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# Index filtrations and Morse decompositions for discrete dynamical systems

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**Abstract.** On a Morse decomposition of an isolated invariant set of a homeomorphism (discrete dynamical system) there are partial orderings defined by the homeomorphism. These are called admissible orderings of the Morse decomposition. We prove the existence of index filtrations for admissible total orderings of a Morse decomposition and introduce the connection matrix in this case.

**Introduction.** One of the methods by which the Conley index theory studies isolated invariant sets is to decompose them into subinvariant sets (Morse sets) and connecting orbits between them. This structure is called a Morse decomposition of an isolated invariant set. A filtration of index pairs associated with a Morse decomposition can be used to find connections between Morse sets. The existence of such a filtration in the case of continuous dynamical systems has been proved in [CoZ] and [Sal] for totally ordered Morse decompositions and in [Fra1] for partially ordered ones. Our purpose is to study the case of a discrete time dynamical system given by a homeomorphism of a locally compact metric space. M. Mrozek [Mr3] has proved that in this case there exist so-called weak index triples for attractor-repeller pairs consisting of $f$-pairs. In many situations they are sufficient, e.g. to obtain the Morse equation. We prove a bit more, the existence of index triples and index filtrations consisting of index pairs. The reason why we prefer index triples is that we can use a simple induction argument then. For this purpose we adapt the proof of existence of index pairs by Mrozek [Mr2].

In [C] and [Fra2] the connection matrix theory for Morse decompositions is developed for flows. The connection matrices are matrices of maps between the homology indices of the sets in the Morse decomposition. They

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provide some information on the structure of the Morse decomposition; in particular, they give an algebraic condition for the existence of connecting orbits between different Morse sets. We wish to investigate the connection matrix theory for a homeomorphism.

Similar results have recently been obtained by David Richeson [Ri]. He defines the analogue of the connection matrix as a pair of matrices corresponding to the functional description of the discrete Conley index developed by A. Szymczak [Szy]. We define it as a single matrix. Even if his approach gives more detailed conditions for the existence of connecting orbits, we think that in several cases it is sufficient to use our method. Moreover, basing on Franzosa's results Richeson concentrates more on the connection matrix theory in his work while we study in detail the properties of Morse decompositions and index filtrations following Salamon and Mrozek's results. In this aspect our proofs are more detailed.

The organization of the paper is as follows. The first section contains preliminaries. In the second section we study properties of Morse decompositions and admissible orderings. In the third section our main result, the theorem on existence of index filtrations is presented. In the last section we introduce the connection matrix for discrete time dynamical systems. The ideas of the proofs of Lemmas 3.5 and 3.6 come from [CoZ]. The proofs of Proposition 2.7 and Lemma 3.7(3) and (5) were motivated by [Sal]. Besides [CoZ] and [Sal], the works of Szymczak [Szy], Mirozek [Mr1, 2, 3] and Reineck [Re] are important references for the index theory presented here.

**1. Preliminaries.** We denote by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ and $\mathbb{N}$ the sets of integers, nonnegative, nonpositive integers and natural numbers, respectively. The usual notation for intervals will refer to intervals in $\mathbb{Z}$, for instance $[n,\infty) := \{m \in \mathbb{Z} : m \geq n\}$.

We assume $X$ is a fixed locally compact metric space. If $A \subset Y \subset X$ the notation $\operatorname{int}_Y A, \operatorname{cl}_Y A, \operatorname{bd}_Y A$ will be used for the interior, closure and boundary of $A$ in $Y$ respectively. If it causes no misunderstanding, we drop the subscript $Y$.

Assume a discrete time dynamical system on $X$ is given, i.e. a fixed homeomorphism $f : X \to X$. We use the convenient notation $xn := f^n(x)$ for any $x \in X$ and $n \in \mathbb{Z}$. If $A \subset X$ and $\Delta \subset \mathbb{Z}$, then $A\Delta := \{xn : x \in A \text{ and } n \in \Delta\}$.

For $N \subset X$ the sets $\operatorname{Inv}^+(N) := \{x \in X : x\mathbb{Z}^+ \subset N\}$, $\operatorname{Inv}^-(N) := x \in X : x\mathbb{Z}^- \subset N\}, \operatorname{Inv}(N) = \operatorname{Inv}^+(N) \cap \operatorname{Inv}^-(N)$ are called the *positively invariant, negatively invariant* and *invariant parts* of $N$, respectively. A set $A$ is called *invariant* iff $\operatorname{Inv}(A) = A$. Similarly one defines *positively invariant* and *negatively invariant* sets.

Proposition 1.1. $\mathrm{Inv}(N)$ *is an invariant set*, *and if* $N$ *is closed then so is* $\mathrm{Inv}(N)$.

The proof is left to the reader.

For $A \subset X$ the sets

$$\Omega^+(A) := \bigcap\{\mathrm{cl}\, A[n,\infty) : n \in \mathbb{N}\}, \qquad \Omega^-(A) := \bigcap\{\mathrm{cl}\, A(-\infty,n] : n \in \mathbb{N}\}$$

are called the *positive* and *negative limit sets* of $A$.

The following statement follows immediately from the definitions.

Proposition 1.2. *If* $I$ *is a closed invariant subset of* $X$ *and* $A \subset I$, *then* $\Omega^+(A)$ *and* $\Omega^-(A)$ *are closed invariant subsets of* $I$.

Definition 1.3. Let $Y$ be a compact, positively (resp. negatively) invariant subset of $X$. A set $A \subset Y$ is called an *attractor* (resp. a *repeller*) relative to $Y$ iff there exists a neighbourhood $U$ of $A$ in $Y$ such that $\Omega^+(U) = A$ (resp. $\Omega^-(U) = A$).

From Proposition 1.2 it follows that attractors and repellers are compact and if $Y$ is invariant then so are every attractor and repeller relative to $Y$. For $A, B \subset X$ we define the *connecting orbit set* from $A$ to $B$ by

$$C(A,B;X) := \{x \in X : \Omega^-(x) \subset A \text{ and } \Omega^+(x) \subset B\}.$$

Proposition 1.4 (see [Mr3], Prop. 3.4). *Let* $I \subset X$ *be a compact invariant set. If* $A$ *is an attractor in* $I$, *then* $A^* := \{x \in I : \Omega^+(x) \cap A = \emptyset\}$ *is a repeller in* $I$. *Similarly if* $A^*$ *is a repeller in* $I$, *then* $A := \{x \in I : \Omega^-(x) \cap A^* = \emptyset\}$ *is an attractor in* $I$.

We call them respectively the *complementary repeller* of $A$ in $I$ and the *complementary attractor* of $A^*$ in $I$. A pair $(A, A^*)$ is called an *attractor-repeller pair* in $I$.

The following proposition gives a useful characterization of attractors and repellers.

Proposition 1.5. *Let* $I \subset X$ *be a compact invariant set. Then for any compact invariant subset* $A \subset I$, $A$ *is an attractor* (*resp. a repeller*) *in* $I$ *if and only if there exists a neighbourhood* $U$ *of* $A$ *in* $I$ *such that for all* $x \in U - A$ *we have* $x\mathbb{Z}^- \not\subset U$ (*resp.* $x\mathbb{Z}^+ \not\subset U$).

Proof. The necessity of the condition is clear since $x\mathbb{Z}^- \subset U$ implies $x \in \Omega^+(U)$.

Let $U'$ be an open neighbourhood of $A$ in $I$ such that $x\mathbb{Z}^- \not\subset U'$ for all $x \in U' - A$ and let $U$ be an open neighbourhood of $A$ in $U'$ such that $A \subset U \subset \mathrm{cl}\, U \subset U'$. Then there exists an $n^* \in \mathbb{N}$ such that $x[-n^*, -1] \not\subset \mathrm{cl}\, U$ for all $x \in I - U$. Now choose a neighbourhood $V$ of $A$ such that $V[0, n^*] \subset U$. Then $V[0,\infty) \subset U$ and therefore $\Omega^+(V) = A$. ■

Proposition 1.6. *If $A'$ is an attractor in $A$ and $A$ is an attractor in $I$, then $A'$ is an attractor in $I$.*

Proof. Let $U$ be a neighbourhood of $A$ in $I$ such $\Omega^+(U) = A$ and let $U'$ be a neighbourhood of $A'$ such that $A' \subset U' \subset U \subset I$ and $U'$ is open in $U$ and $\Omega^+(U' \cap A) = A'$. Let $x \in U'$ be such that $x\mathbb{Z}^- \subset U' \subset U$. From Proposition 1.5 we obtain $x \in \Omega^+(U) = A$. Hence $x\mathbb{Z}^- \subset U' \cap A$ and therefore $x \in \Omega^+(U' \cap A) = A'$. By Proposition 1.5 this implies that $A'$ is an attractor in $I$. ∎

Later on, we will make use of the following

Proposition 1.7. *If $\{K_n\}$ is a decreasing sequence of compact subsets of a topological space $X$ and $f : X \to Y$ is a continuous map, then*

$$f\Big(\bigcap_{n\in\mathbb{N}} K_n\Big) = \bigcap_{n\in\mathbb{N}} f(K_n).$$

Proof. Suppose that $x \in \bigcap_{n\in\mathbb{N}} f(K_n)$. Let $F_n = f^{-1}(x) \cap K_n$. Clearly, $F_n$ is a decreasing sequence of nonempty compact sets. Thus,

$$\bigcap_{n\in\mathbb{N}} F_n = f^{-1}(x) \cap \bigcap_{n\in\mathbb{N}} K_n \neq \emptyset.$$

It follows that $x \in f(\bigcap_{n\in\mathbb{N}} K_n)$. Since the reverse inclusion is obvious, the proof is finished. ∎

## 2. Morse decompositions

Definition 2.1. Let $I$ be a compact invariant subset of $X$. A *Morse decomposition of* $I$ is a finite collection $\{M_p\}_{p\in P}$ of subsets $M_p \subset I$ which are mutually disjoint, compact and invariant, and which can be ordered as $(M_1, \ldots, M_n)$ so that for every $x \in I - \bigcup_{j=1}^n M_j$ there are indices $1 \le i < j \le n$ such that $\Omega^+(x) \subset M_i$ and $\Omega^-(x) \subset M_j$.

Remark 2.2. Such an ordering will then be called an *admissible ordering*. There may be several admissible orderings of the same decomposition. The elements $M_j$ of a Morse decomposition of $I$ will be called *Morse sets* of $I$.

For an admissible ordering $(M_1, \ldots, M_n)$ of a Morse decomposition of $I$ we define the subsets $M_{ji} \subset I$ $(j \ge i)$ as follows:

$$M_{ji} := \{x \in I : \Omega^+(x) \cup \Omega^-(x) \subset M_i \cup M_{i+1} \cup \ldots \cup M_j\}.$$

In particular, $M_{jj} = M_j$.

Proposition 2.3. *Let $(M_1, \ldots, M_n)$ be an admissible ordering of a Morse decomposition of $I$. If $i \le j$, then $(M_1, \ldots, M_{i-1}, M_{ji}, M_{j+1}, \ldots, M_n)$ is an admissible ordering of a Morse decomposition of $I$. Moreover, $(M_i, M_{i+1}, \ldots, M_j)$ is an admissible ordering of a Morse decomposition of $M_{ji}$.*

P r o o f. It is sufficient to prove that $M_{ji}$ is invariant and compact. It is evident that $\Omega^+(x) = \Omega^+(xt)$ for all $t \in \mathbb{Z}$. Let $x \in M_{ji}$ and $k \in \mathbb{Z}$. Since $x \in M_{ji} \subset I$, we have $x\mathbb{Z} \subset I$. Hence $xk \in I$ and $\Omega^+(xk) \cup \Omega^-(xk) = \Omega^+(x) \cup \Omega^-(x) \subset M_i \cup \ldots \cup M_j$ and therefore $xk \in M_{ji}$. Consequently, $x\mathbb{Z} \subset M_{ji}$.

The second assertion is proved in four steps.

STEP 1. *$M_n$ is a repeller in $I$.*

Let $U$ be a neighbourhood of $M_n$ in $I$ such that $\operatorname{cl} U \cap M_i = \emptyset$ for $i < n$. Let $x \in U - M_n \subset I$. Then $\Omega^+(x) \subset M_i$ for some $i < n$ and therefore $\Omega^+(x) \cap \operatorname{cl} U = \emptyset$. We have $x\mathbb{Z}^+ \not\subset U$, for otherwise $\operatorname{cl} x[n,\infty) \subset \operatorname{cl} U$ for all $n \in \mathbb{N}$, and consequently $\Omega^+(x) \subset \operatorname{cl} U$, a contradiction. In view of Proposition 1.5, $M_n$ is a repeller in $I$.

STEP 2. $M_{n-1,1}$ is *an attractor* in $I$.

Indeed,

$$M_{n-1,1} = \{x \in I : \Omega^+(x) \cup \Omega^-(x) \subset M_1 \cup \ldots \cup M_{n-1}\}.$$

By the definition of a Morse decomposition, $M_{n-1,1} = \{x \in I : \Omega^-(x) \cap M_n = \emptyset\}$. Therefore $M_{n-1,1}$ is an attractor in $I$ by Proposition 1.4.

STEP 3. *$M_{j1}$ is an attractor in $I$ for $j = 1, \ldots, n$.*

The proof is by induction on $j$. We give it for $j = n-2$. Analysis similar to that in the proof of Step 2 shows that $M_{n-2,1}$ is an attractor in $M_{n-1,1}$. Since $M_{n-1,1}$ is an attractor in $I$ (by Step 2), we conclude that $M_{n-2,1}$ is an attractor in $I$ by Proposition 1.6.

STEP 4. *$M_{ni}$ is a repeller in $I$ for $i = 1, \ldots, n$.*

$$\begin{aligned} M_{ni} &= \{x \in I : \Omega^+(x) \cup \Omega^-(x) \subset M_i \cup \ldots \cup M_n\} \\ &= \{x \in I : \Omega^+(x) \not\subset M_1 \cup M_2 \cup \ldots \cup M_{i-1}\} \\ &= \{x \in I : \Omega^+(x) \cap M_{i-1,1} = \emptyset\}. \end{aligned}$$

By Proposition 1.4 the last set is the complementary repeller of the attractor $M_{i-1,1}$ in $I$, which proves Step 4.

The set $M_{ji} = M_{j1} \cap M_{ni}$ is compact since it is the intersection of an attractor and a repeller in $I$. ∎

DEFINITION 2.4 (Isolated invariant set). Let $N$ be a compact subset of $X$. If $\operatorname{Inv}(N) \subset \operatorname{int}_X N$, then $N$ is called an *isolating neighbourhood* (in $X$) and $\operatorname{Inv}(N)$ is called an *isolated invariant set*.

PROPOSITION 2.5. *Let $S$ be an isolated invariant set in $X$ and let $\{M_p\}_{p\in P}$ be a Morse decomposition of $S$. Then the sets $M_p$ are also isolated invariant sets in $X$.*

Proof. By assumption there is a compact set $N$ such that $\mathrm{Inv}(N) = S \subset \mathrm{int}_X N$. By the definition of a Morse decomposition, the $M_p$ are compact, invariant and mutually disjoint. Pick any compact neighbourhood $N_p$ of $M_p$ in $X$ which is disjoint from the remaining Morse sets and is contained in $N$. Then $N_p$ is an isolating neighbourhood of $M_p$. It is clear that $M_p = \mathrm{Inv}(M_p) \subset \mathrm{Inv}(N_p)$. Let $x \in \mathrm{Inv}(N_p)$ so that $x\mathbb{Z} \subset N_p \subset N$ and consequently $x \in S$. Since $x[n,\infty) \subset N_p$ and therefore $\mathrm{cl}\, x[n,\infty) \subset \mathrm{cl}\, N_p = N_p$ for all $n \in \mathbb{N}$, we see that $\Omega^+(x) \subset N_p$. Similarly, $\Omega^-(x) \subset N_p$. From the definition of a Morse decomposition it now follows that $x \in M_p$ and thus $\mathrm{Inv}(N_p) = M_p \subset \mathrm{int}_X N_p$. ∎

Remark 2.6. In the same manner we can see that if $(M_1,\ldots,M_n)$ is an admissible ordering of $\{M_p\}_{p\in P}$ then $M_{ji}$ is an isolated invariant set for $i \le j$.

Proposition 2.7. *Let $N$ be an isolating neighbourhood for $S$ and let $(M_1,\ldots,M_n)$ be an admissible ordering of a Morse decomposition of $S$. If $x\mathbb{Z}^+ \subset N$ then $\Omega^+(x) \subset M_i$ for some $i \in \{1,\ldots,n\}$.*

Proof. (a) We first prove the proposition for $n=2$. Proposition 1.2 shows that $\Omega^+(x)$ is a compact invariant subset of $N$ and therefore $\Omega^+(x)\subset S$. From this we can see that either:

1. $\Omega^+(x) \subset M_1$,
2. $\Omega^+(x) \subset M_2$,
3. $\Omega^+(x) \subset M_1 \cup M_2$ and $\Omega^+(x) \not\subset M_1$ and $\Omega^+(x) \not\subset M_2$, or
4. there exists an $x' \in \Omega^+(x) \subset S$ such that $x' \notin M_1 \cup M_2$, and then from the definition of a Morse decomposition, $\Omega^+(x') \subset M_1$ and $\Omega^-(x') \subset M_2$. Since $\Omega^+(x)$ is invariant, $x'\mathbb{Z}^+ \subset \Omega^+(x)$. Let $y$ be a limit point of $\{x'k\}_{k\in\mathbb{Z}^+}$ (it exists because $x'\mathbb{Z}^+ \subset S$ and $S$ is compact). We have $y \in \Omega^+(x')\subset M_1$ and $y \in \Omega^+(x)$ because $x'\mathbb{Z}^+ \subset \Omega^+(x)$ and $\Omega^+(x)$ is closed. Hence $\Omega^+(x) \cap M_1 \neq \emptyset$. Similarly, $\Omega^+(x) \cap M_2 \neq \emptyset$.

It follows from the above that either the proposition holds, or $\Omega^+(x) \cap M_1 \neq \emptyset$ and $\Omega^+(x) \cap M_2 \neq \emptyset$. Suppose the latter holds. Let $U$ be a neighbourhood of $M_1$ in $N$ such that $\mathrm{cl}\, U \cap M_2 = \emptyset$. There is a sequence $\{t_n\} \subset \mathbb{N}$ with $t_n \to \infty$ such that $xt_n \in U$ and $x_0 = \lim xt_n \in M_1$ and $x[t_n, t_{n+1}] \not\subset U$. Hence there exists a sequence $\{t'_n\} \subset \mathbb{N}$ with $t'_n \in [t_n, t_{n+1}]$ such that $x[t_n, t'_n] \subset \mathrm{cl}\, U$ and $x(t'_n+1) \notin U$. Let $x_1$ be any limit point of $\{x(t'_n+1)\}$. We have $x_1 \in N - U$ and $x_1 \in \Omega^+(x) \subset S$.

The rest of the proof is divided into 3 steps.

Step 1. *The sequence $\{t'_n - t_n\}$ is unbounded.*

Suppose on the contrary that $\{t'_n - t_n\}$ is bounded and let $t^*$ be any limit point of it. Take a subsequence $t'_{n_m} - t_{n_m}$ such that $t'_{n_m} - t_{n_m} = t^*$ and therefore $x(t'_{n_m}+1) = xt_{n_m}(t^*+1)$. Letting $m \to \infty$ we obtain $x_1 = x_0(t^*+1)$ and consequently $x_1 \in x_0\mathbb{Z} \subset M_1$. This contradicts the fact that $x_1 \in N - U$.

STEP 2. $x_1(-\infty, -1] \subset \operatorname{cl} U$.

Suppose that $x_1(-\infty, -1] \not\subset \operatorname{cl} U$, i.e. there is a $k \in \mathbb{N}$ such that $x_1(-k) \notin \operatorname{cl} U$. Since $x_1 = \lim_{n\to\infty} x(t'_n + 1)$ for some subsequence, we have $\lim_{n\to\infty} x(t'_n + 1 - k) = x_1(-k) \notin \operatorname{cl} U$ and therefore there exists an $n^* \in \mathbb{N}$ such that $x(t'_n + 1 - k) \notin \operatorname{cl} U$ for $n > n^*$. On the other hand, $\{t'_n - t_n\}$ is unbounded by Step 1 and therefore there is an $\widetilde{n} > n^*$ such that $t'_n - t_n \geq k$ and in consequence $t'_n + 1 - k \geq t_n$ for $n > \widetilde{n}$. Hence $t'_n \geq t'_n + 1 - k \geq t_n$, which gives $x(t'_n + 1 - k) \in x[t_n, t'_n] \subset \operatorname{cl} U$, a contradiction.

STEP 3. We have $\Omega^-(x_1) \subset \operatorname{Inv}(\operatorname{cl} U) = M_1$ and $x_1 \notin M_1$, which contradicts the definition of a Morse decomposition and completes the proof of (a).

(b) *The general case.* Observe that if $n > 2$, then we obtain the two-decomposition $(M_{n-1,1}, M_n)$ of $S$. From (a) we conclude that either $\Omega^+(x) \subset M_n$ or $\Omega^+(x) \subset M_{n-1,1}$. If $\Omega^+(x) \subset M_{n-1,1}$, we consider the Morse decomposition $(M_{n-2,1}, M_{n-1})$ of $M_{n-1,1}$ and replacing $S$ by $M_{n-1,1}$ in (a) we get $\Omega^+(x) \subset M_{n-1}$ or $\Omega^+(x) \subset M_{n-2,1}$. We continue in this fashion obtaining $i \in \{1, \ldots, n\}$ such that $\Omega^+(x) \subset M_i$. ■

**3. Index filtrations for Morse decompositions.** A subset $A$ of $N$ is called *positively invariant with respect to* $N$ provided $A \cap f^{-1}(N) \subset f^{-1}(A)$.

DEFINITION 3.1 (Index pair). Let $S$ be an isolated invariant set. A pair $(N_1, N_0)$ of compact subsets of $X$ is called an *index pair* for $S$ in $X$ if:

(1) $N_0 \subset N_1$,
(2) $S = \operatorname{Inv}(\operatorname{cl}(N_1 - N_0)) \subset \operatorname{int}(N_1 - N_0)$,
(3) $N_0$ is positively invariant with respect to $N_1$,
(4) $N_1 - N_0 \subset f^{-1}(N_1)$ ($N_0$ is an exit set for $N_1$).

M. Mrozek (see [Mr2], Thm. 2.3) has proved the following

THEOREM 3.2 (Existence of index pairs). *Assume $S \subset X$ to be an isolated invariant set. Then for each neighbourhood $\mathcal{O}$ of $S$ there exists an index pair $(N_1, N_0)$ for $S$ such that $N_1 \subset \mathcal{O}$.*

We can now present the main results of this paper.

THEOREM 3.3 (Existence of index triples). *Let $S \subset X$ be an isolated invariant set and let $(M_1, M_2)$ be an admissible ordering of a Morse decomposition of $S$, i.e. $(M_1, M_2)$ is an attractor-repeller pair in $S$. Then there exists a triple $N_0 \subset N_1 \subset N_2$ of compact sets such that*:

(1) $(N_2, N_0)$ *is an index pair for* $S$,
(2) $(N_2, N_1)$ *is an index pair for* $M_2$,
(3) $(N_1, N_0)$ *is an index pair for* $M_1$.

The next result is a consequence of the above by induction on $n$.

THEOREM 3.4 (Existence of index filtrations). *Let* $S \subset X$ *be an isolated invariant set and let* $(M_1, \ldots, M_n)$ *be an admissible ordering of a Morse decomposition of* $S$. *Then there exists a filtration* $N_0 \subset N_1 \subset \ldots \subset N_{n-1} \subset N_n$ *of compact sets such that*, *for any* $i \le j$, *the pair* $(N_j, N_{i-1})$ *is an index pair for* $M_{ji}$. *In particular*, $(N_n, N_0)$ *is an index pair for* $S$, *and* $(N_j, N_{j-1})$ *is an index pair for* $M_j$.

The rest of this section is devoted to the proofs of these theorems. We have divided the proof of Theorem 3.3 into a sequence of lemmas. First we choose any isolating neighbourhood $N$ of $S$, i.e. $\mathrm{Inv}(N) = S \subset \operatorname{int} N$, and define, for $j = 1, 2$, the following subsets of $N$:

$$I_j^+ := \{x \in N : x\mathbb{Z}^+ \subset N \text{ and } \Omega^+(x) \subset M_j \cup M_2\},$$
$$I_j^- := \{x \in N : x\mathbb{Z}^- \subset N \text{ and } \Omega^-(x) \subset M_1 \cup M_j\}.$$

LEMMA 3.5. $I_i^+ \cap I_j^- = M_{ji}$.

Proof. It is obvious that $M_{ji} \subset I_i^+ \cap I_j^-$. If $x \in I_i^+ \cap I_j^-$, then $x\mathbb{Z} \subset N$ and hence $x \in S$. Furthermore $\Omega^+(x) \subset M_i \cup M_2$ and $\Omega^-(x) \subset M_1 \cup M_j$. The claim now follows from the definition of a Morse decomposition. ■

LEMMA 3.6. *The sets* $I_j^+, I_j^-$ *are compact.*

Proof. (a) *The sets* $I_1^+$ *and* $I_2^-$ *are compact.*

Observe that $I_1^+ = \{x \in N : x\mathbb{Z}^+ \subset N\}$ by Proposition 2.7. We prove that $N - I_1^+$ is open relative to $N$. If $x \in N - I_1^+$ then there exists an $n \in \mathbb{N}$ such that $xn \notin N$. By the compactness of $N$ there exists an open neighbourhood $V \subset X$ of $xn$ such that $V \cap N = \emptyset$. Let $U = f^{-n}(V)$ and $\widetilde{U} = U \cap N$. Then $\widetilde{U}$ is a neighbourhood of $x$ in $N$ such that if $y \in \widetilde{U}$ then $y \in N - I_1^+$. Consequently, $N - I_1^+$ is open relative to $N$ and hence $I_1^+$ is compact. The proof that $I_2^-$ is compact is similar.

(b) Let $(M_1, M_2)$ be an admissible ordering of a Morse decomposition of $S$. By definition $I_2^+ \subset I_1^+$ and by (a) the set $I_1^+$ is compact. It remains to show that $I_2^+$ is closed. Let $x = \lim x_n$, $x_n \in I_2^+$. Then $x \in I_1^+$ and hence $\Omega^+(x) \subset M_1 \cup M_2$. We have to show that $\Omega^+(x) \subset M_2$. Assume by contradiction that $\Omega^+(x) \subset M_1$. Since $M_1$ and $M_2$ are disjoint and compact, we can choose open neighbourhoods $U_1$ and $U_2$ of $M_1$ and $M_2$ with $\operatorname{cl} U_1 \cap \operatorname{cl} U_2 = \emptyset$. Observe that $\Omega^+(x_n) \subset M_2$ for all $n \in \mathbb{N}$, because $x_n \in I_2^+$.

STEP 1. *There exists a sequence* $\{t''_n\} \subset \mathbb{N}$ *such that* $x_n t''_n \in U_1$ *and* $x_n[t''_n + 1, \infty) \subset N - U_1$.

There is a $t^* \in \mathbb{N}$ such that $xt^* \in U_1$, because $\Omega^+(x) \subset M_1 \subset U_1$. Let $V$ be a neighbourhood of $xt^*$ in $U_1$ and $U = f^{-t^*}(V)$. Then $U$ is a neighbourhood of $x$ such that $yt^* \in U_1$ for all $y \in U$ and $x_n \in U$ for almost all $n \in \mathbb{N}$. Since $\Omega^+(x_n) \subset M_2 \subset U_2$, we have $x_n[a_n, \infty) \subset U_2$ for $a_n \in \mathbb{N}$ large enough. From this we can define $t''_n := \max\{t^* : x_n t^* \in U_1\}$.

STEP 2. *There exists a sequence* $\{t'_n\} \subset \mathbb{N}$ *with* $t'_n \to \infty$ *such that* $x_n[t'_n, \infty) \subset U_2$ *and* $x_n(t'_n - 1) \notin U_2$.

Suppose it were false. Then we could find $k \in \mathbb{N}$ such that $x_k[t, \infty) \not\subset U_2$ for all $t \in \mathbb{N}$ and, in consequence, there is a sequence $\{\widetilde{t}_l\} \subset \mathbb{N}$ with $\widetilde{t}_l \to \infty$ such that $x_k \widetilde{t}_l \notin U_2$. Consider the sequence $\{x_k \widetilde{t}_l\}_{l \in \mathbb{N}}$ and let $\widetilde{x}$ be any limit point of it. We obtain $\widetilde{x} \in N - U_2$, because $x_k \mathbb{Z}^+ \subset N$ and $x_k \widetilde{t}_l \notin U_2$ for all $l \in \mathbb{N}$. This contradicts the fact that $\widetilde{x} \in \Omega^+(x_k) \subset M_2$. We have proved that there is a sequence $\{t'_n\} \subset \mathbb{N}$ such that $t'_n \to \infty$ and $x_n[t'_n, \infty) \subset U_2$.

In fact, any sequence $\{t'_n\} \subset \mathbb{N}$ such that $x_n[t'_n, \infty) \subset U_2$ is unbounded. To see this, suppose that there is a $t^* \in \mathbb{N}$ such that $t'_n \leq t^*$ for all $n \in \mathbb{N}$. Then $x_n[t^*, \infty) \subset U_2$ for all $n \in \mathbb{N}$. Consider $x[t^*, \infty)$. We obtain $\lim_{n\to\infty} x_n t = xt \in \operatorname{cl} U_2$ for $t \geq t^*$ and so $x[t^*, \infty) \subset \operatorname{cl} U_2$. Hence $\Omega^+(x) \subset \operatorname{cl} U_2$ and $\Omega^+(x) \subset M_1$, a contra- diction.

The above remark and Step 1 show that $\{t'_n\}$ can be chosen such that $x_n(t'_n - 1) \notin U_2$.

STEP 3. *There exists a sequence* $\{t_n\} \subset \mathbb{N}$ *such that* $x_n t_n \in N - (U_1 \cup U_2)$ *and* $x_n[t_n, \infty) \subset N - U_1$.

Observe that if $\{t''_n\}$ is bounded then $t_n = t'_n - 1$ is as required, by Step 2.

Suppose that $\{t''_n\}$ is unbounded and $f(x_n t''_n) \in U_2$ for almost all $n \in \mathbb{N}$. We first choose from $\{t''_n\}$ a subsequence tending to $\infty$. We use the same notation for it. Then we take a subsequence of $\{x_n t''_n\}$ such that $x^* = \lim x_n t''_n$ exists. For any $t \in \mathbb{Z}^+$ we have $x^*[-t, 0] = \lim x_n t_n[-t, 0] = \lim x_n[t_n - t, t_n] \subset N$ since $x_n t''_n \in I_1^+$ and $I_1^+$ is closed. Thus $x^*\mathbb{Z} \subset N$ and so $x^* \in S$. Since $x^* \in \operatorname{cl} U_1$, it follows that either $x^* \in M_1$ or $x^* \in C(M_2, M_1; S)$ by the definition of a Morse decomposition. If $x^* \in M_1$ then $f(x^*) \in M_1$, contrary to $f(x^*) = f(\lim x_n t''_n) = \lim f(x_n t''_n) \in \operatorname{cl} U_2$. Consequently, $x^* \in C(M_2, M_1; S)$. But $\Omega^+(x^*) \subset M_2$, since $f^k(x^*) = \lim f^k(x_n t''_n) \in N - U_1$ for $k \in \mathbb{N}$, a contradiction. This completes the proof of Step 3.

Let $\{x_n t_n\}$ be as in Step 3. Take a subsequence of $\{x_n t_n\}$ such that $x^* = \lim x_n t_n$ exists. We have $x^* \notin M_1 \cup M_2$ and $x^*[0, \infty) \subset N - U_1$ and hence $\Omega^+(x^*) \subset M_2$. Consider again two cases:

1. $\{t_n\} \subset \mathbb{N}$ is bounded and therefore $t_n = t^*$ for infinitely many $n$. Then $x^* = \lim x_n t_n = \lim x_n t^* = xt^*$, which implies that $x^* \in x\mathbb{Z}$. Hence $\Omega^+(x^*) = \Omega^+(x) \subset M_1$, contradicting $\Omega^+(x^*) \subset M_2$.

2. $\{t_n\} \subset \mathbb{N}$ is unbounded. Since $x_n\mathbb{Z}^+ \subset N$, we have $x^*[-t,0] = \lim x_n t_n[-t,0] = \lim x_n[t_n - t, t_n] \subset N$ for all $t \in \mathbb{N}$. Hence $x^*\mathbb{Z}^- \subset N$ and thus $x^*\mathbb{Z} \subset N$. Recalling that $\Omega^+(x^*) \subset M_2$, we conclude that $x^* \in M_2$ by the definition of a Morse decomposition. But this contradicts $x^* \notin M_1 \cup M_2$. This completes the proof of Lemma 3.6. ∎

For any subset $K \subset N$ we define the maximal positively invariant set in $N$ which contains $K$ by

$$P(K,N) := \{x \in N : \exists t \in \mathbb{Z}^+ \text{ such that } x[-t,0] \subset N \text{ and } x(-t) \in K\}.$$

In the next lemma formulations and proofs of (1), (2), (4) come from [Mr2].

Lemma 3.7. *Let $M$ be an isolating neighbourhood for the isolated invariant set $S$ and let*

$$I_1^+ = \{x \in M : x\mathbb{Z}^+ \subset M\}, \quad I_2^- = \{x \in M : x\mathbb{Z}^- \subset M\}.$$

*In (3), (5) and (8) we assume additionally that $(M_1, M_2)$ is an admissible ordering of a Morse decomposition of $S$. Then:*

(1) *If $B \subset M$ is compact and disjoint from $I_1^+$ then so is $P(B,M)$.*

(2) *If $I_2^- \subset B$ and $B$ is compact then $P(B,M)$ is compact.*

(3) *If $U$ is a neighbourhood of $I_1^-$ in $X$ and $W$ is a compact neighbourhood of $I_2^+$ in $M$ such that $W \cap I_1^- = \emptyset$, and $K \subset M$ is a compact set such that*

$$I_1^- \subset K \subset P(K,M) \subset U \cap (M - W),$$

*then $P(K,M)$ is compact.*

(4) *If $U$ is a neighbourhood of $I_2^-$ in $M$ then there exists a compact neighbourhood $K$ of $I_2^-$ in $M$ such that $P(K,M) \subset U$.*

(5) *If $V'$ is a neighbourhood of $I_1^-$ in $X$ then there exists a compact neighbourhood $L$ of $I_1^-$ in $M$ such that $P(L,M) \subset V'$ and $P(L,M)$ is compact.*

(6) *If $x \in P(B,M)$ and $f(x) \in M$ then $f(x) \in P(B,M)$.*

(7) *There exist open neighbourhoods $U, V$ of $I_1^+$ and $I_2^-$ in $M$ such that $f(U \cap V) \subset M$.*

(8) *Let $N \subset M$ also be an isolating neighbourhood for $S$. We can choose simultaneously $U$, $V$ as in (7) and open neighbourhoods $U'$, $V'$ of the sets*

$$\widetilde{I}_1^+ = \{x \in N : x\mathbb{Z}^+ \subset N\}, \quad \widetilde{I}_1^- = \{x \in N : x\mathbb{Z}^- \subset N \text{ and } \Omega^-(x) \subset M_1\}$$

*in $N$ such that $U' = U \cap N$, $V' \cap M_2 = \emptyset$, $V' \subset V$ and $f(U' \cap V') \subset N$.*

Proof. (1) See [Mr2, Lemma 5.6].

(2) See [Mr2, Lemma 5.7].

(3) Let $x_n \in P(K,M)$ converge to $x$ and let $\{t_n\} \subset \mathbb{Z}^+$ be such that $x_n[-t_n, 0] \subset M$ and $x_n(-t_n) \in K$. Then we have $x_n[-t_n, 0] \subset P(K,M) \subset U \cap (M - W)$ for all $n \in \mathbb{N}$. Consider two cases:

(a) $\{t_n\}$ is unbounded. Then $x(-t) = \lim x_n(-t) \in \mathrm{cl}(U \cap (M - W))$ for all $t \in \mathbb{Z}^+$, because $x_n(-t) \in x_n[-t_n, 0]$ for $n$ large enough. Hence $\Omega^-(x) \subset M_1$ and therefore $x \in I_1^- \subset P(K,M)$.

(b) $\{t_n\}$ is bounded with a limit point $t \in \mathbb{Z}^+$. Then we conclude that $x[-t, 0] = \lim x_n[-t, 0] = \lim x_n[-t_n, 0] \subset M$ and $x(-t) = \lim x_n(-t) = \lim x_n(-t_n) \in K$, and therefore $x \in P(K,M)$.

(4) See [Mr2, Lemma 5.8].

(5) We prove this statement in four steps.

STEP 1. *There is a compact neighbourhood $W$ of $I_2^+$ in $M$ such that $W \cap I_1^- = \emptyset$.*

Since $I_1^-$ and $I_2^+$ are compact and $I_1^- \cap I_2^+ = \emptyset$, it follows that for every $x \in I_2^+$ there exists a neighbourhood $U_x$ in $M$ such that $\mathrm{cl}\, U_x$ is compact and $\mathrm{cl}\, U_x \cap I_1^- = \emptyset$. Then $\{U_x\}_{x \in I_2^+}$ is an open covering of the compact set $I_2^+$. We choose a finite covering $\{U_{x_1}, \ldots, U_{x_m}\}$ and define $W = \mathrm{cl}\, U_{x_1} \cup \ldots \cup \mathrm{cl}\, U_{x_m}$.

STEP 2. *There exists a $t^* \in \mathbb{N}$ such that for every $x \in M$ if $x[-t^*, -1] \subset \mathrm{cl}(M - W)$ then $x \in V' \cap (M - W)$.*

If this implication did not hold, then there would exist sequences $\{x_n\} \subset M$ and $\{t_n\} \subset \mathbb{N}$ with $t_n \to \infty$ such that $x_n[-t_n, -1] \subset \mathrm{cl}(M - W)$ and $x_n \notin V' \cap (M - W)$. Any limit point $x$ of $\{x_n\}$ would then satisfy $x(-\infty, -1] \subset \mathrm{cl}(M - W)$ and $x \notin V' \cap (M - W)$. But this would imply $\Omega^+(x) \subset M_1$ and therefore $x \in I_1^- \subset V' \cap (M - W)$, in contradiction to $x \notin V' \cap (M - W)$.

STEP 3. *Construction of $L$.* Define $A = \{x \in I_1^- : x[0, t^*] \subset M\}$ and $B = \{x \in I_1^- : x[0, t^*] \not\subset M\}$. For every $x \in A$ there exists an open neighbourhood $U(x)$ of $x$ in $X$ such that $U(x)[0, t^*] \subset V' \cap (X - W)$. For every $x \in B$ there exists $t(x) \in \mathbb{N}$ such that $x[0, t(x)] \subset V' \cap (X - W)$ and $xt(x) \notin M$. Hence for every $x \in B$ there exists an open neighbourhood of $x$ in $X$ such that $U(x)[0, t(x)] \subset V' \cap (X - W)$ and $U(x)t(x) \cap M = \emptyset$. Since $I_1^-$ is compact, there exist finitely many $x_1, \ldots, x_k \in I_1^-$ such that the sets $U(x_i)$, $i = 1, \ldots, k$, cover $I_1^-$. We choose a compact neighbourhood $L$ of $I_1^-$ such that $L \subset \bigcup_{i=1}^k U(x_i)$.

STEP 4. $P(L, M) \subset V' \cap (M - W)$.

Let $x \in P(L,M)$ and let $t \in \mathbb{Z}^+$ with $x[-t,0] \subset M$ and $x(-t) \in L$. Then $x(-t) \in U(x_i)$ for some $i \in \{1,\dots,k\}$. Suppose that $x \notin V' \cap (X-W)$ and consider two cases.

1. If $x_i \in A$ then $x[-t, t^*-t] \subset V' \cap (X-W)$ and therefore $t^* - t < 0$. Hence there exists a $t' \in [0, t-t^*]$ such that $x[-t,-t'] \subset V' \cap (M-W)$ and $x(-t') \notin V' \cap (M-W)$. This implies $x[-t'-t^*, -t'-1] = x(-t')[-t^*, -1] \subset \mathrm{cl}(M-W)$ and $x(-t') \notin V' \cap (M-W)$, contrary to Step 2.

2. If $x_i \in B$ then $x(-t)[0, t(x_i)] \subset V' \cap (X-W)$ and $x(-t)t(x_i) \notin M$, i.e. $x[-t, -t+t(x_i)] \subset V' \cap (X-W)$ and $x(t(x_i)-t) \notin M$. From $x[-t,0] \subset M$ we obtain $t(x_i) > t$, and from $x \notin V' \cap (X-W)$ we get $t(x_i) < t$, a contradiction.

We conclude that $x \in M \cap (V' \cap (X-W)) = V' \cap (M-W)$, which proves the assertion of Step 4.

Step 4 implies that $P(L,M) \subset V'$. From (3) it follows that $P(L,M)$ is compact, which completes the proof of (5).

(6) The proof is immediate.

(7) For the proof, take two decreasing sequences $\{U_n\}_{n\in\mathbb{N}}$, $\{V_n\}_{n\in\mathbb{N}}$ of compact neighbourhoods of $I_1^+$ and $I_2^-$ in $M$ intersecting in $I_1^+$ and $I_2^-$ respectively. Proposition 1.7 now leads to

$$\bigcap_{n\in\mathbb{N}} f(U_n \cap V_n) = f\Big(\bigcap_{n\in\mathbb{N}} (U_n \cap V_n)\Big) = f(I_1^+ \cap I_2^-) = f(S) = S \subset \operatorname{int} M.$$

By compactness, $f(U_n \cap V_n) \subset \operatorname{int} M$ for some $n \in \mathbb{N}$. Obviously, the sets $U = \operatorname{int}_M U_n$ and $V = \operatorname{int}_M V_n$ satisfy $f(U \cap V) \subset \operatorname{int} M$.

(8) Let $N \subset M$ be an isolating neighbourhood for $S$. Consider

$$\widetilde{I}_1^+ = \{x \in N : x\mathbb{Z}^+ \subset N\} \subset I_1^+,$$
$$\widetilde{I}_1^- = \{x \in N : x\mathbb{Z}^- \subset N \text{ and } \Omega^-(x) \subset M_1\} \subset I_2^-.$$

Let $\{U_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of compact neighbourhoods of $I_1^+$ in $M$ such that $\bigcap_{n\in\mathbb{N}} U_n = I_1^+$. Let $\widetilde{U}_n = U_n \cap N$. Then $\bigcap_{n\in\mathbb{N}} \widetilde{U}_n = I_1^+ \cap N$.

Let $\{V_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of compact neighbourhoods of $I_2^-$ in $M$ such that $\bigcap_{n\in\mathbb{N}} V_n = I_2^-$ , and let $\{\widetilde{V}_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of compact neighbourhoods of $\widetilde{I}_1^-$ in $N$ such that $\widetilde{V}_n \subset V_n$ and $\widetilde{V}_1 \cap M_2 = \emptyset$, and $\bigcap_{n\in\mathbb{N}} \widetilde{V}_n = \widetilde{I}_1^-$. Using Proposition 1.7 we get:

$$\text{(a)} \qquad \bigcap_{n\in\mathbb{N}} f(\widetilde{U}_n \cap \widetilde{V}_n) = f\Big(\bigcap_{n} (\widetilde{U}_n \cap \widetilde{V}_n)\Big) = f(I_1^+ \cap N \cap \widetilde{I}_1^-)$$
$$= f(M_1) = M_1 \subset \operatorname{int} N.$$

By compactness, $f(\widetilde{U}_n \cap \widetilde{V}_n) \subset \operatorname{int} N$ for some $n_1 \in \mathbb{N}$.

(b) $$\bigcap_{n\in\mathbb{N}} f(U_n\cap V_n) = f\Big(\bigcap_{n}(U_n\cap V_n)\Big) = f(I_1^+\cap I_2^-) = f(S) = S\subset \operatorname{int} M.$$

By compactness, $f(U_n\cap V_n)\subset \operatorname{int} M$ for some $n_2\in\mathbb{N}$.

Put $n = \max(n_1, n_2)$. Then the sets $U = \operatorname{int}_M U_n$, $V = \operatorname{int}_M V_n$, $U' = \operatorname{int}_N \widetilde{U}_n$ and $V' = \operatorname{int}_N \widetilde{V}_n$ satisfy our claim. ∎

*Proof of Theorem 3.2* (Construction of an index pair). Let $M$ be an isolating neighbourhood for $S$, contained in $\mathcal{O}$. Set $I^+=\{x\in M : x\mathbb{Z}^+\subset M\}$ and $I^- = \{x\in M : x\mathbb{Z}^-\subset M\}$. By Lemma 3.7(7), there exist open neighbourhoods $U$, $V$ of $I^+$ and $I^-$ in $M$ such that $f(U\cap V)\subset M$. By Lemma 3.7(5), there exists a compact neighbourhood $K$ of $I^-$ in $M$ such that $P(K,M)\subset V$. We put $N_0 := P(M-U, M)$ and

$$N_1 := N_0\cup P(K\cup(M-U), M) = N_0\cup P(K,M).$$

Let us check the conditions defining an index pair.

(o) $N_0\subset N_1$ and by Lemma 3.7(1), (2), $N_0$ and $N_1$ are compact.

(i) $\operatorname{cl}(N_1-N_0)$ *is an isolating neighbourhood for* $S$.

By Lemma 3.7(1), $N_0$ is compact and disjoint from $S$. Since $S\subset \operatorname{int} K\subset \operatorname{int} P(K,M)$, we conclude that $S\subset \operatorname{int} P(K,M) - N_0 = \operatorname{int}(P(K,M)-N_0) \subset M$. This gives $\operatorname{Inv}(\operatorname{cl}(N_1-N_0)) = S\subset \operatorname{int}(N_1-N_0)$.

(ii) $N_0$ *is positively invariant with respect to* $N_1$.

Assume that $x\in N_0 = P(M-U,M)$ and $f(x)\in N_1\subset M$. By Lemma 3.7(6), $f(x)\in P(M-U,M) = N_0$.

(iii) $N_0$ *is an exit set for* $N_1$, *i.e.* $x\in N_1-N_0$ *implies that* $f(x)\in N_1$.

Assume that $x\in N_1-N_0\subset P(K\cup(M-U),M)$. Then $f(x)\in f(N_1-N_0) = f(P(K,M)-P(M-U,M))\subset f(V-(M-U)) = f(U\cap V)\subset M$. Thus, by Lemma 3.7(6), $f(x)\in P(K\cup(M-U),M)\subset N_1$. ∎

*Proof of Theorem 3.3* (Construction of an index triple). Let $M$ be an isolating neighbourhood for $S$ and let $(N_2, N_0)$ be an index pair for $S$ as in the proof of Theorem 3.2. Then $N = \operatorname{cl}(N_2-N_0)$ is an isolating neighbourhood for $S$ and $N\subset M$. Recall that

$$\begin{aligned}
\widetilde{I}_1^+ &= \{x\in N : x\mathbb{Z}^+\subset N\},\\
\widetilde{I}_2^+ &= \{x\in N : x\mathbb{Z}^+\subset N \text{ and } \Omega^+(x)\subset M_2\},\\
\widetilde{I}_1^- &= \{x\in N : x\mathbb{Z}^-\subset N \text{ and } \Omega^-(x)\subset M_1\},\\
\widetilde{I}_2^- &= \{x\in N : x\mathbb{Z}^-\subset N\}.
\end{aligned}$$

By Lemma 3.7(8), there exist open neighbourhoods $U, V$ of $I_1^+$ and $I_2^-$ in $M$ and open neighbourhoods $U', V'$ of $\widetilde{I}_1^+$ and $\widetilde{I}_1^-$ in $N$ such that $U' = U\cap N$, $V'\cap M_2=\emptyset$, $V'\subset V$ and $f(U'\cap V')\subset N$. By Lemma 3.7(5) , there exists a compact neighbourhood $L$ of $\widetilde{I}_1^-$ in $N$ such that $P(L,N) \subset V'$ and $P(L,N)$ is compact. We put

$$N_1 := N_0 \cup P(L,N) = N_0 \cup P(L \cup (N-U'),N).$$

Let us first check that $(N_1,N_0)$ is an index pair for $M_1$.

(o) $N_0 \subset N_1$ and from the definition of an index pair, $N_0$ is compact. By Lemma 3.7(5), $P(L,N)$ is compact and so is $N_1$.

(i) $\mathrm{cl}(N_1 - N_0)$ *is an isolating neighbourhood for* $M_1$.

We have $M_1 \subset \widetilde{I}_1^- \subset \mathrm{int}\, L \subset \mathrm{int}\, P(L,N) \subset \mathrm{int}\, N_1$ and $N_0 \cap M_1 = \emptyset$, since $N_0 \cap S = \emptyset$. Hence $M_1 \subset \mathrm{int}\, N_1 - N_0 = \mathrm{int}(N_1 - N_0)$. Since $M_2 \cap V' = \emptyset$ and $P(L,N) \subset V'$, we obtain $N_1 \cap M_2 = \emptyset$. We thus get $\mathrm{Inv}(\mathrm{cl}(N_1 - N_0)) = M_1 \subset \mathrm{int}(N_1 - N_0)$.

(ii) $N_0$ *is positively invariant with respect to* $N_1$.

Let $x \in N_0$ and $f(x) \in N_1 \subset N_2$. Since $N_0$ is positively invariant in $N_2$, we see that $f(x) \in N_0$.

(iii) $N_0$ *is an exit set for* $N_1$.

Let $x \in N_1 - N_0$ and therefore $x \in P(L \cup (N-U'),N)$, and $f(x) \in f(N_1 - N_0) = f(P(L,N) - N_0)$. We have $P(N-U',N) \subset P(M-U,M) = N_0$ and hence $P(L,N) - N_0 \subset P(L,N) - P(N-U',N) \subset V' - (N-U') = U' \cap V'$. Consequently, $f(x) \in f(P(L,N) - N_0) \subset f(U' \cap V') \subset N$. By Lemma 3.7(6), $f(x) \in P(L \cup (N-U'),N) \subset N_1$.

Let us now check that $(N_2,N_1)$ is an index pair for $M_2$.

(o) $N_1 = N_0 \cup P(L,N) \subset N_0 \cup N \subset N_2$ and $N_1, N_2$ are compact.

(i) $\mathrm{cl}(N_2 - N_1)$ *is an isolating neighbourhood for* $M_2$.

We have $N_0 \cap M_2 = \emptyset$ and $P(L,N) \cap M_2 = \emptyset$, because $P(L,N) \subset V'$ and $V' \cap M_2 = \emptyset$. Therefore $N_1 \cap M_2 = (N_0 \cup P(L,N)) \cap M_2 = \emptyset$. We see at once that $M_2 \subset S \subset \mathrm{int}(N_2 - N_0) \subset \mathrm{int}\, N_2$. Hence $M_2 \subset \mathrm{int}\, N_2 - N_1 = \mathrm{int}(N_2 - N_1)$. Observe that $M_1 \subset \mathrm{int}\, L \subset P(L,N) \subset \mathrm{int}\, N_1$ and therefore $M_1 \cap \mathrm{cl}(N_2 - \mathrm{int}\, N_1) = \emptyset$. But this implies $M_1 \cap \mathrm{cl}(N_2 - N_1) = \emptyset$ and clearly forces $\mathrm{Inv}(\mathrm{cl}(N_2 - N_1)) = M_2 \subset \mathrm{int}(N_2 - N_1)$.

(ii) $N_1$ *is positively invariant in* $N_2$.

Let $x \in N_1$ and $f(x) \in N_2$. One of two cases holds:

1. $x \in N_0$ and $f(x) \in N_2$. Since $N_0$ is positively invariant with respect to $N_2$, it follows that $f(x) \in N_0 \subset N_1$.

2. $x \in P(L,N)$ and $f(x) \in N_2$, and then either $f(x) \in N_0 \subset N_1$, or $f(x) \in N_2 - N_1 \subset \mathrm{cl}(N_2 - N_0) \subset N$ and therefore $f(x) \in P(L,N) \subset N_1$ by Lemma 3.7(6).

(iii) $N_1$ *is an exit set for* $N_2$.

Let $x \in N_2 - N_1 \subset N_2 - N_0$. Since $N_0$ is an exit set for $N_2$, we conclude that $f(x) \in N_2$. ■

*Proof of Theorem 3.4.* The proof is by induction. Assume that the theorem holds for $k \le n$; we will prove it for $n+1$.

Let $(M_1, \ldots, M_{n+1})$ be an admissible ordering of a Morse decomposition of $S$.

Let $N_0 \subset N_n \subset N_{n+1}$ be an index filtration for the admissible ordering of the two-decomposition $(M_{n1}, M_{n+1})$ of $S$. Then $(N_{n+1}, N_0)$ is an index pair for $S$, $(N_n, N_0)$ is an index pair for $M_{n1}$, and $(N_{n+1}, N_n)$ is an index pair for $M_{n+1}$.

Let $N_0 \subset N_1 \subset \ldots \subset N_n$ be an index filtration for the admissible ordering of the $n$-decomposition $(M_1, \ldots, M_n)$ of $M_{n1}$. Then, for any $1 \le i \le n$, $(N_j, N_{i-1})$ is an index pair for $M_{ji}$.

It remains to prove that, for any $1 < i \le n$, $(N_{n+1}, N_{i-1})$ is an index pair for $M_{n+1,i}$.

(o) $N_{i-1} \subset N_{n+1}$ and $N_{i-1}, N_{n+1}$ *are compact*.

(i) $\mathrm{cl}(N_{n+1} - N_{i-1})$ *is an isolating neighbourhood for* $M_{n+1,i}$.

We have $N_{i-1} \cap M_l = \emptyset$ for $i \le l \le n+1$, because $(N_l, N_{l-1})$ is an index pair for $M_l$ and $N_{i-1} \subset N_{l-1}$. Suppose that $x \in C(M_m, M_l; S)$ for $i \le l < m \le n+1$ and $x \in N_{i-1}$. We obtain $\Omega^+(x) \subset M_l \subset \mathrm{int}(N_l - N_{i-1}) = \mathrm{int}\, N_l - N_{i-1}$ and therefore $xt \in \mathrm{int}\, N_l - N_{i-1}$ for every sufficiently large $t \in \mathbb{N}$. Since $x\mathbb{Z} \subset \mathrm{cl}(N_m - N_0)$, it follows that there exists $y \in N_{i-1}$ such that $f(y) \in N_l - N_{i-1}$. This contradicts the fact that $N_{i-1}$ is positively invariant with respect to $N_l$. From what has already been proved, we conclude that $N_{i-1} \cap M_{n+1,i} = \emptyset$. But $M_{n+1,i} \subset S \subset \mathrm{int}(N_{n+1} - N_0) \subset \mathrm{int}\, N_{n+1}$ and therefore $M_{n+1,i} \subset \mathrm{int}\, N_{n+1} - N_{i-1} = \mathrm{int}(N_{n+1} - N_{i-1})$. In addition, if $x \in S$ and $\Omega^+(\mathrm{x}) \subset M_l$ for $l < i$, then $x\mathbb{Z} \cap \mathrm{int}\, N_{i-1} \ne \emptyset$. Hence $\mathrm{Inv}(\mathrm{cl}(N_{n+1} - N_{i-1})) = M_{n+1,i} \subset \mathrm{int}(N_{n+1} - N_{i-1})$.

(ii) $N_{i-1}$ *is positively invariant with respect to* $N_{n+1}$.

Let $x \in N_{i-1} \subset N_n$ and $f(x) \in N_{n+1}$. Since $N_n$ is positively invariant in $N_{n+1}$, we get $f(x) \in N_n$. Thus $x \in N_{i-1}$ and $f(x) \in N_n$, and consequently $f(x) \in N_{i-1}$, because $N_{i-1}$ is positively invariant in $N_n$.

(iii) $N_{i-1}$ *is an exit set for* $N_{n+1}$.

Let $x \in N_{n+1} - N_{i-1} \subset N_{n+1} - N_0$. Since $N_0$ is an exit set for $N_{n+1}$, we obtain $f(x) \in N_{n+1}$. ■

**4. The discrete Conley index and connection matrices.** We recall the notion of the Leray functor introduced by Mrozek ([Mr2], [Mr3]).

Denote by $\mathcal{E}$ the category of graded vector spaces and linear maps of degree zero. A new category $\mathrm{Endo}(\mathcal{E})$ of graded vector spaces with distinguished endomorphism is defined as follows. Objects are pairs $(E,e)$, where $E\in\mathcal{E}$ and $e\in\mathcal{E}(E,E)$. Morphisms from $(E,e)$ to $(F,f)$ are all maps $\Phi\in\mathcal{E}(E,F)$ such that $\Phi\circ e=f\circ\Phi$. $\mathrm{Auto}(\mathcal{E})$ is the full subcategory of $\mathrm{Endo}(\mathcal{E})$ consisting of graded vector spaces with a distinguished isomorphism. The full subcategory of $\mathrm{Endo}(\mathcal{E})$ consisting of all objects with finite-dimensional components and their morphisms will be denoted by $\mathrm{Endo}_0(\mathcal{E})$.

For $(E,e)\in\mathrm{Endo}(\mathcal{E})$ we define the *generalized kernel* of $e$ as

$$\mathrm{gker}(e):=\bigcup\{e^{-n}(0)\mid n\in\mathbb{N}\}.$$

Put

$$L(E,e):=(E/\mathrm{gker}(e),e')$$

where $e':E/\mathrm{gker}(e)\ni[x]\mapsto[e(x)]\in E/\mathrm{gker}(e)$ is the induced endomorphism. Assume that $\Phi:(E,e)\to(F,f)$ is a morphism. Let

$$\Phi':E/\mathrm{gker}(e)\ni[x]\mapsto[\Phi(x)]\in f/\mathrm{gker}(f)$$

denote the induced morphism. We then put $L(\Phi):=\Phi'$. Thus we have defined a covariant functor $L:\mathrm{Endo}_0(\mathcal{E})\to\mathrm{Auto}(\mathcal{E})$ called the *Leray functor*.

Let $H_*$ be the singular homology functor with rational coefficients. If we consider an index pair $N=(N_1,N_0)$, then the map $f_N:N_1/N_0\to N_1/N_0$ given by

$$f_N([x]):=\begin{cases}[f(x)] & \text{if } x,f(x)\in N_1\backslash N_0,\\ [N_0] & \text{otherwise},\end{cases}$$

is continuous (see e.g. [Szy], Lemma 4.3), and it induces an endomorphism $f_*:H_*(N_1,N_0)\to H_*(N_1,N_0)$. Therefore $(H_*(N_1,N_0),f_*)\in\mathrm{Endo}(\mathcal{E})$. We also denote by $H_*$ the extension of the homology functor to this category.

DEFINITION 4.1. The *homology Conley index* of an isolated invariant set $S$ is defined as

$$CH_*(S):=LH_*(N),$$

where $N$ is any index pair for $S$ in $X$.

Due to [Mr2], Thm. 2.6, the above definition makes sense.

Let $P=\{1,\dots,n\}$ be a finite totally ordered set. A subset $I\subseteq P$ is an *interval* if $i,j\in I$ and $i<k<j$ imply $k\in I$. Two elements $i,j\in P$ are *adjacent* if $\{i,j\}$ is an interval. Similarly, a pair of disjoint intervals $(I,J)$ is called *adjacent* if

(1) $I\cup J$ is an interval,
(2) $i\in I$ and $j\in J$ imply $i<j$.

Let $(M_1, \ldots, M_n)$ be an admissible ordering of a Morse decomposition of an isolated invariant set $S$. For an interval $I \subseteq P$, define

$$M(I) := \Big(\bigcup_{i\in I} M_i\Big) \cup \Big(\bigcup_{i,j\in I} C(M_i, M_j; S)\Big).$$

From Proposition 2.3 we see that $M(I)$ is an isolated invariant set and we define

$$CH_*(I) := CH_*(M(I)).$$

If $(A, A^*)$ is an attractor-repeller pair in an isolated invariant set $S$ such that $CH_*(S)$, $CH_*(A^*)$ and $CH_*(A)$ are graded vector spaces with finite-dimensional components (this assumption is satisfied e.g. when $X$ is a compact ANR), then we can construct a long exact sequence relating the homology indices of $S, A^*$ and $A$ (see [Mr3]). Namely, there is a long exact sequence

$$\ldots \to H_1(N_2, N_1) \xrightarrow{\partial} H_0(N_1, N_0) \to H_0(N_2, N_0) \to H_0(N_2, N_1) \to 0$$

where $(N_2, N_1, N_0)$ is the filtration given by Theorem 3.3. Applying the Leray functor we obtain an exact sequence of homology Conley indices

$$\ldots \to CH_1(A^*) \xrightarrow{\partial} CH_0(A) \to CH_0(S) \to CH_0(A^*) \to 0.$$

This sequence, called the *homology index sequence of the attractor-repeller pair*, provides an algebraic condition for the existence of connecting orbits. The map $\partial$ is called the *connection map*. Exactness implies that if $CH_*(S) = 0$, then $\partial$ is an isomorphism. If $C(A^*, A; S) = \emptyset$, then $CH_*(S) \simeq CH_*(A^*) \oplus CH_*(A)$ and it follows that $\partial = 0$. So we have

THEOREM 4.2. *If the connection map is nontrivial then $C(A^*, A; S)$ is nonempty.*

Since we need the Leray functor to maintain exactness of homological sequences, from now on we assume that $X$ is a compact ANR, which is sufficient according to [Mr3].

Given a Morse decomposition $\{M_p\}_{p\in P}$, if $(I, J)$ is an adjacent pair of intervals, then $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$, where $IJ := I \cup J$. So there is an exact sequence

$$\ldots \to CH_q(I) \to CH_q(IJ) \to CH_q(J) \xrightarrow{\partial(I,J)} CH_{q-1}(I) \to \ldots \tag{4.3}$$

The connection matrix condenses the Morse-theoretic information contained in the maps $\partial(I, J)$ into maps defined between the sets $\{M_p\}_{p\in P}$. To do this, for an interval $I \subseteq P$ define

$$C\Delta(I) := \bigoplus_{i\in I} CH_*(i)$$

and let $C\Delta$ denote $C\Delta(P)$. A $\mathbb{Q}$-linear map $\Delta : C\Delta \to C\Delta$ can be thought of as a matrix

$$[\Delta(i,j) : CH_*(j) \to CH_*(i) \mid i,j \in P].$$

We say that $\Delta = \Delta(P)$ is *upper triangular* if $\Delta(i,j) = 0$ for $j \leq i$, and $\Delta$ is a *boundary map* if each $\Delta(i,j)$ has degree $-1$ and $\Delta \circ \Delta = 0$.

It is not difficult to show that if $\Delta$ is an upper triangular boundary map, then so is the restriction $\Delta(I) : C\Delta(I) \to C\Delta(I)$. If $I$ and $J$ are adjacent intervals, then there is an obvious exact sequence of chain complexes

$$0 \to C\Delta(I) \to C\Delta(IJ) \to C\Delta(J) \to 0,$$

which gives a long exact homology sequence

$$\ldots \to H_q\Delta(I) \to H_q\Delta(IJ) \to H_q\Delta(J) \to H_{q-1}\Delta(I) \to \ldots \tag{4.4}$$

Definition 4.5. We say that the upper triangular boundary map $\Delta : C\Delta \to C\Delta$ is a *connection matrix* if for each interval $I \subseteq P$ there exists a homomorphism $\Phi(I) : H\Delta(I) \to CH_*(I)$ such that

(1) for $i \in P$, $\Phi(i) : H\Delta(i) = CH_*(i) \to CH_*(i)$ is the identity,

(2) for each adjacent pair of intervals $(I, J)$ the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_q\Delta(I) & \longrightarrow & H_q\Delta(IJ) & \longrightarrow & H_q\Delta(J) & \longrightarrow & H_{q-1}\Delta(I) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \Phi(I)} & & \downarrow{\scriptstyle \Phi(IJ)} & & \downarrow{\scriptstyle \Phi(J)} & & \downarrow{\scriptstyle \Phi(I)} & \\
\cdots \longrightarrow & CH_q(I) & \longrightarrow & CH_q(IJ) & \longrightarrow & CH_q(J) & \longrightarrow & CH_{q-1}(I) & \longrightarrow \cdots
\end{array}$$

where the top row is (4.4) and the bottom row is (4.3).

We denote the collection of all connection matrices of the admissible ordering $M = \{M_i\}_{i=1}^n$ of the Morse decomposition of $S$ by $\mathcal{CM}(M)$.

The existence of connection matrices was shown by Franzosa in the case of continuous dynamical systems (see [Fra2]). The same conclusion can be drawn for discrete dynamical systems.

Theorem 4.6. *The set $\mathcal{CM}(M)$ is nonempty.*

Proof. First observe that if $X$ is a compact ANR then the homology functor and the Leray functor commute (see [Mr3]). Now the theorem is an easy consequence of Thm. 3.4 and [Fra2], Thm. 3.8. ∎

We can now state the analogue of Theorem 4.2.

Theorem 4.7. *If $\Delta \in \mathcal{CM}(M)$, $i$ and $j$ are adjacent and $\Delta(i,j) \neq 0$, then $C(M_j, M_i; S) \neq \emptyset$.*

Proof. It is sufficient to observe that the first condition of Definition 4.5 implies that if $i$ and $j$ are adjacent, then $\Delta(i,j) = \partial(i,j)$. ∎

REMARK 4.8. Using induction and the five-lemma, the second condition of Definition 4.5 implies that $H\Delta(I) \simeq CH_*(I)$ for any interval $I$.

EXAMPLE 4.9. This example is adapted from [Mr3]. Let $D \subset \mathbb{R}^2$ be a square and let $f_0 : D \to D$ be a continuous map as indicated in Fig. 1. Extend $f_0$ to a homeomorphism $f : S^2 \to S^2$ with a repelling point $r$ outside $D$.

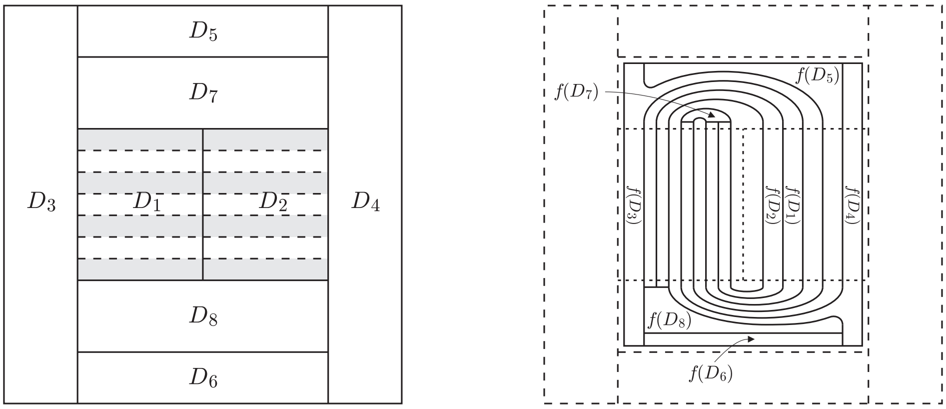


Fig. 1

Take $M_1 := \text{Inv}(D_7 \cup D_8)$, $M_2 := \text{Inv}(D_1 \cup D_2)$, $M_3 := \{r\}$. It is easy to check that $M = \{M_1, M_2, M_3\}$ is a Morse decomposition of $S = S^2$ with admissible ordering $(1 < 2 < 3)$ and $\mathcal{N} = \{N_i\}_{i=0}^3$ with $N_0 = \emptyset$, $N_1 = D_7 \cup D_8 \cup P$ ($P$ is the union of the shaded areas), $N_2 = D_1 \cup D_2 \cup D_7 \cup D_8$, $N_3 = S^2$ is an index filtration for $M$. Moreover, a simple verification shows

$$CH_k(M_1) = \begin{cases} Q^2 & \text{for } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$
$$CH_k(M_2) = \begin{cases} Q & \text{for } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$
$$CH_k(M_3) = \begin{cases} Q & \text{for } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let us compute the connection matrix of the above Morse decomposition. Because we have chosen field coefficients, the connection matrix is upper triangular. $CH_1(M_{21})$ is easily seen to be trivial; therefore so is $H_1\Delta(12)$. Since the homology indices $CH_1(M_2)$ and $CH_0(M_1)$ are nontrivial, it follows that $\Delta(1,2) \neq 0$. It is not difficult to see that it is the only nonzero entry of $\Delta$. Thus $\mathcal{CM}(M)$ consists of one matrix of the form

$$\begin{bmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $*$ indicates the only nonzero entry. Then since $M_2$ and $M_1$ are adjacent in the admissible ordering it follows that $C(M_2, M_1; S)$ is nonempty.

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