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A characterization of linear automorphisms of the Euclidean ball

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Abstract. Let B be the open unit ball for a norm on $\mathbb{C}^n$. Let $f : B \to B$ be a holomorphic map with $f(0) = 0$. We consider a condition implying that f is linear on $\mathbb{C}^n$. Moreover, in the case of the Euclidean ball $\mathbb{B}$, we show that f is a linear automorphism of $\mathbb{B}$ under this condition.

1. Introduction. Let $\Delta=\{z \in \mathbb{C} : |z|<1\}$ denote the open unit disc in the complex plane $\mathbb{C}$. Let $f : \Delta \to \Delta$ be a holomorphic map with $f(0) = 0$. By the classical Schwarz lemma, if there exists a single point $z_0 \in \Delta \setminus \{0\}$ such that $|f(z_0)| = |z_0|$, then $f(z) = \lambda z$ with a complex number λ such that $|\lambda| = 1$ for all $z \in \Delta$. That is, f is a linear automorphism of Δ.

Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. It is natural to consider a generalization of the above classical Schwarz lemma to the open unit ball $B = \{z \in \mathbb{C}^n : \|z\| < 1\}$ in $\mathbb{C}^n$. Let $f : B \to B$ be a holomorphic map with $f(0) = 0$.

J. P. Vigué [13], [14] proved that if every boundary point of B in $\mathbb{C}^n$ is a complex extreme point of $\overline{B}$ and

$$(1.1) \qquad C_B(f(0), f(w)) = C_B(0, w) \quad \text{or equivalently} \quad \|f(w)\| = \|w\|$$

holds on an open subset U of B, then f is a linear automorphism of $\mathbb{C}^n$, where C_B denotes the Carathéodory distance on the open set B. The first author [4], [5] generalized the above classical Schwarz lemma to the case where (1.1) holds on some local complex submanifold of codimension 1. We note that a single point $z_0 \in \Delta\setminus\{0\}$ is a complex submanifold of codimension 1 in $\mathbb{C}$. The second author [7], [8] extended those results to the case where (1.1) holds on a subset mapped onto a non-pluripolar subset in the projective space. We note that an open set is non-pluripolar.

In this paper, we show the following theorems.

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Theorem A. *Let* $\|\cdot\|$ *be a norm on* $\mathbb{C}^n$ *and let* $B = \{z \in \mathbb{C}^n : \|z\| < 1\}$ *be the open unit ball. Assume that every boundary point* $p \in \partial B$ *is a complex extreme point of the closure* $\overline{B}$ *of* B. *Let* $f : B \to B$ *be a holomorphic map with* $f(0) = 0$. *Assume that there exist an open subset* U *of* B *and a totally real*, *real-analytic* $(n-1)$-*dimensional submanifold* X *of* U *such that there exists a point* $a \in X$ *with* $0 \notin a + T_a(X) \oplus iT_a(X)$. *If* $C_B(f(0), f(w)) = C_B(0, w)$ *or equivalently* $\|f(w)\| = \|w\|$ *for every* $w \in X$, *then* f *is linear on* $\mathbb{C}^n$.

Theorem B. *Let* $\|\cdot\|_2$ *be the Euclidean norm on* $\mathbb{C}^n$. *Let* $\mathbb{B} = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \|z\|_2 = (\sum_{j=1}^n |z_j|^2)^{1/2} < 1\}$ *be the Euclidean unit ball. If* U, X, f *are as in the assumption of Theorem* A, *then* $f : \mathbb{B} \to \mathbb{B}$ *is a linear automorphism of* $\mathbb{B}$.

2. Preliminaries. Let Δ be the open unit disc in the complex plane $\mathbb{C}$. The *Poincaré distance* ϱ on Δ is defined by

$$\varrho(z, w) = \frac{1}{2} \log \frac{1 + \left|\frac{z - w}{1 - z\overline{w}}\right|}{1 - \left|\frac{z - w}{1 - z\overline{w}}\right|} \qquad (z, w \in \Delta).$$

Let D be a domain in $\mathbb{C}^n$. The *Carathéodory distance* C_D on D is defined by

$$C_D(p, q) = \sup\{\varrho(f(p), f(q)) : f \in \mathrm{Hol}(D, \Delta)\} \qquad (p, q \in D).$$

A holomorphic map $\varphi : \Delta \to D$ is called a *complex geodesic* on D if

$$C_D(\varphi(z), \varphi(w)) = \varrho(z, w) \qquad (\text{for all } z, w \in \Delta).$$

The following proposition is well known (cf. S. Dineen [3], M. Jarnicki and P. Pflug [9], E. Vesentini [11], [12]).

Proposition 2.1. *Let* E *be a complex Banach space with norm* $\|\cdot\|$. *Let* B *be the open unit ball of* E *for the norm* $\|\cdot\|$. *Then* $C_B(0, x) = C_\Delta(0, \|x\|)$ *for all* $x \in B$.

This proposition implies that the conditions $\|f(x)\| = \|x\|$ and $C_B(f(0), f(x)) = C_B(0, x)$ are equivalent.

We recall the definition of a complex extreme point. Let V be a convex subset of $\mathbb{C}^n$. A point $x \in V$ is called a *complex extreme point* of V if $y = 0$ is the only vector in $\mathbb{C}^n$ such that the function $\zeta \mapsto x + \zeta y$ maps Δ into V. For example, C^2-smooth strictly pseudoconvex boundary points are complex extreme points (cf. p. 257 of M. Jarnicki and P. Pflug [9]).

Using the uniqueness of complex geodesics on B at the origin in the direction of complex extreme points, we obtain the following proposition (cf. H. Hamada [4], [5], T. Honda [7], [8], J. P. Vigué [13], [14]).

PROPOSITION 2.2. *Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$ and let $B = \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the open unit ball. Assume that every $p \in \partial B$ is a complex extreme point of $\overline{B}$. Let $f : B \to B$ be a holomorphic map with $f(0) = 0$. Let $f(z) = \sum_{m=1}^{\infty} P_m(z)$ be the development of f by m-homogeneous polynomials P_m in a neighborhood of 0 in $\mathbb{C}^n$. If $C_B(f(0), f(w)) = C_B(0, w)$ or equivalently $\|f(w)\| = \|w\|$ at a point $w \in B \setminus \{0\}$, then $P_m(w) = 0$ for all $m \geq 2$.*

3. Totally real submanifolds. Let X be a real submanifold of an open subset $U \subset \mathbb{C}^n$. Then X is said to be *totally real* if $T_p(X) \cap iT_p(X) = \{0\}$ for all $p \in X$, where $T_p(X)$ denotes the tangent space of X at p. The following lemma is proved in H. Hamada and J. Kajiwara [6], when $k = 0$ (cf. A. Andreotti and G. A. Fredricks [1]).

LEMMA 3.1. *Let U be an open subset of $\mathbb{C}^n$. Let X be a totally real, real-analytic $(n-k)$-dimensional submanifold of U, where $0 \leq k \leq n-1$. Then for every $a \in X$, there exist an open subset $\widetilde{U}$ of U, an $(n-k)$-dimensional complex submanifold M of $\widetilde{U}$, a connected open subset W of $\mathbb{C}^{n-k}$ and an injective holomorphic map $\psi : W \to \widetilde{U}$ such that $a \in \psi(\mathbb{R}^{n-k} \cap W) = X \cap \widetilde{U} \subset M = \psi(W)$.*

Proof. From the condition on X, for every $a \in X$, there exist an open neighborhood $\widetilde{U}$ of a in $\mathbb{C}^n = \{(w_1, \ldots, w_n) : w_j \in \mathbb{C}\}$ and an open neighborhood V of 0 in $\mathbb{R}^{n-k} = \{(x_1, \ldots, x_{n-k}) : x_j \in \mathbb{R}\}$ and real-analytic functions ψ_j $(1 \leq j \leq n)$ on V such that $\psi = (\psi_1, \ldots, \psi_n) : V \to X \cap \widetilde{U}$ is bijective with $\psi(0) = a$. Since ψ is real-analytic, there exists a neighborhood W of 0 in $\mathbb{C}^{n-k} = \{(z_1, \ldots, z_{n-k}) : z_j \in \mathbb{C}\}$ such that ψ is holomorphic on W. Then

$$\operatorname{rank} \frac{\partial(\psi_1, \ldots, \psi_n, \overline{\psi}_1, \ldots, \overline{\psi}_n)}{\partial(x_1, \ldots, x_{n-k})}(0) = n-k. \tag{3.1}$$

We set $M = \{\psi(z') : z' = (z_1, \ldots, z_{n-k}) \in W\} = \psi(W)$. We will show that M is an $(n-k)$-dimensional complex submanifold of $\widetilde{U}$, upon shrinking M and $\widetilde{U}$ if necessary.

Now we have $(\psi_*(\partial/\partial x_1))(a), \ldots, (\psi_*(\partial/\partial x_{n-k}))(a) \in T(X) \otimes \mathbb{C}_a$ and

$$\psi_*\left(\frac{\partial}{\partial x_j}\right) = \sum_{\beta=1}^{n} \frac{\partial \psi_\beta}{\partial x_j} \frac{\partial}{\partial w_\beta} + \sum_{\beta=1}^{n} \frac{\partial \overline{\psi}_\beta}{\partial x_j} \frac{\partial}{\partial \overline{w}_\beta}.$$

We put

$$\sum_{j=1}^{n-k} \alpha_j \frac{\partial \psi_\beta}{\partial x_j}(0) = 0 \quad \text{ for } \alpha_j \in \mathbb{C},\ 1 \leq \beta \leq n.$$

Then

$$\begin{aligned}
&\sum_{j=1}^{n-k} \overline{\alpha}_j \left(\psi_* \left(\frac{\partial}{\partial x_j} \right) \right) (a) \\
&\quad = \sum_{j=1}^{n-k} \overline{\alpha}_j \left(\sum_{\beta=1}^{n} \frac{\partial \psi_\beta}{\partial x_j}(0) \frac{\partial}{\partial w_\beta} + \sum_{\beta=1}^{n} \frac{\partial \overline{\psi}_\beta}{\partial x_j}(0) \frac{\partial}{\partial \overline{w}_\beta} \right)(a) \\
&\quad = \sum_{\beta=1}^{n} \left(\sum_{j=1}^{n-k} \overline{\alpha}_j \frac{\partial \psi_\beta}{\partial x_j}(0) \right) \frac{\partial}{\partial w_\beta}(a) + \sum_{\beta=1}^{n} \overline{\left(\sum_{j=1}^{n-k} \alpha_j \frac{\partial \psi_\beta}{\partial x_j}(0) \right)} \frac{\partial}{\partial \overline{w}_\beta}(a) \\
&\quad = \sum_{\beta=1}^{n} \left(\sum_{j=1}^{n-k} \overline{\alpha}_j \frac{\partial \psi_\beta}{\partial x_j}(0) \right) \frac{\partial}{\partial w_\beta}(a) \\
&\quad \in \mathrm{HT}(X, \mathbb{C}^n)_a.
\end{aligned}$$

Since X is totally real, $\mathrm{HT}(X, \mathbb{C}^n) = \{0\}$. So

$$\sum_{j=1}^{n-k} \overline{\alpha}_j \left(\psi_* \left(\frac{\partial}{\partial x_j} \right) \right)(a) = 0.$$

From (3.1), $\{(\psi_*(\partial/\partial x_j))(a)\}_{j=1}^{n-k}$ is linearly independent over $\mathbb{C}$. Then $\overline{\alpha}_j = 0$, $1 \le j \le n-k$. Therefore

$$\operatorname{rank} \frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(x_1, \ldots, x_{n-k})}(0) = n - k.$$

Since $\psi_1, \ldots, \psi_n$ are holomorphic, we have

$$\operatorname{rank} \frac{\partial(\psi_1, \ldots, \psi_n)}{\partial(z_1, \ldots, z_{n-k})}(0) = n - k.$$

Hence $M = \psi(W)$ is an $(n-k)$-dimensional complex submanifold of $\widetilde{U}$, upon shrinking M, $\widetilde{U}$ and W if necessary. ∎

The following lemma is proved in H. Hamada [5].

LEMMA 3.2. *Let U be an open subset of $\mathbb{C}^n$. Let M be a complex submanifold of U of dimension $n-1$. Assume that there exists a point a in M such that $a + T_a(M)$ does not contain the origin. Then there exists a neighborhood U_1 of a in $\mathbb{C}^n$ such that $U_1 \subset \mathbb{C}M = \{tx : t \in \mathbb{C},\ x \in M\}$.*

Proof of Theorem A. By Lemma 3.1, there exists an $(n-1)$-dimensional complex submanifold M of an open subset $\widetilde{U} \subset \mathbb{C}^n$ such that $a \in X \cap \widetilde{U} \subset M \cap \widetilde{U} = M$. Let $f(z) = \sum_{m=1}^{\infty} P_m(z)$ be the development of f by m-homogeneous polynomials P_m in a neighborhood of 0 in $\mathbb{C}^n$. By Proposition 2.2, $P_m \equiv 0$ on X for all $m \ge 2$. Since $P_m|_M$ is holomorphic, we have

$P_m \equiv 0$ on M for all $m \geq 2$. Since $0 \notin a + T_a(X) \oplus iT_a(X) = a + T_a(M)$, by Lemma 3.2, there exists a neighborhood Ω of a in $\mathbb{C}^n$ such that $\Omega \subset \mathbb{C}M$. Then $\|P_m(tz)\| = |t|^m \|P_m(z)\| = 0$ for all $z \in M$ and $t \in \mathbb{C}$. So $P_m \equiv 0$ on $\mathbb{C}M \supset \Omega$. By the identity theorem, $P_m \equiv 0$ on $\mathbb{C}^n$ for all $m \geq 2$. Therefore $f = P_1$, i.e. f is linear on $\mathbb{C}^n$. ∎

4. Non-pluripolar subsets. Let Ω be a complex manifold. A subset $S \subset \Omega$ is said to be *pluripolar* in Ω if there exists a non-constant plurisubharmonic function u on Ω such that $S \subset u^{-1}(-\infty)$.

By the definition of a pluripolar set, we have the following lemma.

LEMMA 4.1. *Let Ω be a connected complex manifold. Let Σ be a subset of Ω. Then Σ is a non-pluripolar subset of Ω if and only if all plurisubharmonic functions u on Ω with $u \equiv -\infty$ on Σ satisfy $u \equiv -\infty$ on Ω.*

Let k be a positive number. A non-negative function $u : \mathbb{C}^n \to [0, +\infty)$ is said to be *complex homogeneous of order k* if $u(\lambda x) = |\lambda|^k u(x)$ for all $\lambda \in \mathbb{C}$, $x \in \mathbb{C}^n$.

The following lemma is proved in T. Honda [8] (cf. T. J. Barth [2]).

LEMMA 4.2. *Let $u : \mathbb{C}^n \to [0, +\infty)$ be an upper semicontinuous function. If u is a complex homogeneous function of order k, then the following conditions are equivalent*:

(1) *u is plurisubharmonic on $\mathbb{C}^n$*;
(2) *$\log u$ is plurisubharmonic on $\mathbb{C}^n$.*

Proof of Theorem B. By Theorem A, f is linear. By Lemma 3.1, for $a \in X$, there exist an open subset $\widetilde{U}$ of U, an $(n-1)$-dimensional complex submanifold M of $\widetilde{U}$, a connected open subset W of $\mathbb{C}^{n-1}$ and an injective holomorphic map $\psi : W \to \widetilde{U}$ such that $\psi(\mathbb{R}^{n-1} \cap W) = X \cap \widetilde{U} \subset M \cap \widetilde{U} = \psi(W)$.

We will show $X \cap \widetilde{U}$ is non-pluripolar in $M \cap \widetilde{U}$. Let u be a plurisubharmonic function on $M \cap \widetilde{U}$ with $u \equiv -\infty$ on $X \cap \widetilde{U}$. Then $u \circ \psi \equiv -\infty$ on $\mathbb{R}^{n-1} \cap W$. By Lemma 3.5 of K. H. Shon [10], $\mathbb{R}^{n-1} \cap W$ is a non-pluripolar subset of W. So, by Lemma 4.1, we have $u \circ \psi \equiv -\infty$ on W, i.e. $u \equiv -\infty$ on $M \cap \widetilde{U}$. Hence $X \cap \widetilde{U}$ is non-pluripolar in $M \cap \widetilde{U}$.

By Proposition 2.1 and the distance decreasing property of the Carathéodory distances, we have for all $z \in \mathbb{B}$,

$$C_\Delta(0, \|z\|_2) = C_\mathbb{B}(0, z) \geq C_\mathbb{B}(0, f(z)) = C_\Delta(0, \|f(z)\|_2).$$

Since $C_\Delta(0, r)$ is strictly increasing for $0 \leq r < 1$, we obtain $\|f(z)\|_2 \leq \|z\|_2$ for all $z \in \mathbb{B}$. Since f is linear on $\mathbb{C}^n$, $\|f(z)\|_2 \leq \|z\|_2$ for all $z \in \mathbb{C}^n$. So we define a non-negative function

$$g(z) = \|z\|_2^2 - \|f(z)\|_2^2 \geq 0 \quad \text{for } z \in \mathbb{C}^n.$$

Since f is linear, there exists an $n\times n$ matrix A such that

$$f(z)=Az=\Big(\sum_{k=1}^{n}a_{jk}z_k\Big).$$

So, for $\zeta=(\zeta_1,\dots,\zeta_n)\in\mathbb{C}^n$,

$$\begin{aligned}\sum_{\alpha,\beta=1}^{n}\frac{\partial^2 g}{\partial z_\alpha\partial\overline{z}_\beta}\zeta_\alpha\overline{\zeta}_\beta &= \sum_{\alpha,\beta=1}^{n}\frac{\partial^2(\|z\|_2^2-\|f(z)\|_2^2)}{\partial z_\alpha\partial\overline{z}_\beta}\zeta_\alpha\overline{\zeta}_\beta\\ &=\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\|z\|_2^2}{\partial z_\alpha\partial\overline{z}_\beta}\zeta_\alpha\overline{\zeta}_\beta-\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\|Az\|_2^2}{\partial z_\alpha\partial\overline{z}_\beta}\zeta_\alpha\overline{\zeta}_\beta\\ &=\|\zeta\|_2^2-\|A\zeta\|_2^2=\|\zeta\|_2^2-\|f(\zeta)\|_2^2\ge 0.\end{aligned}$$

Therefore g is plurisubharmonic on $\mathbb{C}^n$. Since g is complex homogeneous of order 2, by Lemma 4.2, $\log g$ is plurisubharmonic on $\mathbb{C}^n$. So $\log g$ is plurisubharmonic on $M\cap\widetilde{U}$. Since $\|w\|_2=\|f(w)\|_2$ for every $w\in X$, $\log g\equiv-\infty$ on $X\cap\widetilde{U}\subset M\cap\widetilde{U}$. Since $X\cap\widetilde{U}$ is non-pluripolar in $M\cap\widetilde{U}$, by Lemma 4.1 $\log g\equiv-\infty$ on $M\cap\widetilde{U}$, i.e. $g\equiv 0$ on $M\cap\widetilde{U}$. Therefore $\|f(w)\|_2=\|w\|_2$ for all $w\in M\cap\widetilde{U}$. Since $M\cap\widetilde{U}$ is an $(n-1)$-dimensional complex submanifold of $\widetilde{U}$ and $0\notin a+T_a(M)=a+T_a(X)\oplus iT_a(X)$, by Corollary 1 of H. Hamada [5], f is a linear automorphism of $\mathbb{B}$. ■

Remark. We set $f(z)=(z_1,\dots,z_{n-1},z_n^2)$. Then f maps $\mathbb{B}$ into itself and $f(0)=0$.

(1) Let $X=\{(x_1+iy_1,\dots,x_n+iy_n)\in\mathbb{B}\colon\ y_1=b,\ x_n=y_2=\dots=y_n=0\}$, where $0<|b|<1$. Then X is a totally real, real-analytic $(n-1)$-dimensional submanifold of $\mathbb{B}$. Moreover, $0\notin a+T_a(X)$ and $0\in a+T_a(X)\oplus iT_a(X)$ for any $a\in X$. We have $\|f(w)\|=\|w\|$ for every $w\in X$. However, f is not linear. So the condition that $0\notin a+T_a(X)\oplus iT_a(X)$ cannot be weakened to $0\notin a+T_a(X)$ in our theorems.

(2) Let $X_{n-k}=\{x_{n-k+1}=b,\ x_{n-k+2}=\dots=x_n=y_1=\dots=y_n=0\}$ for $k\ge 2$, where $0<|b|<1$. Then X_{n-k} is a totally real, real-analytic $(n-k)$-dimensional submanifold of $\mathbb{B}$, and $0\notin a+T_a(X_{n-k})\oplus iT_a(X_{n-k})$ for any $a\in X_{n-k}$. We have $\|f(w)\|=\|w\|$ for every $w\in X_{n-k}$. However, f is not linear. So the condition that the real dimension of X is $n-1$ cannot be omitted in our theorems.

(3) In the case $n=3$, let $X=\{(x_1+iy_1,x_2+iy_2,x_3+iy_3)\in\mathbb{C}^3 : x_2=b,\ x_3=y_2=y_3=0\}\cong\mathbb{R}^2$, where $0<|b|<1$. Then $X\cap\mathbb{B}$ is a real-analytic 2-dimensional submanifold, and $0\notin a+T_a(X)+iT_a(X)$ for any $a\in X$. We have $\|f(w)\|=\|w\|$ for every $w\in X$. However, f is not linear. So the condition that X is totally real cannot be omitted either.

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