C_\omega + C_{\kappa-\omega}) + C_\kappa B] \le h$$

and $Fu\in D$. The set D is a convex and compact subset of E_q for every $q>0$. We now prove that the operator F is continuous on D with respect to the norm $\|\cdot\|_q$, provided q is sufficiently large. For any $u,v\in D$, we have

$$\begin{aligned}&|(Fu)(t,z)-(Fv)(t,z)|\\ &\quad\le \int_0^{|t|} |a(s,z,u(\alpha(s,z))) - a(s,z,v(\alpha(s,z)))||D_z^k u(\beta(s,z))|^n\,|ds|\\ &\qquad + \int_0^{|t|} |a(s,z,v(\alpha(s,z)))||(D_z^k u(\beta(s,z)))^n - (D_z^k v(\beta(s,z)))^n|\,|ds|\\ &\qquad + \int_0^{|t|} |b(s,z,u(\gamma(s,z))) - b(s,z,v(\gamma(s,z)))|\,|ds|\\ &\quad\le \int_0^{|t|} |a(s,z,u(\alpha(s,z))) - a(s,z,v(\alpha(s,z)))|\\ &\qquad \times (C_{q,k-1}\|D_z u\|_q d(\beta(s,z))^{-q-k+1})^n\,|ds|\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{|t|} A d(s,z)^{-\omega} n[C_{q,k-1}\max\{\|D_z u\|_q, \|D_z v\|_q\} d(\beta(s,z))^{-q-k+1}]^{n-1} \\
&\times |D_z^k u(\beta(s,z)) - D_z^k v(\beta(s,z))|\,|ds| \\
&+\int_0^{|t|} |b(s,z,u(\gamma(s,z))) - b(s,z,v(\gamma(s,z)))|\,|ds| \\
\le{}& (C_{q,k-1}\|D_z u\|_q)^n \sup_{\mu\in[0,1]} |a(\mu t, z, u(\alpha(\mu t,z))) - a(\mu t, z, v(\alpha(\mu t, z)))| \\
&\times d(\mu t, z)\int_0^{|t|} d(s,z)^{-q-1}\frac{d(s,z)^q}{d(\beta(s,z))^{n(q+k-1)}}\,|ds| \\
&+ An[C_{q,k-1}\max\{\|D_z u\|_q, \|D_z v\|_q\}]^{n-1} \\
&\times \int_0^{|t|} d(s,z)^{-\omega} d(\beta(s,z))^{-(n-1)(q+k-1)} C_{q,k}\|u-v\|_q d(\beta(s,z))^{-q-k}\,|ds| \\
&+\int_0^{|t|} |b(s,z,u(\gamma(s,z))) - b(s,z,v(\gamma(s,z)))|\,|ds|.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sup_{(s,z)\in G_\eta} \frac{d(s,z)^q}{d(\beta(s,z))^{nq+m}}
&\le \lambda^{q/(\kappa-\omega)} \sup_{(s,z)\in G_\eta} \frac{d(\beta(s,z))^{qn(\kappa+k-1)/(\kappa-\omega)}}{d(\beta(s,z))^{nq+m}} \\
&= \lambda^{q/(\kappa-\omega)} \sup_{(s,z)\in G_\eta} d(\beta(s,z))^{qn(k-1+\omega)/(\kappa-\omega)-m} < +\infty
\end{aligned}
$$

for any $m > 0$ and for sufficiently large q, there exists a constant c such that

$$
\|Fu - Fv\|_q \le c\|u-v\|_q + c \sup_{(t,z)\in G_\eta} \Delta_{u,v}(t,z) d(t,z)
$$

for some $q > 0$, where

$$
\begin{aligned}
\Delta_{u,v}(s,z) = {}& |a(s,z,u(\alpha(s,z))) - a(s,z,v(\alpha(s,z)))| \\
&+ |b(s,z,u(\gamma(s,z))) - b(s,z,v(\gamma(s,z)))|.
\end{aligned}
$$

Fix $u \in D$. Let $d_0 > 0$ and $G(d_0) = \{(t,z) \in G_\eta : d(t,z) \ge d_0\}$. Then we get

$$
\begin{aligned}
\|Fu - Fv\|_q \le{}& c\|u-v\|_q + c \sup_{(t,z)\in G_\eta\setminus G(d_0)} \Delta_{u,v}(t,z) d(t,z) \\
&+ c \sup_{(t,z)\in G(d_0)} \Delta_{u,v}(t,z) d(t,z) = S_1 + S_2 + S_3.
\end{aligned}
$$

We prove that $S_1+S_2+S_3$ tends to zero if v tends to u in the norm $\|\cdot\|_q$. Since $u,v\in D$, we have

$$\varDelta_{u,v}(t,z)d(t,z)\le 2Ad(t,z)^{1-\omega}+2Bd(t,z)^{1-\kappa},$$

hence S_2 becomes small when d_0 is small enough. Given any fixed d_0, we observe that the functions a,b are uniformly continuous on $G(d_0)\times\overline{K}(0,r)$ and the functions α,γ are uniformly continuous on $G(d_0)$. Therefore, $S_3\to 0$ as $\|v-u\|_q\to 0$. This proves the continuity of F on D. The Schauder fixed point theorem completes the proof of the first assertion.

Applying conditions (4) with $u,v\in D$, $(t,z)\in G_\eta$, we have

$$\begin{aligned}
&|D_t[(Fu)-(Fv)](t,z)|\\
&\quad\le |a(t,z,u(\alpha(t,z)))-a(t,z,v(\alpha(t,z)))||D_z^k u(\beta(t,z))|^n\\
&\qquad+|a(t,z,v(\alpha(t,z)))||(D_z^k u(\beta(t,z)))^n-(D_z^k v(\beta(t,z)))^n|\\
&\qquad+|b(t,z,u(\gamma(t,z)))-b(t,z,v(\gamma(t,z)))|\\
&\quad\le A'd(t,z)^{-\omega'}|u(\alpha(t,z))-v(\alpha(t,z))|d(\alpha(t,z))^p d(\alpha(t,z))^{-p}\\
&\qquad\times(C_{\kappa,k-1}hd(\beta(t,z))^{-\kappa-k+1})^n\\
&\qquad+Ad(t,z)^{-\omega}n(C_{\kappa,k-1}hd(\beta(t,z))^{-\kappa-k+1})^{n-1}\\
&\qquad\times|D_z^k u(\beta(t,z))-D_z^k v(\beta(t,z))|\\
&\qquad+B'd(t,z)^{-\kappa'}|u(\gamma(t,z))-v(\gamma(t,z))|d(\gamma(t,z))^p d(\gamma(t,z))^{-p}\\
&\quad\le A'(C_{\kappa,k-1}h)^n\|u-v\|_p d(t,z)^{-\omega'}d(\alpha(t,z))^{-p}d(\beta(t,z))^{-n(\kappa+k-1)}\\
&\qquad+An(C_{\kappa,k-1}h)^{n-1}C_{p,k}\|u-v\|_p\\
&\qquad\times d(t,z)^{-\omega}d(\beta(t,z))^{-(n-1)(\kappa+k-1)}d(\beta(t,z))^{-k-p}\\
&\qquad+B'\|u-v\|_p d(t,z)^{-\kappa'}d(\gamma(t,z))^{-p}\\
&\quad\le[A'(C_{\kappa,k-1}h)^n\lambda\lambda_1+An(C_{\kappa,k-1}h)^{n-1}C_{p,k}\lambda\lambda_2^{p+1-\kappa}+B'\lambda_3]\\
&\qquad\times\|u-v\|_p d(t,z)^{-p-1},
\end{aligned}$$

hence $\|Fu-Fv\|_p\le L\|u-v\|_p$ and F is contractive on D with respect to the norm $\|\cdot\|_p$. The Banach contraction principle completes the proof.

Remark 2. Theorem 1 only gives a local existence (and uniqueness) result. Assume that $|\alpha_0(t,z)|,|\beta_0(t,z)|,|\gamma_0(t,z)|<|t|$ for $0<|t|<T$ (α_0, β_0, γ_0 are the time-coordinates of α,β,γ respectively), and a,b are analytic on $\varOmega\times K(0,T)\times\mathbb{C}$. Then we can extend any local solution of (2) to the set $\varOmega\times K(0,T)$ by a step-by-step method. Assumption (3) of Theorem 1 is

essential, and it is satisfied when there exists $d_0 > 0$ such that $d(\beta(t,z)) \geq d_0$ for $(t,z) \in G_\eta$. Such a condition is assumed in [6], [7]. One may expect that (3) cannot be satisfied when

$$\inf\{d(\beta(t,z)) : (t,z) \in G_\eta\} = 0.$$

We demonstrate in the Example below that, taking any k, n, κ, ω, there exists a deviating argument β which is not separated from the lateral boundary of the Haar pyramid, but (3) is satisfied. Moreover, the assumptions of Theorem 1 require η to be small enough. The deviating argument in the Example transforms G_η into itself for any $\eta, t_0 > 0$ sufficiently small.

Example. Take $r \geq 2^{m/(m-1)}$, $m > 1$. Define

$$\Omega = \{z \in \mathbb{C} : |z| < r,\ |\arg z| < \pi/2\}.$$

Take further

$$a \in \mathbb{C}, \quad a \neq 0, \quad 0 < \eta < \frac{2m-1}{m^2|a|}, \quad 0 < t_0 \leq \frac{1}{|a|b}, \quad b = r^{(m-1)/m}.$$

We have

$$d(z) = \min\{\operatorname{Re} z, r - |z|\}.$$

Define

$$\beta(t,z) = (at^2, z^{1/m}), \qquad |\arg z^{1/m}| < \frac{\pi}{2m}.$$

We prove that $\beta(G_\eta) \subset G_\eta$. Since $r > 1$, it is easily seen that $\beta(0,z) \in \Omega$ if $z \in \Omega$. Let

$$z^{1/m} = x\exp(i\phi), \qquad x \in (0, r^{1/m}), \qquad |\phi| < \frac{\pi}{2m}.$$

We get $r \geq 2r^{1/m} \geq x(1+\cos\phi)$, so $r - |z^{1/m}| = r - x \geq x\cos\phi = \operatorname{Re} z^{1/m}$, hence

$$d(z^{1/m}) = \operatorname{Re} z^{1/m}, \qquad z \in \Omega,$$

and

$$\frac{d(z)}{d(z^{1/m})^m} \leq \frac{\operatorname{Re} z}{(\operatorname{Re} z^{1/m})^m} = \frac{x^m \cos m\phi}{(x\cos\phi)^m} = \frac{\cos m\phi}{\cos^m \phi} \leq 1.$$

In particular,

$$d(z) \leq \sup_{y\in\Omega}(d(y^{1/m}))^{m-1} d(z^{1/m}) \leq b d(z^{1/m}),$$

$$d(\beta(t,z)) = d(z^{1/m}) - \frac{|at^2|}{\eta} \geq \frac{1}{b}d(z) - \frac{1}{b}\frac{|t|}{\eta} = \frac{d(t,z)}{b} > 0,$$

if $(t,z) \in G_\eta$. This implies $\beta(G_\eta) \subset G_\eta$. Now we prove that

$$A(t,z) = \frac{d(t,z)}{d(\beta(t,z))^m} \leq 1.$$

We have

$$A(t,z) \le \frac{\operatorname{Re} z - |t|/\eta}{(\operatorname{Re} z^{1/m} - |at^2|/\eta)^m}.$$

The estimate $\eta < (2m-1)/(m^2|a|)$ and the inequality $\cos m\phi \le \cos\phi \le 1$ imply that the right-hand side of the above inequality is decreasing in $|t| \in [0, \eta \operatorname{Re} z)$, thus its maximum is reached at $|t| = 0$, hence

$$A(t,z) \le \frac{\operatorname{Re} z}{(\operatorname{Re} z^{1/m})^m} \le 1. \tag{5}$$

Estimate (5) is optimal. Indeed, $A(0,z) = 1$ if $\operatorname{Im} z = 0$ and $\operatorname{Re} z < 1$. It follows from (5) that, if $m(\kappa-\omega) \ge n(\kappa+k-1)$, then

$$d(t,z)^{\kappa-\omega} \le d(\beta(t,z))^{m(\kappa-\omega)} \le \widehat{d}^{m(\kappa-\omega)-n(\kappa+k-1)} d(\beta(t,z))^{n(\kappa+k-1)},$$

therefore (3) is satisfied.

We generalize Theorem 1 to the equation

$$D_t u(t,z) = \sum_{n=1}^{N} \sum_{|k_n| \le K} a_{k_n}(t,z,u(\alpha_{k_n}(t,z))) \prod_{i=1}^{n} D_z^{k_{ni}} u(\beta_{k_n,i}(t,z)) + b(t,z,u(\gamma(t,z))),$$

where $k_n = (k_{n1}, \ldots, k_{nn})$ is such that $k_{ni} \ge 1$ and $|k_n| = k_{n1} + \ldots + k_{nn}$.

If all coefficients a_{k_n} vanish but one ($k_{n_0} = (k, \ldots, k)$) and $\beta_{k_n,i} = \beta$, $i = 1, \ldots, n$, then the above equation becomes equation (2).

THEOREM 2. *Suppose that there are $r, h > 0$, $\kappa \in (0,1)$, and $\omega_{k_n} \in [0,\kappa)$, $\eta, \lambda_{k_n} > 0$, $A_{k_n}, B \ge 0$ such that a_{k_n}, b are analytic functions on $G_\eta \times \overline{K}(0,r)$, and the functions $\alpha_{k_n}, \beta_{k_n,i}, \gamma$ map G_η into itself. Assume that, for $(t,z) \in G_\eta$, $|u| \le r$, we have*

$$\|a_{k_n}(\cdot,u)\|_{\omega_{k_n}} \le A_{k_n}, \qquad \|b(\cdot,u)\|_\kappa \le B,$$

$$d(t,z)^{\kappa-\omega_{k_n}} \le \lambda_{k_n} \prod_{i=1}^{n} d(\beta_{k_n,i}(t,z))^{\kappa+k_{ni}-1},$$

$$\frac{\eta}{1-\kappa} \widehat{d}^{1-\kappa} \Big[B + \sum_{n=1}^{N} \sum_{|k_n| \le K} A_{k_n} h^n \prod_{i=1}^{n} C_{\kappa,k_{ni}-1} \Big] \le r,$$

$$\frac{\eta}{\kappa} \Big[C_\kappa B + \sum_{n=1}^{N} \sum_{|k_n| \le K} A_{k_n} h^n \lambda_{k_n} (C_{\omega_{k_n}} + C_{\kappa-\omega_{k_n}}) \prod_{i=1}^{n} C_{\kappa,k_{ni}-1} \Big] \le h.$$

Then there exists an analytic solution to the homogeneous Cauchy problem for equation (6) in the set D. Moreover, if there exist constants $p, \lambda_{k_n}^{(1)}, \lambda^{(3)} > 0$, $A'_{k_n}, B', \omega'_{k_n}, \kappa' \ge 0$ such that

$$|a_{k_n}(t,z,u) - a_{k_n}(t,z,v)| \le A'_{k_n} d(t,z)^{-\omega'_{k_n}} |u-v|,$$

$$|b(t,z,u)-b(t,z,v)| \le B' d(t,z)^{-\kappa'}|u-v|,$$

$$d(t,z)^{p+1-\kappa+\omega_{k_n}-\omega'_{k_n}} \le \lambda^{(1)}_{k_n} d(\alpha_{k_n}(t,z))^p, \qquad d(t,z)^{p+1-\kappa'} \le \lambda^{(3)} d(\gamma(t,z))^p,$$

$$\frac{\eta}{p}\Big\{\sum_{n=1}^{N}\sum_{|k_n|\le K}\lambda_{k_n}\Big[A'_{k_n}h^n\Big(\prod_{i=1}^{n}C_{\kappa,k_{ni}-1}\Big)\lambda^{(1)}_{k_n}$$
$$+A_{k_n}h^{n-1}\sum_{j=1}^{n}\Big(\prod_{i=1,i\ne j}^{n}C_{\kappa,k_{ni}-1}\Big)C_{p,k_{nj}}\lambda^{(2)}_{k_n,j}\Big]+B'\lambda^{(3)}\Big\}<1,$$

for $(t,z)\in G_\eta$, $|u|,|v|\le r$, *where*

$$\lambda^{(2)}_{k_n,j} = \sup_{(t,z)\in G_\eta}\left(\frac{d(t,z)}{d(\beta_{k_n,j}(t,z))}\right)^{p+1-\kappa},$$

then the solution is unique in D.

We omit the proof, because its idea is similar to that of the proof of Theorem 1.

The results of this paper can be easily generalized for a multidimensional variable z and a strongly coupled system of equations. Moreover, the results hold true in the real case, i.e. for functions u of variables $(t,z)\in G_\eta\subset \mathbb{R}\times\mathbb{C}$ of class C^1 in t and analytic in z. It suffices to assume that the first coordinates of the deviating arguments of the unknown function are independent of z.

References

[1] A. Augustynowicz, *Existence and uniqueness of solutions for partial differential-functional equations of the first order with deviating arguments of the derivative of unknown function*, Serdica Math. J. 23 (1997), 203–210.

[2] —, *Analytic solutions to the first order partial differential equations with time delays at the derivatives*, Funct. Differ. Equations 6 (1999), 19–29.

[3] A. Augustynowicz and H. Leszczyński, *On the existence of analytic solutions of the Cauchy problem for first-order partial differential equations with retarded variables*, Comment. Math. Prace Mat. 36 (1996), 11–25.

[4] —, —, *On x-analytic solutions to the Cauchy problem for partial differential equations with retarded variables*, Z. Anal. Anwendungen 15 (1996), 345–356.

[5 —, —, *Periodic solutions to the Cauchy problem for PDEs with retarded variables*, submitted.

[6] A. Augustynowicz, H. Leszczyński and W. Walter, *Cauchy–Kovalevskaya theory for equations with deviating variables*, Aequationes Math. 58 (1999), 143–156.

[7] —, —, —, *Cauchy–Kovalevskaya theory for nonlinear equations with deviating variables*, Nonlinear Anal., to appear.

[8] S. von Kowalevsky, *Zur Theorie der partiellen Differentialgleichungen*, J. Reine Angew. Math. 80 (1875), 1–32.

[9] H. Leszczyński, *Fundamental solutions to linear first-order equations with a delay at derivatives*, Boll. Un. Mat. Ital. A (7) 10 (1996), 363–375.
[10] M. Nagumo, *Über das Anfangswertproblem partieller Differentialgleichungen*, Japan. J. Math. 18 (1942), 41–47.
[11] R. M. Redheffer and W. Walter, *Existence theorems for strongly coupled systems of partial differential equations over Bernstein classes*, Bull. Amer. Math. Soc. 82 (1976), 899–902.
[12] W. Walter, *An elementary proof of the Cauchy–Kovalevsky Theorem*, Amer. Math. Monthly 92 (1985), 115–126.
[13] —, *Functional differential equations of the Cauchy–Kovalevsky type*, Aequationes Math. 28 (1985), 102–113.
[14] T. Yamanaka, *A Cauchy–Kovalevskaja type theorem in the Gevrey class with a vector-valued time variable*, Comm. Partial Differential Equations 17 (1992), 1457–1502.
[15] T. Yamanaka and H. Tamaki, *Cauchy–Kovalevskaya theorem for functional partial differential equations*, Comment. Math. Univ. St. Paul. 29 (1980), 55–64.

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Reçu par la Rédaction le 30.11.1998
Révisé le 30.4.1999