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On asymptotic cyclicity of doubly stochastic operators

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Abstract. It is proved that a doubly stochastic operator P is weakly asymptotically cyclic if it almost overlaps supports. If moreover P is Frobenius–Perron or Harris then it is strongly asymptotically cyclic.

1. Introduction. Let $(X,\mathcal{A},\mu)$ be a (complete) σ-finite measure space. The Banach lattice of real $\mathcal{A}$-measurable functions f such that $|f|^p$ is μ-integrable (resp. $\operatorname{ess\,sup}|f| < \infty$) is denoted by $L^p(\mu)$ (resp. $L^\infty(\mu)$). $\|\cdot\|_p$ stands for the relevant norm. Functions equal μ-almost everywhere are identified. A linear operator $P : L^1(\mu) \to L^1(\mu)$ is called *Markov* if $Pf \ge 0$ and $\|Pf\|_1 = \|f\|_1$ for all $f \ge 0$, $f \in L^1(\mu)$. By $\mathcal{D} = \mathcal{D}(X,\mathcal{A},\mu)$ we denote the set of all (normalized) *densities* on X, that is,

$$\mathcal{D} = \{f \in L^1(\mu) : f \ge 0,\ \|f\|_1 = 1\}.$$

We say that $f_* \in \mathcal{D}$ is *stationary* if $Pf_* = f_*$. If $(X,\mathcal{A},\mu)$ is a probability space and $P\mathbf{1} = \mathbf{1}$ then a Markov operator P is called *doubly stochastic* (or *doubly markovian*). An important basic property of doubly stochastic operators is that together with their adjoints, they are positive linear contractions on each $L^p(\mu)$, where $1 \le p \le \infty$ (see Proposition 1.1 in [Br] for the details). In particular, instead of studying the convergence on L^1 we may pass to L^2 if necessary. It is a routine trick to identify a Markov operator P possessing a stationary, strictly positive density f_* with its rescaled version $\overline{P}f = P(ff_*)/f_*$, which is defined on $L^1(f_* d\mu)$. Clearly $\overline{P}$ is doubly stochastic. Therefore our results are formulated only for doubly stochastic operators. Their generalizations to Markov operators with strictly positive

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stationary densities are obvious. The strict positivity assumption may be removed if P is a Frobenius–Perron operator (see [Z]).

Let $S : X \to X$ be a nonsingular (i.e. $\mu(S^{-1}(A)) = 0$ if $\mu(A) = 0$) measurable transformation of $(X, \mathcal{A}, \mu)$. Recall that the corresponding *Frobenius–Perron operator* $P_S : L^1(\mu) \to L^1(\mu)$ is defined by $\int_A P_S f\, d\mu = \int_{S^{-1}(A)} f\, d\mu$. Clearly P_S is a Markov operator and its adjoint $P_S^* : L^\infty(\mu) \to L^\infty(\mu)$ is the composition operator $P_S^* h = h \circ S$ and is called the *Koopman operator*. If S preserves μ then the Koopman operator P_S^* is isometric on each $L^p(\mu)$, noninvertible in general.

Definition 1. A Markov operator $P : L^1(\mu) \to L^1(\mu)$ is called *strongly asymptotically cyclic* if there exist a finite family of densities $g_1, \ldots, g_r$ and linear functionals $\Lambda_1, \ldots, \Lambda_r$ such that

$$\lim_{n\to\infty} \Big\| P^n f - \sum_{j=1}^{r} \Lambda_j(f) g_{(j+n) \bmod r} \Big\|_1 = 0 \tag{1}$$

for all $f \in \mathcal{D}$. If $r = 1$ then P is called *asymptotically stable*. We also say that P is *weakly asymptotically cyclic* (w.a.c.) if the convergence (1) holds for the weak topology only.

Asymptotic properties of iterates of doubly stochastic operators have been extensively studied (see [B1], [B2], [B3], [BB], [K1], [K2], [R1], [R2], [Z]). For a comprehensive review of the subject and many examples the reader is referred to the monograph [LM].

It has been proved in [BB] (see also [R1]) that if P is Harris or Frobenius–Perron then asymptotic stability holds whenever P overlaps supports (i.e. $P^n f_1 \wedge P^n f_2 \neq 0$ for all densities f_1, f_2 and n large enough). In this paper the concept of overlapping is generalized. We discuss how asymptotic properties of iterates are affected. We introduce the following:

Definition 2. We say that a Markov operator $P : L^1(\mu) \to L^1(\mu)$ *almost overlaps supports* (a.o.s. for abbreviation) if there exists $d \geq 0$ such that for all densities $f_1, f_2 \in \mathcal{D}$ there exist $n = n(f_1, f_2)$ and $m = m(f_1, f_2)$ such that $|n - m| \leq d$ and $P^n f_1 \wedge P^m f_2 \neq 0$, where $\wedge$ stands for the ordinary minimum in $L^1(\mu)$.

Definition 3. We say that a Markov operator $P : L^1(\mu) \to L^1(\mu)$ *individually almost overlaps supports* (i.a.o.s. for abbreviation) if there exists $d \geq 1$ such that for every density $f \in \mathcal{D}$ there exist $n = n(f) < m = m(f) \leq n + d$ such that $P^n f \wedge P^m f \neq 0$.

If for every $f \in L^1(X, \mathcal{A}, \mu)$ the iterates $P^n f$ have a norm convergent subsequence (i.e. $\omega_1(f) = \{g : \|P^{n_k} f - g\|_1 \to 0 \text{ for some } n_k \to \infty\} \neq \emptyset$) and if P a.o.s. then P is asymptotically cyclic (see [B3]). Similar results were obtained in [B1] for kernel Markov operators (i.e. $Pf(x) = \int k(x, y) f(y)\, d\mu(y)$

for suitable $k(x,y)$). In this case $\omega_1(f) \neq \emptyset$ is compact due to Krasnosel'skiĭ's theorem (see [L] for a self-contained proof). Our current approach differs from [BB] and is based on ideas of [F], where most of our notation and terminology come from. We briefly recall the necessary ones. A Markov operator $P : L^1(\mu) \to L^1(\mu)$ is said to be *conservative* if for some (equivalently, all) strictly positive $f \in L^1(\mu)$ we have $\sum_{n=0}^{\infty} P^n f(x) = \infty$ μ-a.e. It is well known that if $P^*h \le h$ for some $h \in L^\infty(\mu)$ then $P^*h = h$ whenever P is conservative. Clearly each Markov operator with strictly positive stationary density is conservative. Let us recall that conservative Markov operators P (in particular all doubly stochastic operators) are *nondisappearing*, i.e. if $P^*f = 0$ for some $f \ge 0$ then $f = 0$. Hence (see Lemma 0 in [KL] for the details) if $P^*g = \mathbf{1}_A$ with $0 \le g \le 1$ then there exists a unique $E \in \mathcal{A}$ such that $g = \mathbf{1}_E$. The family of all $A \in \mathcal{A}$ such that for every n there exists $A_n \in \mathcal{A}$ such that $P^{*n}\mathbf{1}_A = \mathbf{1}_{A_n}$ is denoted by $\Sigma_{\mathrm{d}}(P)$. Clearly $\Sigma_{\mathrm{d}}(P)$ is a sub-σ-algebra if P is doubly stochastic, and it is then called a *deterministic σ-algebra*. By $\Sigma_1(P)$ we denote the sub-σ-algebra of $\Sigma_{\mathrm{d}}(P)$ consisting of all A such that for every natural n we have $P^{*n}P^n\mathbf{1}_A = P^nP^{*n}\mathbf{1}_A = \mathbf{1}_A$ (see [F] for the details). By symmetry $\Sigma_1(P) = \Sigma_1(P^*)$.

We start with the following:

PROPOSITION 1. *Let P be a doubly stochastic operator on $L^1(X,\mathcal{A},\mu)$. If P i.a.o.s. then there exists $r \le d!$ such that $P^r\mathbf{1}_A = \mathbf{1}_A$ for all $A \in \Sigma_{\mathrm{d}}(P^*) = \Sigma_1(P)$, where d comes from Definition* 3. *Moreover for every $f \in L^p(X,\mathcal{A},\mu)$, weak $\lim_{n\to\infty} P^{rn}f$ exists and belongs to $L^p(X,\Sigma_{\mathrm{d}}(P^*),\mu)$. If P a.o.s. then $\Sigma_{\mathrm{d}}(P^*) = \Sigma_1(P)$ is finite* (*atomic*) *and consists of at most $d+1$ atoms.*

Proof. Given $A \in \Sigma_{\mathrm{d}}(P^*)$ we consider the maximal natural r_A for which there exists $\Sigma_{\mathrm{d}}(P^*) \ni B \subseteq A$ such that

$$\mathbf{1}_B, P\mathbf{1}_B = \mathbf{1}_{B_1}, \ldots, P^{r-1}\mathbf{1}_B = \mathbf{1}_{B_{r-1}}$$

are pairwise orthogonal. We notice that always $r_A \le d$. In fact, by the i.a.o.s. assumption we can choose $n < m$ with $m - n \le d$ such that $P^m\mathbf{1}_B \wedge P^n\mathbf{1}_B \neq 0$. Then

$$0 \neq P^{*n}P^m\mathbf{1}_B \wedge P^{*n}P^n\mathbf{1}_B = P^{m-n}\mathbf{1}_B \wedge \mathbf{1}_B,$$

and $r_A \le d$ follows.

If $P^{r_A}\mathbf{1}_B \neq \mathbf{1}_B$ then we define $D = B \setminus B_{r_A} \neq \emptyset$. Clearly $D \subseteq A$ and

$$\mathbf{1}_D, P\mathbf{1}_D = \mathbf{1}_{D_1}, \ldots, P^{r_A}\mathbf{1}_D = \mathbf{1}_{D_{r_A}}$$

are pairwise orthogonal, contradicting the maximality of r_A. Hence $P^{r_A}\mathbf{1}_B = \mathbf{1}_B$.

Now let

$$\mathcal{C}_A = \{B \in \Sigma_{\mathrm{d}}(P^*) : B \subseteq A, \text{ and } P^r\mathbf{1}_B = \mathbf{1}_B \text{ for some } 1 \le r \le d\}.$$

It is not hard to see that $A = \bigcup_{j=1}^{d} B_j$, where $P^j \mathbf{1}_{B_j} = \mathbf{1}_{B_j}$ for every j (some B_j may be empty). Finally, define $R_A = \mathrm{LCM}\{j : B_j \neq \emptyset\} \leq d!$. Then $P^{R_A}\mathbf{1}_B = \mathbf{1}_B$ for every $B \in \Sigma_{\mathrm{d}}(P^*) \cap A$. Substituting $X = A$ we get $P^r \mathbf{1}_B = \mathbf{1}_B$ for all $B \in \Sigma_{\mathrm{d}}(P^*)$, where $r = R_X$. In particular $P^r = \mathrm{Id}$ on $L^p(X, \Sigma_{\mathrm{d}}(P^*), \mu) \supseteq L^p(X, \Sigma_1(P), \mu)$. Choose $f \in L^2(X, \Sigma_{\mathrm{d}}(P^*), \mu) \ominus L^2(X, \Sigma_1(P), \mu)$. By Theorem A on page 85 in [F] we have weak $\lim_{n\to\infty} P^n f = 0$. On the other hand $P^{rn} f = f$ for every n. Therefore $f = 0$. This proves that $\Sigma_{\mathrm{d}}(P^*) = \Sigma_1(P^*) \subseteq \Sigma_{\mathrm{d}}(P)$.

Now assume that P a.o.s. and as before let r_X stand for the length of the longest orthogonal sequence $\mathbf{1}_A, P\mathbf{1}_A, \ldots, P^{r_X-1}\mathbf{1}_A$, where $A \in \Sigma_{\mathrm{d}}(P^*)$. We have already noticed that $P^{r_X}\mathbf{1}_A = \mathbf{1}_A$ and $r_X \leq d$.

Suppose that A is not an atom. Choose an arbitrary $\Sigma_{\mathrm{d}}(P^*) \ni B \subsetneq A$. The functions $\mathbf{1}_B, P\mathbf{1}_B, \ldots, P^{r-1}\mathbf{1}_B$ are also pairwise orthogonal. If $P^{r_X}\mathbf{1}_B = \mathbf{1}_B$ then the sequences $\mathbf{1}_B, P\mathbf{1}_B, \ldots, P^{r-1}\mathbf{1}_B, \ldots$ and $\mathbf{1}_{A\setminus B}, P\mathbf{1}_{A\setminus B}, \ldots \ldots, P^{r-1}\mathbf{1}_{A\setminus B}, \ldots$ are disjoint. This contradicts the a.o.s. assumption. On the other hand if $P^{r_X}\mathbf{1}_B \neq \mathbf{1}_B$ we may produce a set $D = B \setminus B_{r_X}$ with $r_D > r_X$, contradicting the maximality of r_X. We conclude that A is an atom. Because of a.o.s. we have $A \cup A_1 \cup \ldots \cup A_{r_X-1} = X$. Clearly all A_j, where $0 \leq j \leq r-1$, are atoms as well. In particular $\Sigma_{\mathrm{d}}(P^*)$ is finite and atomic. We easily get $P^n\mathbf{1}_B = P^s\mathbf{1}_B$, where $s = n \bmod r$ and $r = r_X = R_X$ for simplicity. ∎

The following corollary follows directly from Proposition 1 and Theorem A on page 85 in [F].

Corollary 1. *A doubly stochastic operator P with the a.o.s. property is weakly asymptotically cyclic. In particular for every $f \in L^1(X, \mathcal{A}, \mu)$ we have*

$$\text{weak} \lim_{n\to\infty} \Big(P^n f - \beta \sum_{j=0}^{r-1} \Big(\int_{A_j} f\, d\mu\Big) \mathbf{1}_{A_{(j+n) \bmod r}}\Big) = 0,$$

where $A_0, A_1, \ldots, A_{r-1}$ are the atoms of $\Sigma_{\mathrm{d}}(P^*)$ and $\beta = 1/\mu(A_0)$.

The next result is a generalization of Theorem 2 which was originally proved in [B3] using different methods. The present version has an "individual" character. In [B3] we assume that $\omega_1(f) \neq \emptyset$ for all $f \in \mathcal{D}$.

Theorem 1. *Let P be an a.o.s. doubly stochastic operator. If $f \in L^1(X, \mathcal{A}, \mu)$ is such that $\omega_1(f) \neq \emptyset$ then*

$$\lim_{n\to\infty} \Big\| P^n f - \beta \sum_{j=0}^{r-1} \Big(\int_{A_j} f\, d\mu\Big) \mathbf{1}_{A_{(j+n) \bmod r}} \Big\|_1 = 0,$$

where $A_0, A_1, \ldots, A_{r-1}$ are the atoms of $\Sigma_{\mathrm{d}}(P^*)$ and $\beta = 1/\mu(A_0)$.

Proof. Let $n_k \to \infty$ be such that $P^{n_k} f$ converges in L^1 norm to some g. Since r is finite there exists a subsequence $n_{k_j} = \text{const} = d \bmod r$. By Corollary 1 we have $g = \beta \sum_{j=0}^{r-1} (\int_{A_j} f\,d\mu) \mathbf{1}_{A_{(j+d) \bmod r}}$. Clearly g is P^r-invariant. The convergence of $P^{rn} f$ to g along some subsequence implies the convergence of the whole sequence $P^{rn} f$ as P is a contraction. We get

$$\lim_{n\to\infty} \|P^{rn+d} f - g\|_1 = 0.$$

After a slight reformulation we obtain the strong asymptotic cyclicity of $P^n f$ as all sequences $P^{n_k+j} f$ are norm convergent to $P^j g$. ■

The proof of Proposition 1 shows that $P^r = \mathrm{Id}$ on $L^2(X, \Sigma_{\mathrm{d}}(P^*), \mu)$ whenever P i.a.o.s. However in this case $\Sigma_{\mathrm{d}}(P^*)$ is not necessarily finite (atomic). By [F], for $f \in L^2(X, \mathcal{A}, \mu)$, weak $\lim_{n\to\infty} P^{rn} f = E(f \mid \Sigma_{\mathrm{d}}(P^*))$. In particular all weak limits of $P^n f$ are P^r-invariant. We obtain another generalization of [B3]:

PROPOSITION 2. *Let P be an i.a.o.s. doubly stochastic operator. Then there exists $r \le d!$ such that for every $f \in L^p(X, \mathcal{A}, \mu)$ with $\omega_p(f) \ne \emptyset$, where $1 \le p < \infty$, we have*

$$\lim_{n\to\infty} \|P^{rn} f - E(f \mid \Sigma_{\mathrm{d}}(P^*))\|_p = 0,$$

where $E(\cdot \mid \Sigma_{\mathrm{d}}(P^*))$ stands for the conditional expectation operator with respect to the σ-algebra $\Sigma_{\mathrm{d}}(P^*)$.

Proof. Without loss of generality we may confine our proof to $L^2(X, \mathcal{A}, \mu)$ only. As in the proof of Theorem 1 we show that $P^{rn} f$ converges in L^2 norm to some g. Given $f \in L^2(X, \mathcal{A}, \mu)$ let $f = f_1 + f_2$, where $f_1 \in L^2(X, \Sigma_{\mathrm{d}}(P^*), \mu)$ and $f_2 \perp L^2(X, \Sigma_{\mathrm{d}}(P^*), \mu)$. Since weak $\lim_{n\to\infty} P^{rn} f_2 = 0$ we have $g = f_1$ as f_1 is P^r-invariant. Clearly $f_1 = E(f \mid \Sigma_{\mathrm{d}}(P^*))$ and the proof is complete. ■

If $\Sigma_{\mathrm{d}}(P^*)$ is fully atomic (for instance when P is Harris or simply kernel), then X may be decomposed into disjoint cycles. Namely $X = \bigcup_{k=1} \bigcup_{j=0}^{r_k-1} A_{k,j}$ and $P^n \mathbf{1}_{A_{k,j}} = \mathbf{1}_{A_{k,(j+n) \bmod r_k}}$. This in conjunction with Corollary 1 gives

COROLLARY 2. *Let P be an i.a.o.s. doubly stochastic operator on $L^1(X, \mathcal{A}, \mu)$. If $\Sigma_{\mathrm{d}}(P^*)$ is atomic with atoms $A_{k,j}$ described as above then for every f we have*

$$\text{(2)} \qquad \textit{weak} \lim_{n\to\infty} \Big(P^n f - \sum_{k=1} \sum_{j=0}^{r_k-1} \beta_k \Big(\int\limits_{A_{k,j}} f\,d\mu \Big) \mathbf{1}_{A_{k,(j+n) \bmod r_k}} \Big) = 0,$$

where $\beta_k = 1/\mu(A_{k,j})$. If moreover $\omega_1(f) \ne \emptyset$ then the convergence (2) *is in norm.*

If P is a Frobenius–Perron operator then a.o.s. implies strong asymptotic cyclicity because P^r restricted to the spaces $L^1(A_j, \mathcal{A} \cap A_j, \mu(\cdot \cap A_j))$ is asymptotically stable. This is because the tail σ-algebras of all $P^r|_{A_j}$ are trivial. The same result is obtained in [BB] using functional arguments. We recall that $\lim_{n\to\infty} P^{*n}P^n = Q$ exists in the L^2 strong operator topology. Obviously Q is doubly stochastic. It has been noticed in [BB] that Q is a projection ($Q^2 = Q$) if P is Frobenius–Perron. Clearly Q is a projection if P is strongly asymptotically cyclic. Here we adapt some elements of [BB] to show:

THEOREM 2. *Let P be an a.o.s. doubly stochastic operator on $L^1(X, \mathcal{A}, \mu)$. Then the following conditions are equivalent*:

(i) *P is strongly asymptotically cyclic*,
(ii) *Q and P commute*,
(iii) *Q is a projection.*

P r o o f. (i)$\Rightarrow$(ii). Let $f \in L^2(\mu)$. It follows from Theorem 1 that

$$\lim_{n\to\infty} P^{rn} f = \sum_{j=0}^{r-1} \frac{1}{\mu(A_j)} \Big(\int_{A_j} f \, d\mu\Big) \mathbf{1}_{A_j} = E(f)$$

in L^2 norm (we may switch from L^1 to L^2 because all L^p strong operator topologies, where $1 \le p < \infty$, coincide on the set of doubly stochastic operators; see [Br] for the details). Since $P^{*r}P^r = \mathrm{Id}$ on $L^2(X, \Sigma_{\mathrm{d}}(P^*), \mu)$ and P^* is an L^2 contraction we get

$$P^{*r}E(f) = P^{*r} \lim_{n\to\infty} P^{rn} f = P^{*r}P^r \lim_{n\to\infty} P^{r(n-1)} f = E(f).$$

We have

$$\begin{aligned} \|Qf - E(f)\|_2 &= \lim_{n\to\infty} \|P^{*rn}P^{rn} f - E(f)\|_2 \\ &= \lim_{n\to\infty} \|P^{*rn}(P^{rn} f - E(f))\|_2 \\ &\le \lim_{n\to\infty} \|P^{rn} f - E(f)\|_2 = 0. \end{aligned}$$

This means that $Q = E$. Now (ii) is clear as

$$QPf = E(Pf) = \lim_{n\to\infty} P^{rn} Pf = P \lim_{n\to\infty} P^{rn} f = PE(f) = PQf.$$

(ii)$\Rightarrow$(iii). For every n and $f \in L^2(\mu)$ we have $Qf = P^{*n}QP^n f$. If Q and P commute then

$$Qf = P^{*n}QP^n f = P^{*n}P^n Qf = \lim_{n\to\infty} P^{*n}P^n Qf = Q^2 f.$$

Therefore Q is a projection.

(iii)$\Rightarrow$(i). It follows from Proposition 1 that P is weakly asymptotically cyclic. First we note that the invariant σ-algebra $\Sigma_{\mathrm{i}}(Q)$ coincides with $\Sigma_{\mathrm{d}}(P^*)$. This easily follows from the identity $Q = P^{*n}QP^n$. In fact, given

$A \in \Sigma_{\mathrm{i}}(Q)$ we apply Lemma 0 from [KL] to obtain $P^n \mathbf{1}_A = \mathbf{1}_{A_n}$ for every natural n. This gives $A \in \Sigma_{\mathrm{d}}(P^*)$. On the other hand if $P^n \mathbf{1}_A = \mathbf{1}_{A_n}$ then obviously $P^{*n} P^n \mathbf{1}_A = \mathbf{1}_A$ and passing with n to infinity we obtain $A \in \Sigma_{\mathrm{i}}(Q)$. The equality $\Sigma_{\mathrm{i}}(Q) = \Sigma_{\mathrm{d}}(P^*) = \Sigma_1(P)$ is proved. We get

$$Qf = E(f \mid \Sigma_{\mathrm{d}}(P^*)) = \sum_{j=0}^{r-1} \frac{1}{\mu(A_j)} \Big(\int_{A_j} f \, d\mu \Big) \mathbf{1}_{A_j},$$

where $A_0, A_1, \ldots, A_{r-1}$ are the atoms of $\Sigma_{\mathrm{d}}(P^*)$. In particular we have $Qf = (1/\mu(A_j))(\int f \, d\mu)\mathbf{1}_{A_j}$ if f is concentrated on A_j. Repeating arguments from [BB] for every $f \in \mathcal{D}$ which is concentrated on A_j we get

$$\begin{aligned}
\Big\| P^{rn} f - \frac{1}{\mu(A_j)} \mathbf{1}_{A_j} \Big\|_2^2 &= \int \Big(P^{rn} f - \frac{1}{\mu(A_j)} \mathbf{1}_{A_j} \Big) \Big(P^{rn} f - \frac{1}{\mu(A_j)} \mathbf{1}_{A_j} \Big) \, d\mu \\
&= \int P^{rn} f \cdot P^{rn} f \, d\mu - \frac{1}{\mu(A_j)} \\
&= \int P^{*rn} P^{rn} f \cdot f \, d\mu - \frac{1}{\mu(A_j)} \to \int Qf \cdot f \, d\mu - \frac{1}{\mu(A_j)} \\
&= \int \frac{1}{\mu(A_j)} \cdot f \, d\mu - \frac{1}{\mu(A_j)} = 0.
\end{aligned}$$

Since A_j's cover the whole space X we obtain

$$\lim_{n \to \infty} \Big\| P^{rn} f - \sum_{j=0}^{r-1} \Big(\int_{A_j} f \, d\mu \Big) \frac{1}{\mu(A_j)} \mathbf{1}_{A_j} \Big\|_2 = 0$$

for every $f \in L^2(\mu)$. Clearly the convergence $P^{rn} f \to E(f)$ in $L^2(\mu)$ implies the norm convergence in $L^1(\mu)$, thus P_S is strongly asymptotically cyclic. ∎

Final remarks. It is not generally true that a doubly stochastic operator which overlaps supports is asymptotically stable. A suitable counterexample was supplied by R. Rudnicki and may be found in [R2].

Let P_S be a Frobenius–Perron operator with stationary density f_*. If P_S a.o.s. then it is strongly asymptotically cyclic even if $\mathrm{supp}(f_*) \neq X$. This was proved by R. Zaharopol [Z]. Roughly speaking this is because $\bigcup_{n=1}^{\infty} S^{-n}(\mathrm{supp}(f_*)) = X$, which easily follows from a.o.s.

On the other hand there are kernel Markov operators P with stationary densities and overlapping supports which are not asymptotically stable. For this consider $X = \mathbb{N} \cup \{0\}$ with counting measure μ, and let

$$p_{i,j} = \begin{cases} 1 & \text{if } i = j = 0, \\ 1/2^i & \text{if } j = 0 \text{ and } i \neq 0, \\ 1 - 1/2^i & \text{if } j = i + 1 \text{ and } i \neq 0, \end{cases}$$

be transition probabilities. In the standard way the matrix $[p_{i,j}]$ defines a

Markov operator (chain) on $\ell^1(X)$. Namely we set $Pf(j) = \sum_{i=0}^{\infty} f(i)p_{i,j}$. Clearly P overlaps supports as $Pf(0) > 0$ for any nonnegative nonzero f, and $f_* = \delta_0$ is the only stationary density. On the other hand we have $\lim_{n\to\infty} \int_{\{0\}} P^n f\, d\mu < 1$ for every $f \in \mathcal{D}$ which is not entirely concentrated on $\{0\}$. Hence P is not asymptotically stable.

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