A characterization of linear automorphisms
of the Euclidean ball

by Hidetaka Hamada (Kitakyushu) and Tatsuhiro Honda (Omuta)

Abstract. Let $B$ be the open unit ball for a norm on $\mathbb{C}^n$. Let $f : B \to B$ be a holomorphic map with $f(0) = 0$. We consider a condition implying that $f$ is linear on $\mathbb{C}^n$. Moreover, in the case of the Euclidean ball $B$, we show that $f$ is a linear automorphism of $B$ under this condition.

1. Introduction. Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$. Let $f : \Delta \to \Delta$ be a holomorphic map with $f(0) = 0$. By the classical Schwarz lemma, if there exists a single point $z_0 \in \Delta \setminus \{0\}$ such that $|f(z_0)| = |z_0|$, then $f(z) = \lambda z$ with a complex number $\lambda$ such that $|\lambda| = 1$ for all $z \in \Delta$. That is, $f$ is a linear automorphism of $\Delta$.

Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$. It is natural to consider a generalization of the above classical Schwarz lemma to the open unit ball $B = \{z \in \mathbb{C}^n : \|z\| < 1\}$ in $\mathbb{C}^n$. Let $f : B \to B$ be a holomorphic map with $f(0) = 0$.

J. P. Vigué [13], [14] proved that if every boundary point of $B$ in $\mathbb{C}^n$ is a complex extreme point of $B$ and

$$C_B(f(0), f(w)) = C_B(0, w) \quad \text{or equivalently} \quad \|f(w)\| = \|w\|$$

holds on an open subset $U$ of $B$, then $f$ is a linear automorphism of $\mathbb{C}^n$, where $C_B$ denotes the Carathéodory distance on the open set $B$. The first author [4], [5] generalized the above classical Schwarz lemma to the case where (1.1) holds on some local complex submanifold of codimension 1. We note that a single point $z_0 \in \Delta \setminus \{0\}$ is a complex submanifold of codimension 1 in $\mathbb{C}$. The second author [7], [8] extended those results to the case where (1.1) holds on a subset mapped onto a non-pluripolar subset in the projective space. We note that an open set is non-pluripolar.

In this paper, we show the following theorems.

1991 Mathematics Subject Classification: Primary 32A10.

Key words and phrases: totally real, non-pluripolar, Schwarz lemma, complex extreme point, automorphism.
Theorem A. Let $\| \cdot \|$ be a norm on $\mathbb{C}^n$ and let $B = \{ z \in \mathbb{C}^n : \| z \| < 1 \}$ be the open unit ball. Assume that every boundary point $p \in \partial B$ is a complex extreme point of the closure $\overline{B}$ of $B$. Let $f : B \to B$ be a holomorphic map with $f(0) = 0$. Assume that there exist an open subset $U$ of $B$ and a totally real, real-analytic $(n-1)$-dimensional submanifold $X$ of $U$ such that there exists a point $a \in X$ with $0 \notin a + T_a(X) \oplus iT_a(X)$. If $C_B(f(0), f(w)) = C_B(0, w)$ or equivalently $\| f(w) \| = \| w \|$ for every $w \in X$, then $f$ is linear on $\mathbb{C}^n$.

Theorem B. Let $\| \cdot \|_2$ be the Euclidean norm on $\mathbb{C}^n$. Let $B = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \| z \|_2 = (\sum_{j=1}^n |z_j|^2)^{1/2} < 1 \}$ be the Euclidean unit ball. If $U, X, f$ are as in the assumption of Theorem A, then $f : B \to B$ is a linear automorphism of $B$.

2. Preliminaries. Let $\Delta$ be the open unit disc in the complex plane $\mathbb{C}$. The Poincaré distance $\varrho$ on $\Delta$ is defined by

$$\varrho(z, w) = \frac{1}{2} \log \frac{1 + \frac{z - w}{1 - z \overline{w}}}{1 - \frac{z - w}{1 - z \overline{w}}} \quad (z, w \in \Delta).$$

Let $D$ be a domain in $\mathbb{C}^n$. The Carathéodory distance $C_D$ on $D$ is defined by

$$C_D(p, q) = \sup \{ \varrho(f(p), f(q)) : f \in \text{Hol}(D, \Delta) \} \quad (p, q \in D).$$

A holomorphic map $\varphi : \Delta \to D$ is called a complex geodesic on $D$ if $C_D(\varphi(z), \varphi(w)) = \varrho(z, w)$ (for all $z, w \in \Delta$).

The following proposition is well known (cf. S. Dineen [3], M. Jarnicki and P. Pflug [9], E. Vesentini [11], [12]).

Proposition 2.1. Let $E$ be a complex Banach space with norm $\| \cdot \|$. Let $B$ be the open unit ball of $E$ for the norm $\| \cdot \|$. Then $C_B(0, x) = C_{\Delta}(0, \| x \|)$ for all $x \in B$.

This proposition implies that the conditions $\| f(x) \| = \| x \|$ and $C_B(f(0), f(x)) = C_B(0, x)$ are equivalent.

We recall the definition of a complex extreme point. Let $V$ be a convex subset of $\mathbb{C}^n$. A point $x \in V$ is called a complex extreme point of $V$ if $y = 0$ is the only vector in $\mathbb{C}^n$ such that the function $\zeta \mapsto x + \zeta y$ maps $\Delta$ into $V$. For example, $C^2$-smooth strictly pseudoconvex boundary points are complex extreme points (cf. p. 257 of M. Jarnicki and P. Pflug [9]).

Using the uniqueness of complex geodesics on $B$ at the origin in the direction of complex extreme points, we obtain the following proposition (cf. H. Hamada [4], [5], T. Honda [7], [8], J. P. Vigué [13], [14]).
Proposition 2.2. Let \( \| \cdot \| \) be a norm on \( \mathbb{C}^n \) and let \( B = \{ z \in \mathbb{C}^n : \| z \| < 1 \} \) be the open unit ball. Assume that every \( p \in \partial B \) is a complex extreme point of \( B \). Let \( f : B \to B \) be a holomorphic map with \( f(0) = 0 \). Let \( f(z) = \sum_{m=1}^{\infty} P_m(z) \) be the development of \( f \) by \( m \)-homogeneous polynomials \( P_m \) in a neighborhood of \( 0 \) in \( \mathbb{C}^n \). If \( C_B(f(0), f(w)) = C_B(0, w) \) or equivalently \( \| f(w) \| = \| w \| \) at a point \( w \in B \setminus \{ 0 \} \), then \( P_m(w) = 0 \) for all \( m \geq 2 \).

3. Totally real submanifolds. Let \( X \) be a real submanifold of an open subset \( U \subset \mathbb{C}^n \). Then \( X \) is said to be totally real if \( T_p(X) \cap iT_p(X) = \{ 0 \} \) for all \( p \in X \), where \( T_p(X) \) denotes the tangent space of \( X \) at \( p \). The following lemma is proved in H. Hamada and J. Kajiwara [6], when \( k = 0 \) (cf. A. Andreotti and G. A. Fredricks [1]).

Lemma 3.1. Let \( U \) be an open subset of \( \mathbb{C}^n \). Let \( X \) be a totally real, real-analytic \((n-k)\)-dimensional submanifold of \( U \), where \( 0 \leq k \leq n-1 \). Then for every \( a \in X \), there exist an open set \( \tilde{U} \) of \( U \), an \((n-k)\)-dimensional complex submanifold \( M \) of \( \tilde{U} \), a connected open subset \( W \) of \( \mathbb{C}^{n-k} \) and an injective holomorphic map \( \psi : W \to \tilde{U} \) such that \( a \in \psi(\mathbb{R}^{n-k} \cap W) = X \cap \tilde{U} \subset M = \psi(W) \).

Proof. From the condition on \( X \), for every \( a \in X \), there exist an open neighborhood \( \tilde{U} \) of \( a \) in \( \mathbb{C}^n = \{(w_1, \ldots, w_n) : w_j \in \mathbb{C} \} \) and an open neighborhood \( V \) of \( 0 \) in \( \mathbb{R}^{n-k} = \{(x_1, \ldots, x_{n-k}) : x_j \in \mathbb{R} \} \) and real-analytic functions \( \psi_j \) (\( 1 \leq j \leq n \)) on \( V \) such that \( \psi = (\psi_1, \ldots, \psi_n) : V \to X \cap \tilde{U} \) is bijective with \( \psi(0) = a \). Since \( \psi \) is real-analytic, there exists a neighborhood \( W \) of \( 0 \) in \( \mathbb{C}^{n-k} = \{(z_1, \ldots, z_{n-k}) : z_j \in \mathbb{C} \} \) such that \( \psi \) is holomorphic on \( W \). Then

\[
\text{rank} \left( \frac{\partial (\psi_1, \ldots, \psi_n, \overline{\psi}_1, \ldots, \overline{\psi}_n)}{\partial (x_1, \ldots, x_{n-k})} \right)(0) = n-k.
\]

We set \( M = \{ \psi(z') : z' = (z_1, \ldots, z_{n-k}) \in W \} = \psi(W) \). We will show that \( M \) is an \((n-k)\)-dimensional complex submanifold of \( \tilde{U} \), upon shrinking \( M \) and \( \tilde{U} \) if necessary.

Now we have \((\psi_*(\partial/\partial x_1))(a), \ldots, (\psi_*(\partial/\partial x_{n-k}))(a) \in T(X) \otimes \mathbb{C}_a \) and

\[
\psi_*(\frac{\partial}{\partial x_j}) = \sum_{\beta=1}^{n} \frac{\partial \psi_\beta}{\partial x_j} \frac{\partial}{\partial w_\beta} + \sum_{\beta=1}^{n} \frac{\partial \overline{\psi}_\beta}{\partial x_j} \frac{\partial}{\partial \overline{w}_\beta}.
\]

We put

\[
\sum_{j=1}^{n-k} \alpha_j \frac{\partial \psi_\beta}{\partial x_j}(0) = 0 \quad \text{for } \alpha_j \in \mathbb{C}, \ 1 \leq \beta \leq n.
\]
Then
\[ \sum_{j=1}^{n-k} \pi_j \left( \psi_\ast \left( \frac{\partial}{\partial x_j} \right) \right)(a) = \sum_{j=1}^{n-k} \pi_j \left( \sum_{\beta=1}^{n} \frac{\partial \psi_\beta}{\partial x_j} (0) \frac{\partial}{\partial w_\beta} + \sum_{\beta=1}^{n} \frac{\partial \psi_\beta}{\partial w_\beta} \frac{\partial}{\partial x_j} \right)(a) \]
\[ = \sum_{\beta=1}^{n} \left( \sum_{j=1}^{n-k} \pi_j \frac{\partial \psi_\beta}{\partial x_j} (0) \right) \frac{\partial}{\partial w_\beta} (a) + \sum_{\beta=1}^{n} \left( \sum_{j=1}^{n-k} \pi_j \frac{\partial \psi_\beta}{\partial x_j} (0) \right) \frac{\partial}{\partial w_\beta} (a) \]
\[ = \sum_{\beta=1}^{n} \left( \sum_{j=1}^{n-k} \pi_j \frac{\partial \psi_\beta}{\partial x_j} (0) \right) \frac{\partial}{\partial w_\beta} (a) \]
\[ \in \text{HT}(X, \mathbb{C}^n)_a. \]
Since X is totally real, HT(X, \mathbb{C}^n) = \{0\}. So
\[ \sum_{j=1}^{n-k} \pi_j \left( \psi_\ast \left( \frac{\partial}{\partial x_j} \right) \right)(a) = 0. \]
From (3.1), \{ \left( \psi_\ast \left( \frac{\partial}{\partial x_j} \right) \right)(a) \}_{j=1}^{n-k} \] is linearly independent over \mathbb{C}. Then \pi_j = 0, 1 \leq j \leq n - k. Therefore
\[ \text{rank} \left( \frac{\partial (\psi_1, \ldots, \psi_n)}{\partial (x_1, \ldots, x_{n-k})} \right)(0) = n - k. \]
Since \psi_1, \ldots, \psi_n are holomorphic, we have
\[ \text{rank} \left( \frac{\partial (\psi_1, \ldots, \psi_n)}{\partial (z_1, \ldots, z_{n-k})} \right)(0) = n - k. \]
Hence \( M = \psi \ast (W) \) is an \((n - k)\)-dimensional complex submanifold of \( \tilde{U} \), upon shrinking \( M, \tilde{U} \) and \( W \) if necessary.

The following lemma is proved in H. Hamada [5].

**Lemma 3.2.** Let \( U \) be an open subset of \( \mathbb{C}^n \). Let \( M \) be a complex submanifold of \( U \) of dimension \( n - 1 \). Assume that there exists a point \( a \) in \( M \) such that \( a + T_a(M) \) does not contain the origin. Then there exists a neighborhood \( U_1 \) of \( a \) in \( \mathbb{C}^n \) such that \( U_1 \subset CM = \{ tx : t \in \mathbb{C}, x \in M \} \).

**Proof of Theorem A.** By Lemma 3.1, there exists an \((n - 1)\)-dimensional complex submanifold \( M \) of an open subset \( \tilde{U} \subset \mathbb{C}^n \) such that \( a \in X \cap \tilde{M} \subset M \cap \tilde{U} = M \). Let \( f(z) = \sum_{m=1}^{\infty} P_m(z) \) be the development of \( f \) by \( m \)-homogeneous polynomials \( P_m \) in a neighborhood of \( 0 \) in \( \mathbb{C}^n \). By Proposition 2.2, \( P_m \equiv 0 \) on \( X \) for all \( m \geq 2 \). Since \( P_m|_M \) is holomorphic, we have
\[ P_m \equiv 0 \text{ on } M \text{ for all } m \geq 2. \]

Since \( 0 \notin \mathbb{C} \), by Lemma 3.2, there exists a neighborhood \( \Omega \) of \( a \) in \( \mathbb{C}^n \) such that \( \Omega \subset \mathbb{C}M \).

Then \( \|P_m(tz)\| = |t|^m \|P_m(z)\| = 0 \) for all \( z \in M \) and \( t \in \mathbb{C} \). So \( P_m \equiv 0 \) on \( \mathbb{C}M \supset \Omega \). By the identity theorem, \( P_m \equiv 0 \) on \( \mathbb{C}^n \) for all \( m \geq 2 \). Therefore \( f = P_1 \), i.e. \( f \) is linear on \( \mathbb{C}^n \). \( \blacksquare \)

4. Non-pluripolar subsets. Let \( \Omega \) be a complex manifold. A subset \( S \subset \Omega \) is said to be pluripolar in \( \Omega \) if there exists a non-constant plurisubharmonic function \( u \) on \( \Omega \) such that \( S \subset u^{-1}(-\infty) \).

By the definition of a pluripolar set, we have the following lemma.

**Lemma 4.1.** Let \( \Omega \) be a connected complex manifold. Let \( \Sigma \) be a subset of \( \Omega \). Then \( \Sigma \) is a non-pluripolar subset of \( \Omega \) if and only if all plurisubharmonic functions \( u \) on \( \Omega \) with \( u \equiv -\infty \) on \( \Sigma \) satisfy \( u \equiv -\infty \) on \( \Omega \).

Let \( k \) be a positive number. A non-negative function \( u : \mathbb{C}^n \to [0, +\infty) \) is said to be complex homogeneous of order \( k \) if \( u(\lambda x) = |\lambda|^k u(x) \) for all \( \lambda \in \mathbb{C}, x \in \mathbb{C}^n \).

The following lemma is proved in T. Honda [8] (cf. T. J. Barth [2]).

**Lemma 4.2.** Let \( u : \mathbb{C}^n \to [0, +\infty) \) be an upper semicontinuous function. If \( u \) is a complex homogeneous function of order \( k \), then the following conditions are equivalent:

1. \( u \) is plurisubharmonic on \( \mathbb{C}^n \);
2. \( \log u \) is plurisubharmonic on \( \mathbb{C}^n \).

**Proof of Theorem B.** By Theorem A, \( f \) is linear. By Lemma 3.1, for \( a \in X \), there exist an open subset \( \tilde{U} \) of \( U \), an \((n-1)\)-dimensional complex submanifold \( M \) of \( \tilde{U} \), a connected open subset \( W \) of \( \mathbb{C}^{n-1} \) and an injective holomorphic map \( \psi : W \to \tilde{U} \) such that \( \psi(\mathbb{R}^{n-1} \cap W) = X \cap \tilde{U} \subset M \cap \tilde{U} = \psi(W) \).

We will show \( X \cap \tilde{U} \) is non-pluripolar in \( M \cap \tilde{U} \). Let \( u \) be a plurisubharmonic function on \( M \cap \tilde{U} \) with \( u \equiv -\infty \) on \( X \cap \tilde{U} \). Then \( u \circ \psi \equiv -\infty \) on \( \mathbb{R}^{n-1} \cap W \). By Lemma 3.5 of K. H. Shon [10], \( \mathbb{R}^{n-1} \cap W \) is a non-pluripolar subset of \( W \). So, by Lemma 4.1, we have \( u \circ \psi \equiv -\infty \) on \( W \), i.e. \( u \equiv -\infty \) on \( M \cap \tilde{U} \). Hence \( X \cap \tilde{U} \) is non-pluripolar in \( M \cap \tilde{U} \).

By Proposition 2.1 and the distance decreasing property of the Carathéodory distances, we have for all \( z \in \mathbb{B} \),

\[
C_\Delta(0, \|z\|, z) = C_\Delta(0, z) \geq C_\Delta(0, f(z)) = C_\Delta(0, \|f(z)\|). 
\]

Since \( C_\Delta(0, r) \) is strictly increasing for \( 0 \leq r < 1 \), we obtain \( \|f(z)\| \leq \|z\| \) for all \( z \in \mathbb{B} \). Since \( f \) is linear on \( \mathbb{C}^n \), \( \|f(z)\| \leq \|z\| \) for all \( z \in \mathbb{C}^n \). So we define a non-negative function

\[
g(z) = \|z\|^2 - \|f(z)\|^2 \geq 0 \quad \text{for } z \in \mathbb{C}^n. 
\]
Since $f$ is linear, there exists an $n \times n$ matrix $A$ such that

$$f(z) = Az = \left( \sum_{k=1}^{n} a_{jk}z_k \right).$$

So, for $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$,

$$\sum_{\alpha, \beta=1}^{n} \frac{\partial^2 g}{\partial z_{\alpha} \partial \overline{z}_{\beta}} \zeta_{\alpha} \overline{\zeta}_{\beta} = \sum_{\alpha, \beta=1}^{n} \frac{\partial^2 (\|z\|^2 - \|f(z)\|^2)}{\partial z_{\alpha} \partial \overline{z}_{\beta}} \zeta_{\alpha} \overline{\zeta}_{\beta}$$

$$= \sum_{\alpha, \beta=1}^{n} \frac{\partial^2 \|z\|^2}{\partial z_{\alpha} \partial \overline{z}_{\beta}} \zeta_{\alpha} \overline{\zeta}_{\beta} - \sum_{\alpha, \beta=1}^{n} \frac{\partial^2 \|A\|_{\beta}}{\partial z_{\alpha} \partial \overline{z}_{\beta}} \zeta_{\alpha} \overline{\zeta}_{\beta}$$

$$= \|\zeta\|^2 - \|A\|_{\beta} \geq 0.$$

Therefore $g$ is plurisubharmonic on $\mathbb{C}^n$. Since $g$ is complex homogeneous of order 2, by Lemma 4.2, $\log g$ is plurisubharmonic on $\mathbb{C}^n$. So $\log g$ is plurisubharmonic on $M \cap \tilde{\mathcal{U}}$. Since $\|w\|_2 = \|f(w)\|_2$ for every $w \in X$, $\log g \equiv -\infty$ on $X \cap \tilde{\mathcal{U}} \subset M \cap \tilde{\mathcal{U}}$. Since $X \cap \tilde{\mathcal{U}}$ is non-pluripolar in $M \cap \tilde{\mathcal{U}}$, by Lemma 4.1 $\log g \equiv -\infty$ on $M \cap \tilde{\mathcal{U}}$, i.e. $g \equiv 0$ on $M \cap \tilde{\mathcal{U}}$. Therefore $\|f(w)\|_2 = \|w\|_2$ for all $w \in M \cap \tilde{\mathcal{U}}$. Since $M \cap \tilde{\mathcal{U}}$ is an $(n-1)$-dimensional complex submanifold of $\tilde{\mathcal{U}}$ and $0 \not\in a + T_a(M) = a + T_a(X) \oplus iT_a(X)$, by Corollary 1 of H. Hamada [5], $f$ is a linear automorphism of $\mathbb{B}$. ■

**Remark.** We set $f(z) = (z_1, \ldots, z_{n-1}, z_n^2)$. Then $f$ maps $\mathbb{B}$ into itself and $f(0) = 0$.

1. Let $X = \{x_1 + iy_1, \ldots, x_n + iy_n\} \subset \mathbb{B}$: $y_1 = b$, $x_n = y_2 = \ldots = y_n = 0$, where $0 < |b| < 1$. Then $X$ is a totally real, real-analytic $(n-1)$-dimensional submanifold of $\mathbb{B}$. Moreover, $0 \not\in a + T_a(X)$ and $0 \not\in a + T_a(X) \oplus iT_a(X)$ for any $a \in X$. We have $\|f(w)\| = \|w\|$ for every $w \in X$. However, $f$ is not linear. So the condition that $0 \not\in a + T_a(X) \oplus iT_a(X)$ cannot be weakened to $0 \not\in a + T_a(X)$ in our theorems.

2. Let $X_{n-k} = \{x_{n-k+1} = b, x_{n-k+2} = \ldots = x_n = y_1 = \ldots = y_n = 0\}$ for $k \geq 2$, where $0 < |b| < 1$. Then $X_{n-k}$ is a totally real, real-analytic $(n-k)$-dimensional submanifold of $\mathbb{B}$, and $0 \not\in a + T_a(X_{n-k}) \oplus iT_a(X_{n-k})$ for any $a \in X_{n-k}$. We have $\|f(w)\| = \|w\|$ for every $w \in X_{n-k}$. However, $f$ is not linear. So the condition that the real dimension of $X$ is $n-1$ cannot be omitted in our theorems.

3. In the case $n = 3$, let $X = \{(x_1 + iy_1, x_2 + iy_2, x_3 + iy_3) \in \mathbb{C}^3 : x_2 = b, x_3 = y_2 = y_3 = 0\} \cong \mathbb{R}^2$, where $0 < |b| < 1$. Then $X \cap \mathbb{B}$ is a real-analytic 2-dimensional submanifold, and $0 \not\in a + T_a(X) \oplus iT_a(X)$ for any $a \in X$. We have $\|f(w)\| = \|w\|$ for every $w \in X$. However, $f$ is not linear. So the condition that $X$ is totally real cannot be omitted either.
References