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Projective quartics revisited

by T. Szemberg and H. Tutaj-Gasińska (Kraków)

Abstract. We classify all smooth projective varieties of degree 4 and describe their syzygies.

0. Introduction. The aim of this note is to present a new proof of the classification of projective quartics obtained first by Swinnerton-Dyer in [8]. Our method uses adjunction and follows the pattern set by Ionescu [5], [6] for varieties of degree $5 \le d \le 8$. Whereas the method is not new it becomes particularly transparent in the case of the lower degree and therefore seems worth presenting. Additionally, we completely describe syzygies of projective quartics. We also correct a small mistake made in [8, p. 404].

We prove the following

Theorem. *Let $X \subset \mathbb{P}^N$ be a nondegenerate smooth projective variety of dimension n and degree* 4. *Then*

(a) *either X is linearly normal and it is one of the following*:
- *a hypersurface*,
- *a complete intersection of two quadrics*,
- *the Veronese surface in* $\mathbb{P}^5$,
- *one of the scrolls*: $S_4, S_{1,3}, S_{2,2}, S_{1,1,2}, S_{1,1,1,1}$;

(b) *or X is not linearly normal and it is either*
- *a smooth rational quartic curve in* $\mathbb{P}^3$, *or*
- *a projection of the Veronese surface into* $\mathbb{P}^4$.

In the above Theorem we follow Harris [3] and denote by $S_{i_1,\ldots,i_r}$ the scroll $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(i_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(i_r))$ polarized by the tautological bundle.

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Remark. The correspondence between our notation and that of [8] is the following: $V_1 = S_{1,1,1,1}$, $V_2 =$ the Veronese surface in $\mathbb{P}^5$, $V_3 =$ projection of V_2 into $\mathbb{P}^4$, $V_4 = S_{1,1,2}$, $V_5 = S_{2,2}$, $V_6 = S_{1,3}$.

We work throughout over the field $\mathbb{C}$ of complex numbers. A variety in a projective space is supposed to be nondegenerate unless otherwise stated.

1. Preliminaries. We begin by recalling the following

Definition 1.1. A smooth subvariety X in $\mathbb{P}^N$ is called *linearly normal* if the restriction mapping $H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(\mathcal{O}_X(1))$ is surjective, i.e. if X is embedded by a complete linear system. A variety is called *projectively normal* if the restriction mappings $H^0(\mathcal{O}_{\mathbb{P}^N}(d)) \to H^0(\mathcal{O}_X(d))$ are surjective for all $d \geq 1$.

Clearly, to be projectively normal is a property of the embedding. A line bundle L on X is *normally generated* if it is very ample and X is projectively normal under the embedding $\varphi_L : X \to \mathbb{P} = \mathbb{P}(H^0(L)^*)$ defined by L. Given such an embedding we have

$$H^0(\mathcal{O}_{\mathbb{P}}(d)) \cong \operatorname{Sym}^d H^0(\mathcal{O}_{\mathbb{P}}(1)) \cong \operatorname{Sym}^d H^0(L),$$

so we can reformulate the above definition by requiring that the canonical maps $\operatorname{Sym}^d H^0(L) \to H^0(dL)$ are surjective for $d \geq 2$, the very ampleness of L being automatic (see [7]). Thus a line bundle L is normally generated if the mappings $H^0((d-1)L) \otimes H^0(L) \to H^0(dL)$ are surjective for $d \geq 2$.

Definition 1.2. Let X be a smooth projective variety and let L be a very ample line bundle on X. The line bundle L is said to be *normally presented* if the ideal I_X of X under the embedding φ_L is generated by quadrics.

Now we recall two notions from the theory developed by Fujita.

Definition 1.3. Let (X, L) be a polarized nonsingular variety of dimension n.

(a) The number $\Delta(L) := L^n + n - h^0(L)$ is the *Δ-genus* of L.
(b) The number $g(L) := 1 + \frac{1}{2}(K_X + (n-1)L).L^{n-1}$ is the *sectional genus* of L.

It is well known that these two quantities vanish simultaneously, i.e.

Lemma 1.4 (Fujita, [2], Corollary 1). *$\Delta(L) = 0$ if and only if $g(L) = 0$.*

The following property of adjoint linear systems will be useful.

Proposition 1.5 (Fujita, [2], Theorem 1). *Let (X, L) be a polarized smooth variety of dimension n. The line bundle $K_X + (n+1)L$ is nef, and $K_X + nL$ is nef unless $(X, L) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*

In the sequel we need the following elementary inequality for a nondegenerate subvariety X in a projective space:

$$\operatorname{codim} X + 1 \leq \deg X. \tag{1}$$

We finish this section by the following observation:

LEMMA 1.6. *Let X be a smooth projective surface such that $c_1^2(X) > 0$. Then $3c_2(X) \geq c_1^2(X)$.*

Proof. First we observe that it is enough to prove the lemma for minimal surfaces. Indeed, under blowing up c_1^2 drops by one and c_2 increases by one so if the inequality holds for the minimal model of X it also holds for X. Then the assertion follows from the Miyaoka–Yau inequality for surfaces of general type [1, Theorem VII.4.1] and straightforward checking for other surfaces (cf. [1, p. 188, Table 10]). ■

2. Quartics of codimension 3. Let $X \subset \mathbb{P}^{n+3}$ be a smooth subvariety of dimension n and degree 4 and let $L = \mathcal{O}_X(1)$. From inequality (1) we see that X is linearly normal. It follows that $\Delta(L) = 0$ and consequently $g(L)=0$ by Lemma 1.4. The main idea of this section is to study the adjoint linear system $K_X + nL$.

LEMMA 2.1. *The dimension of $h^0(K_X + nL)$ equals 3.*

Proof. If $n = 1$ then $X \cong \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(4)$ so the assertion is clear. For $n \geq 2$ let $D \in |L|$ be a smooth irreducible and reduced divisor. From the adjunction formula we have

$$(K_X + nL)|_D \cong K_D + (n-1)L|_D.$$

Of course, $L|_D$ is again very ample, $L|_D^{(n-1)} = 4$ and from (1) we obtain $\Delta(L|_D) = 0$. The exact sequence

$$0 \to K_X + (n-1)L \to K_X + nL \to (K_X + nL)|_D \to 0$$

together with Kodaira vanishing implies that

$$h^0(K_X + nL) = h^0(K_D + (n-1)L|_D) + h^0(K_X + (n-1)L).$$

The second summand vanishes since $(K_X + (n-1)L)L^{n-1} < 0$ and L is ample. Thus it is enough to show the assertion for $(D, L|_D)$. Proceeding by induction we are done. ■

LEMMA 2.2. *The linear system $K_X + nL$ is base point free.*

Proof. If $n = 1$ then $X \cong \mathbb{P}^1$ and $K_X + L = \mathcal{O}_{\mathbb{P}^1}(2)$.

Now suppose that $n \geq 2$ and let $x \in X$ be a fixed point. Since L is very ample there is an irreducible reduced smooth divisor $D \in |L|$ passing through x. Exactly as in the proof of the previous lemma we get

$$H^0(K_X + nL) \cong H^0(K_D + (n-1)L|_D),$$

so it is enough to show that $K_D+(n-1)L|_D$ is base point free. By induction we again reduce the situation to the curve case. ∎

PROPOSITION 2.3. *Let $X \subset \mathbb{P}^{n+3}$ be a smooth quartic of dimension n. Then X is either the Veronese surface or a rational normal scroll.*

Proof. Let φ be the morphism defined by the linear system $K_X + nL$. Its image $\varphi(X)$ is either $\mathbb{P}^2$ or a curve in $\mathbb{P}^2$. Since $g(L) = 0$ this curve must be a smooth conic. We now study these two cases.

CASE 1. Suppose that $\varphi(X) = \mathbb{P}^2$. Then $n \ge 2$. If $n = 2$ then φ is a generically finite morphism and we have

$$0 < (K_X + 2L)^2 = K_X^2 + 4(K_X + L)L = K_X^2 - 8.$$

Applying Riemann–Roch to the line bundle $-L$ we get $\chi(\mathcal{O}_X) = 1$. Since $K_X^2 > 8$ it follows from Lemma 1.6 that $c_2(X) \ge 3$, which in turn implies $K_X^2 = 9$ by the Noether formula. Hence $(K_X+2L)^2 = 1$ and φ is a birational morphism. If X were not isomorphic to $\mathbb{P}^2$ there would be a curve $C \subset X$ contracted by φ. Since $g(L) = 0$ we have $(K_X + L)L = -2$, which implies $K_X.L = -6$. Thus in the Hodge inequality

$$((K_X + 2L).L)^2 \ge (K_X + 2L)^2 L^2$$

we have equality, which implies that $K_X + 2L$ and L are algebraically proportional. This shows that $K_X + 2L$ is ample, hence it cannot contract any curves. Thus $X \cong \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(2)$.

We want to show that this case is not possible for $n \ge 3$. Indeed, if $n \ge 3$ then $(K_X + nL)^n = 0$ but $(K_X + nL)^2$ is effective—the class of a fiber of φ. This implies

$$(K_X + nL)^2 L^{n-2} > 0. \tag{2}$$

Let $D \in |L|$ be a smooth irreducible and reduced divisor. Since

$$(K_X + nL)^2 L^{n-2} = (K_D + (n-1)L|_D)^2 (L|_D)^{n-3}$$

and $L|_D$ is very ample with $g(L|_D) = \Delta(L|_D) = 0$ we can restrict our considerations to the case $n = 3$. In this situation a smooth divisor $D \in |L|$ must be isomorphic to $\mathbb{P}^2$ by the same argument as in the case $n = 2$. Thus $L|_D \cong \mathcal{O}_D(2)$. From the formula for the sectional genus of L we get $(K_X + 2L)L^2 = -2$, hence $K_X L^2 = -10$. Since $(K_X + 3L)^3 = 0$ we obtain $K_X^3 + 9K_X^2 L = 162$. Now, from (2) we get $K_X^2 L > 24$.

Now consider the line bundle K_X+4L. It is nef according to Proposition 1.5 and it is also big since

$$\begin{aligned}(K_X + 4L)^3 &= K_X^3 + 9K_X^2 L + 3K_X^2 L + 48K_X L^2 + 64L^3\\ &> 162 + 3 \cdot 24 - 480 + 256 > 0.\end{aligned}$$

In the exact sequence

$$0 \to 2K_X + 4L \to 2K_X + 5L \to (2K_X + 5L)|_D \to 0$$

we have $h^1(2K_X+4L) = h^1(K_X+(K_X+4L)) = 0$ by Kodaira vanishing and $2K_X+5L|_D \cong \mathcal{O}_D$. Hence either there is an effective divisor $F \in |2K_X+5L|$ or $2K_X+5L$ is trivial. The latter possibility is easily excluded by computing the selfintersection. Thus we have

$$0 < L^2.F = L^2.(2K_X + 5L) = 0,$$

a contradiction.

CASE 2. Suppose that $\varphi(X) \cong \mathbb{P}^1$. Let F be a fiber of φ. Then we have $\mathcal{O}_F(F) = \mathcal{O}_F$ and by the adjunction formula $K_F \cong K_X|_F$. Moreover, $(K_X + nL)|_F \cong \mathcal{O}_F$ implies that $K_F \cong -nL|_F$ with $L|F$ ample. Applying Proposition 1.5 we obtain $F \cong \mathbb{P}^{n-1}$. This means that X is a $\mathbb{P}^{n-1}$-bundle over $\mathbb{P}^1$. Since every such bundle arises as a projectivization of a vector bundle and every vector bundle on $\mathbb{P}^1$ decomposes, we get $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$. Since L is very ample it follows that $a_i > 0$ for $i = 1, \ldots, n$ and $L^n = 4 = a_1 + \ldots + a_n$. Thus we are left with a few possibilities of scrolls: $S_4, S_{2,2}, S_{1,3}, S_{1,1,2}$ and $S_{1,1,1,1}$. ■

3. Quartics of codimension 2. Let $X \subset \mathbb{P}^{n+2}$ be a smooth quartic of dimension n and let $L = \mathcal{O}_X(1)$. First we suppose that X is linearly normal.

PROPOSITION 3.1. *Let $X \subset \mathbb{P}^{n+2}$ be a smooth linearly normal quartic of dimension n. Then X is a complete intersection of two quadrics.*

Proof. We observe that $\Delta(L) = 1$ and by Lemma 1.4, $g(L) \geq 1$.

If X is a curve then the assertion follows either by a straightforward computation or from the classification of curves in $\mathbb{P}^3$ as in [4, Ex. IV.6.4.2]. Now we suppose that $n \geq 2$. Let H be a hyperplane in $\mathbb{P}^{n+2}$ such that $Y = X \cap H$ is smooth and irreducible. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_X(1) & \to & \mathcal{O}_{\mathbb{P}^{n+2}}(1) & \to & L & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_X(2) & \to & \mathcal{O}_{\mathbb{P}^{n+2}}(2) & \to & \mathcal{O}_X(2) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{I}_Y(2) & \to & \mathcal{O}_H(2) & \to & \mathcal{O}_Y(2) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Computing the cohomology, in the upper row we get an isomorphism $H^0(\mathcal{O}_{\mathbb{P}^{n+2}}(1)) \cong H^0(L)$ and by the snake lemma an isomorphism

$$H^0(\mathcal{I}_X(2)) \cong H^0(\mathcal{I}_Y(2)).$$

Since $g(L) = g(L|_Y)$ we can reduce the whole situation to the curve case. ∎

If X is not linearly normal then we have the following

PROPOSITION 3.2. *Let $X \subset \mathbb{P}^{n+2}$ be a smooth nonlinearly normal quartic of dimension n. Then X is either a rational quartic curve in $\mathbb{P}^3$ or the Veronese surface in $\mathbb{P}^4$.*

Proof. From (1) it follows that X is a projection of a smooth quartic Z in $\mathbb{P}^{n+3}$. We listed all these quartics in the previous section. As rational varieties they may be easily parametrized and explicit calculations show that only S_4 and the Veronese surface admit a smooth projection. We omit these simple but lengthy computations here. ∎

4. The free resolution of quartics. In this section we compute the free resolution of all smooth quartics. For a hyperplane in $\mathbb{P}^N$ we have $\mathcal{I}_X \cong \mathcal{O}_{\mathbb{P}^N}(-4)$ and for a complete intersection of two quadrics we have

$$0 \leftarrow \mathcal{I}_X \leftarrow \mathcal{O}_{\mathbb{P}^N}(-2)^2 \leftarrow \mathcal{O}_{\mathbb{P}^N}(-4) \leftarrow 0.$$

The vector bundle maps

$$\begin{array}{ccccccc} & & & & \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) & & \\ & & & \nearrow & & \searrow & \\ \mathcal{O}_{\mathbb{P}^1}(1)^4 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(1)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(2) & & & & \mathcal{O}_{\mathbb{P}^1}(4) \\ & & & \searrow & & \nearrow & \\ & & & & \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3) & & \end{array}$$

induce inclusions of scrolls:

$$\begin{array}{ccccccc} & & & & S_{2,2} & & \\ & & & \swarrow & & \nwarrow & \\ S_{1,1,1,1} & \longleftarrow & S_{1,1,2} & & & & S_4 \\ & & & \nwarrow & & \swarrow & \\ & & & & S_{1,3} & & \end{array}$$

as hyperplane sections. They all have the same free resolution:

$$0 \leftarrow \mathcal{I}_X \leftarrow \mathcal{O}(-2)^6 \leftarrow \mathcal{O}(-3)^8 \leftarrow \mathcal{O}(-4)^3 \leftarrow 0. \tag{3}$$

We also get the same free resolution for the Veronese surface.

For nonlinearly normal quartics we get

$$0 \leftarrow \mathcal{I}_X \leftarrow \mathcal{O}(-3)^7 \leftarrow \mathcal{O}(-4)^{10} \leftarrow \mathcal{O}(-5)^5 \leftarrow \mathcal{O}(-6) \leftarrow 0$$

for the Veronese surface in $\mathbb{P}^4$ and

$$0 \leftarrow \mathcal{I}_X \leftarrow \mathcal{O}(-2) \oplus \mathcal{O}(-3)^3 \leftarrow \mathcal{O}(-4)^4 \leftarrow \mathcal{O}(-5) \leftarrow 0$$

for the rational quartic curve in $\mathbb{P}^3$.

COROLLARY 4.1. *Let X be a smooth linearly normal quartic. Then X is projectively normal. Moreover, X is normally presented unless it is a hypersurface.*

From the resolution (3) it clearly follows that the argument provided by Swinnerton-Dyer to distinguish between the scrolls $S_{1,3}$ and $S_{2,2}$ is incorrect: both surfaces are contained in ∞^5 quadrics. Although it can be deduced from the general statements on Hirzebruch surfaces that these scrolls are not isomorphic we give here a simple direct argument.

PROPOSITION 4.2. *The scrolls $S_{1,3}$ and $S_{2,2}$ are not isomorphic.*

Proof. Suppose that there is an isomorphism $\varphi : S_{1,3} \to S_{2,2}$. First we note that $S_{2,2}$ is a join variety of two smooth conics sitting in disjoint planes in $\mathbb{P}^5$. There is an isomorphism

$$\mathbb{P}^1 \times \mathbb{P}^1 \ni ((s:t),(x:y)) \to (sx^2 : sxy : sy^2 : tx^2 : txy : ty^2) \in S_{2,2} \subset \mathbb{P}^5.$$

Let F_1, F_2 be generators of $H^2(S_{2,2},\mathbb{Z})$ such that $F_1^2 = F_2^2 = 0$ and $F_1.F_2 = 1$.

The scroll $S_{1,3}$ is in turn a join variety of a line $D_1 \subset \mathbb{P}^5$ and a rational normal cubic D_2 sitting in the complementary $\mathbb{P}^3 \subset \mathbb{P}^5$. Let F denote the class of the ruling on $S_{1,3}$. Since $F^2 = 0$ and it is effective we can assume that $F = \varphi^*(F_1)$. We denote the second generator by $G = \varphi^*(F_2)$. Then $D_i = \alpha_i F + \beta_i G$ for $i = 1, 2$. Since $D_i.F = 1$ we have $\beta_i = 1$. Since D_1 and D_2 are disjoint we have

$$0 = D_1.D_2 = \alpha_1 + \alpha_2.$$

This combined with $D_i^2 = 2\alpha_i$ implies $\alpha_1 = \alpha_2 = 0$, since there are no curves on $\mathbb{P}^1 \times \mathbb{P}^1$ with negative selfintersection. Thus D_1 and D_2 are homologous, which is absurd because their intersection number with a hyperplane in $\mathbb{P}^5$ is 1 and 3 respectively. ■

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Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: szemberg@im.uj.edu.pl

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