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Hypersurfaces with parallel affine curvature tensor R^*

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Abstract. In [OV] we introduced an affine curvature tensor R^*. Using it we characterized some types of hypersurfaces in the affine space $\mathbb{R}^{n+1}$. In this paper we study hypersurfaces for which R^* is parallel relative to the induced connection.

1. Let M be an n-dimensional connected manifold and $f : M \to \mathbb{R}^{n+1}$ its immersion into the standard affine space $\mathbb{R}^{n+1}$. Denote by D the standard connection in $\mathbb{R}^{n+1}$. If ξ is an equiaffine transversal vector field for f, that is, $D\xi$ is tangential to f, then the formulas of Gauss and Weingarten can be written as follows:

$$D_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi, \tag{1.1}$$

$$D_X \xi = -f_* SX, \tag{1.2}$$

where X, Y are tangent vector fields on M, ∇ is the induced connection on M, h the second fundamental form and S the shape operator. It is known that the rank of h is independent of the choice of a transversal vector field. If the rank is equal to n everywhere on M, then f is called *nondegenerate*. For a nondegenerate hypersurface there exists a unique (up to a constant) equiaffine transversal vector field such that

$$\mathrm{tr}_h(\nabla_X h)(\cdot, \cdot) = 0 \tag{1.3}$$

for every $X \in TM$. This transversal vector field is called the *affine normal*.

Throughout the paper we shall study nondegenerate hypersurfaces endowed with equiaffine transversal vector fields. The relationship between ∇, h and S is given by the fundamental equations

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY \qquad \text{(Gauss)}, \tag{1.4}$$

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$$\nabla h(X,Y,Z) = \nabla h(Y,X,Z) \qquad \text{(Codazzi I)}, \tag{1.5}$$

$$\nabla S(X,Y) = \nabla S(Y,X) \qquad \text{(Codazzi II)}, \tag{1.6}$$

$$h(SX,Y) = h(X,SY) \qquad \text{(Ricci)}, \tag{1.7}$$

where R is the curvature tensor of ∇. The tensor field R^* given by $R^*(X,Y)Z = R(X,Y)SZ$ is a curvature tensor relative to h, that is, $h(R^*(X,Y)Z,W)$ is skew-symmetric for Z and W. Note that R is not, in general, a curvature tensor relative to h.

One can study various conditions imposed on R^*. For instance, in [OV] we proved that $R^*_x = 0$ if and only if $\operatorname{rk} S_x \le 1$. If R^* constantly vanishes on M, then $M = M_1 \cup M_2$, where $M_1 = \{x \in M : S_x = 0\}$ and $M_2 = \{x \in M : \operatorname{rk} S_x = 1\}$. A transversal vector field ξ is a curve in M_2, that is, around each point of M_2 there is a coordinate system $(u_1,\ldots,u_n)$ such that ξ depends only on one variable. Surfaces in $\mathbb{R}^3$ with affine normals which are curves are described in [O] and [OS]. For instance, such surfaces with nondiagonalizable shape operator are characterized as follows:

THEOREM 1. *Let $f : M \to \mathbb{R}^3$ be a nondegenerate surface with affine normal ξ. The following conditions are equivalent*:

(a) *ξ is a curve and the affine shape operator S is nondiagonalizable.*
(b) *f is a minimal ruled surface.*
(c) *f is a ruled surface with planar generators.*

Surfaces with diagonalizable shape operator are characterized by differential equations. For instance, in the case of surfaces with parallel image of the shape operator we have

THEOREM 2. *Let $f : M \to \mathbb{R}^3$ be a nondegenerate surface equipped with the affine normal ξ inducing ∇, h and S. If S is diagonalizable*, $\operatorname{im} S$ *is* 1-*dimensional and parallel relative to ∇, then for every $x \in M$ there is a coordinate system (u,v) around x and functions $\phi(u,v)$, $a(u)$ such that ϕ is positive valued*, *ϕ and a satisfy the equation*

$$\varepsilon_1\phi_{uu} + \varepsilon_1\frac{a'}{2}\phi_u + \varepsilon_2 e^{-a}\phi_{vv} = -\phi$$

and

$$\begin{aligned}
\nabla_{\partial_u}\partial_u &= \left((\log\phi)_u - \frac{a'}{2}\right)\partial_u, & h(\partial_u,\partial_u) &= \varepsilon_1\phi e^{-a},\\
\nabla_{\partial_u}\partial_v &= (\log\phi)_v\partial_u, & h(\partial_u,\partial_v) &= 0,\\
\nabla_{\partial_v}\partial_v &= -\varepsilon(\log\phi)_u e^a\partial_u, & h(\partial_v,\partial_v) &= \varepsilon_2\phi,
\end{aligned}$$
$$S\partial_u = e^a\phi^{-1}\partial_u, \qquad S\partial_v = 0,$$

where $\varepsilon_1,\varepsilon_2 = \pm1$ and $\varepsilon = \varepsilon_1\varepsilon_2$. The immersion f is equal modulo the special affine group $SA(3,\mathbb{R})$ to

$$f(u,v) = p(v) + q(u,v)$$

where $p(v)$ and $q(u,v)$ are obtained in the following way. Let $U = I \times J$ be a domain of a coordinate system (u,v) and $\xi(u) : I \to \mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ be a centroaffine curve satisfying the equation

$$\xi'' = -\varepsilon_1 \xi + \frac{a'}{2}\xi'.$$

Let $q(u,v)$ and $p(v)$ be arbitrary functions satisfying the equations

$$q_u = -e^{-a}\phi\xi', \qquad p'' = -q_{vv} + \varepsilon\phi_u\xi' + \varepsilon_2\phi\xi$$

and the condition $p'(v) \notin \mathbb{R}^2$ for every $v \in J$. The vector field $\xi(u,v) = \xi(u)$ is the affine normal for $f(u,v)$ up to a constant.

The case where $\operatorname{im} S$ is not ∇-parallel is more complicated and we refer to [OS] for information about it.

In [OV] we also introduced the Ricci and Weyl tensors determined by R^*. We proved that for a quasi-umbilical hypersurface the Weyl tensor vanishes. The converse (for manifolfds of dimension greater than 3) is proved in $[\mathrm{D}]_1$. In $[\mathrm{D}]_1$ and $[\mathrm{D}]_2$ other conditions on R^* are studied.

2. In this paper we prove the following result.

THEOREM 3. *Let $f : M \to \mathbb{R}^{n+1}$ be a nondegenerate hypersurface equipped with an equiaffine transversal vector field ξ inducing a connection ∇. If $\nabla R^* = 0$, then R^* constantly vanishes on M or f is a nondegenerate central quadric in $\mathbb{R}^{n+1}$ and ξ is its affine normal* (*up to a constant*).

Proof. If there exists a point $x \in M$ such that $R^*_x = 0$, then $R^* = 0$ on the whole M because R^* is parallel relative to a connection. From now on we assume that $R^* \neq 0$ everywhere on M. In this case $\operatorname{rk} S_x > 1$ for every $x \in M$.

We first consider the condition $R \cdot R^* = 0$. We have

$$\begin{aligned}
(2.1)\qquad (R(X,Y)\cdot R^*)(Z,V)W
&= (h(V,SW)h(Y,SZ) - h(Z,SW)h(Y,SV))SX \\
&\quad + (h(Z,SW)h(X,SV) - h(V,SW)h(X,SZ))SY \\
&\quad + (h(Y,V)h(Z,SW) - h(Y,Z)h(V,SW))S^2X \\
&\quad + (h(X,Z)h(V,SW) - h(X,V)h(Z,SW))S^2Y \\
&\quad + (h(X,V)h(SY,SW) - h(Y,V)h(SX,SW) \\
&\qquad + h(X,W)h(V,S^2Y) - h(Y,W)h(V,S^2X))SZ \\
&\quad + (h(Y,Z)h(SX,SW) - h(X,Z)h(SY,SW) \\
&\qquad + h(Y,W)h(Z,S^2X) - h(X,W)h(Z,S^2Y))SV.
\end{aligned}$$

We consider a few cases. In all the cases x denotes any point of M.

Case I. Assume first that S_x is diagonalizable. By Lemma 1 of [VV] we know that S_x and h_x are simultaneously diagonalizable. Let $e_1, \ldots, e_n$ be an h-orthonormal basis of T_xM consisting of eigenvectors of S_x. Let $\varrho_1, \ldots, \varrho_n$ be the eigenvalues of S_x corresponding to $e_1, \ldots, e_n$ respectively. If $\dim M = 2$, then, by (2.1), we get

$$0 = (R(e_1, e_2) \cdot R^*)(e_1, e_2)e_1 = \varrho_1\varrho_2(\varrho_2 - \varrho_1)h(e_1, e_1)h(e_2, e_2)e_1$$

and consequently $\varrho_1 = \varrho_2$. If $\dim M > 2$ and i, j, k are mutually distinct, then $(R(e_j, e_i) \cdot R^*)(e_k, e_i)e_k = 0$ yields

$$\varrho_k\varrho_j(\varrho_j - \varrho_i) = 0. \tag{2.2}$$

Since $\operatorname{rk} S_x > 1$, we can assume that ϱ_1 and ϱ_2 are not zero. If we put $\varrho_k = \varrho_1$ and $\varrho_j = \varrho_2$ in (2.2), then we get $\varrho_2 = \varrho_i$ for every $i \geq 3$. By taking $k = 2$, $j = 1$ and $i \geq 3$ we obtain $\varrho_1 = \varrho_i$. Consequently, $\varrho_1, \ldots, \varrho_n$ are all equal, that is, $S_x = \varrho I_x$ for some nonzero ϱ where I_x is the identity endomorphism of T_xM.

In the next cases we assume that S_x is not diagonalizable. In particular, h_x is not definite.

Case II. Assume that $\dim M = 2$. Let X, Y be an h-orthonormal basis of T_xM, i.e. $h(X, Y) = 0$, $h(X, X) = 1 = -h(Y, Y)$. By (2.1) we get

$$\begin{aligned}
0 &= (R(X, Y) \cdot R^*)(X, Y)X \\
&= (h(Y, SX)^2 - h(X, SX)h(Y, SY) + h(SX, SX) + h(SY, SY))SX \\
&\quad - 2h(SX, SY)SY - h(X, SX)S^2X + h(Y, SX)S^2Y.
\end{aligned}$$

Since X, Y is an h-orthonormal basis, we obtain

$$\begin{aligned}
0 &= h((R(X, Y) \cdot R^*)(X, Y)X, Y) \\
&= h(Y, SX)^2 h(SX, Y) - h(X, SX)h(Y, SY)h(SX, Y) \\
&\quad + h(SX, X)^2 h(SX, Y) - h(SX, Y)^2 h(SX, Y) \\
&\quad + h(SY, X)^2 h(SX, Y) - h(SY, Y)^2 h(SX, Y) \\
&\quad - 2h(SX, X)h(SY, X)h(SY, Y) + 2h(SX, Y)h(SY, Y)^2 \\
&\quad - h(X, SX)^2 h(SY, X) + h(X, SX)h(SX, Y)h(SY, Y) \\
&\quad + h((Y, SX)h(SY, X)^2 - h(Y, SX)h(SY, Y)^2 \\
&= 2h(X, SY)(h(X, SY)^2 - h(SX, X)h(SY, Y)).
\end{aligned}$$

Since S_x is not diagonalizable, we have $h(SX, Y) \neq 0$. Thus

$$h(X, SY)^2 - h(SX, X)h(Y, SY) = 0.$$

This means that $\det S_x = 0$, which contradicts the assumption $\operatorname{rk} S_x > 1$.

In cases III and IV the dimension of M is assumed to be greater than 2.

Case III. Assume that $\operatorname{rk} S_x < n$. We first show that $S_x^2 = 0$ on T_xM. By (2.1) applied to any $X, Y, Z, V \in T_xM$ and $0 \neq W \in \ker S_x$ we get

$$(2.3)\quad (h(X,W)h(S^2V,Y) - h(Y,W)h(V,S^2X))SZ + (h(Y,W)h(Z,S^2X) - h(X,W)h(Z,S^2Y))SV = 0,$$

Let $V \in T_xM \setminus \ker S_x$. Since $\operatorname{rk} S_x > 1$, there is Z such that SV, SZ are linearly independent. Then, by (2.3), we have

$$h(X,W)h(S^2V,Y) = h(Y,W)h(V,S^2X)$$

for every X, Y. It follows that

$$(2.4)\qquad h(S^2V,Y)W = h(Y,W)S^2V$$

for every $Y \in T_xM$. Hence

$$(2.5)\qquad h(S^2V,Y) = 0$$

for every $Y \in \langle W\rangle^\perp$, where $\langle W\rangle^\perp$ is the subspace h-orthogonal to W. If there is a vector $W \in \ker S_x$ such that $h(W,W) \neq 0$, then the space $\langle W\rangle^\perp$ is an algebraic complement to $\operatorname{span}\{W\}$ in T_xM. Therefore, by (2.5) and the obvious fact $h(S^2V,W) = h(SV,SW) = 0$, we get $S^2 = 0$ on T_xM. Assume now that $h(W,W) = 0$ for every $W \in \ker S_x$. To get a contradiction we also assume that S^2 is not identically zero on T_xM. If there exist $W \in \ker S$ and V such that S^2V is not parallel to W, then (by (2.4)) $h(Y,W) = 0$ for every $Y \in T_xM$, that is, $W = 0$, which is a contradiction. Hence for every $V \notin \ker S$ and every $W \in \ker S$ the vector S^2V is parallel to W. It follows that

$$(2.6)\qquad \dim\ker S_x = 1$$

and

$$(2.7)\qquad \operatorname{im} S_x^2 = \ker S_x.$$

Assume that $n > 3$. Let $\mathcal{L}$ be an algebraic complement to $\ker S_x$ in T_xM. Then $\dim\mathcal{L} = n-1$ and $S_{|\mathcal{L}}$ is an injection. Since $\operatorname{rk} S_x = n-1$ and $n > 3$, we have $\dim(\mathcal{L}\cap\operatorname{im} S_x) \geq 2$. Since S_x restricted to $\mathcal{L}\cap\operatorname{im} S_x$ is an injection, we have $\operatorname{rk} S_x^2 \geq 2$, which contradicts (2.6) and (2.7).

Assume $n = 3$. Then $\operatorname{rk} S_x = 2$. We now take $0 \neq X \in \ker S_x$. By (2.7) there is Y such that $X = S^2Y$. We set $Z = SY$ and choose V such that $X = SZ$ and SV are linearly independent. By (2.1) we get

$$(2.8)\quad (h(X,Z)h(V,SW) + h(X,V)h(X,W))X - 2h(X,Z)h(X,W)SV = 0$$

for every $W \in T_xM$. It follows that $h(X,Z) = 0$, i.e. $Z \in \langle X\rangle^\perp$. Hence, by (2.8), $h(X,V)h(X,W) = 0$ for every W, that is, $V \in \langle X\rangle^\perp$. Since Z, V are linearly independent, they span $\langle X\rangle^\perp$. But $X \in \langle X\rangle^\perp$ and consequently $X = x_1Z + x_2V$ for some numbers x_1, x_2. Therefore $0 = SX = x_1SZ + x_2SV$, i.e. SZ, SV are linearly dependent, which is a contradiction. Consequently, $S^2 = 0$ on T_xM.

Let Y, X be such that SY, SX are linearly independent. By (2.1) we have

$$h(V, SW)h(Y, SZ) - h(Z, SW)h(Y, SV) = 0 \tag{2.9}$$

for every $V, Z, W \in T_xM$. Since $\dim\langle Y\rangle^{\perp} = n-1$ and $\operatorname{rk} S_x \geq 2$, we have $\langle Y\rangle^{\perp} \cap \operatorname{im} S_x \neq \{0\}$. It follows that there is Z such that $SZ \neq 0$ and $h(Y, SZ) = 0$. For such a Z, by (2.9), we get $h(SZ, W)h(Y, SV) = 0$ for every W and V, i.e. $h(SY, V) = 0$ for every V, contrary to $SY \neq 0$.

Case IV. Assume that $\operatorname{rk} S_x = n$. Since $n \geq 3$ and S_x is an isomorphism, there are vectors $e_1, e_2, \widetilde{e}_3 \in T_xM$ such that $e_1, e_2, S\widetilde{e}_3$ are h-orthonormal. We put $X = e_2$, $Y = \widetilde{e}_3$, $Z = e_1$, $V = e_2$, $W = \widetilde{e}_3$. By (2.1) we get

$$h(e_2, e_2)h(S\widetilde{e}_3, S\widetilde{e}_3) = h(\widetilde{e}_3, \widetilde{e}_3)h(Se_2, Se_2) \tag{2.10}$$

and

$$h(\widetilde{e}_3, e_1)h(Se_2, S\widetilde{e}_3) + h(\widetilde{e}_3, \widetilde{e}_3)h(e_1, S^2e_2) - h(\widetilde{e}_3, e_2)h(e_1, S^2\widetilde{e}_3) = 0. \tag{2.11}$$

Since $h(e_2, e_2) \neq 0$ and $h(S\widetilde{e}_3, S\widetilde{e}_3) \neq 0$, we obtain $h(\widetilde{e}_3, \widetilde{e}_3) \neq 0$. By (2.10) and the fact that $h(e_2, e_2) = \pm 1$ we get

$$h(e_2, e_2)h(Se_2, Se_2) = \frac{h(S\widetilde{e}_3, S\widetilde{e}_3)}{h(\widetilde{e}_3, \widetilde{e}_3)}. \tag{2.12}$$

If we take $X = e_1$, $Y = \widetilde{e}_3$, $Z = e_1$, $V = e_2$, $W = e_3$, then (2.1) yields

$$-h(\widetilde{e}_3, e_2)h(Se_1, S\widetilde{e}_3) + h(e_1, \widetilde{e}_3)h(e_2, S^2\widetilde{e}_3) - h(\widetilde{e}_3, \widetilde{e}_3)h(e_2, S^2e_1) = 0 \tag{2.13}$$

and

$$h(e_1, e_1)h(Se_1, Se_1) = \frac{h(S\widetilde{e}_3, S\widetilde{e}_3)}{h(\widetilde{e}_3, \widetilde{e}_3)}. \tag{2.14}$$

Formulas (2.12) and (2.14) imply

$$h(e_1, e_1)h(Se_1, Se_1) = h(e_2, e_2)h(Se_2, Se_2).$$

Since e_1, e_2 can be an arbitrary h-orthonormal pair, for any h-orthonormal basis $e_1, \ldots, e_n$ of T_xM we have

$$h(e_i, e_i)h(S^2e_i, e_i) = h(e_j, e_j)h(S^2e_j, e_j) \tag{2.15}$$

for every $i, j = 1, \ldots, n$. By comparing (2.11) and (2.13) we obtain

$$h(\widetilde{e}_3, \widetilde{e}_3)h(S^2e_1, e_2) = 0,$$

that is,

$$h(S^2e_1, e_2) = 0. \tag{2.16}$$

Therefore, if $e_1, \ldots, e_n$ is an h-orthonormal basis of T_xM, then

$$h(S^2 e_i, e_j) = 0 \tag{2.17}$$

for any $i \neq j$, $i, j = 1, \ldots, n$. Formulas (2.15), (2.17) imply that $S_x^2 = \alpha I_x$. Of course $\alpha \neq 0$. Suppose $\alpha < 0$. Then $S/\sqrt{-\alpha}$ is a complex structure on T_xM. In particular, n is even. Hence $n \geq 4$. Let $e_1, \ldots, e_{n-1}, S\widetilde{e}_n$ be an h-orthonormal basis of T_xM and let $X = e_1$, $Y = e_2$, $V = \widetilde{e}_n$, $W = e_3$ and Z be such that $h(Se_3, Z) \neq 0$. Then

$$\begin{aligned} h(S^2Y, W) &= h(S^2X, W) = h(X, W) = h(Y, W) \\ &= h(SV, X) = h(SV, Y) = h(SV, W) = 0. \end{aligned}$$

Consequently, by (2.1), $h(e_1, \widetilde{e}_n)h(Se_3, Z) = 0$, i.e. $h(e_1, \widetilde{e}_n) = 0$. Since the order of $e_1, \ldots, e_{n-1}$ is not important, we have

$$h(\widetilde{e}_n, e_i) = 0 \tag{2.18}$$

for every $i = 1, \ldots, n-1$, i.e. $S\widetilde{e}_n$ is parallel to $\widetilde{e}_n$. Let $S\widetilde{e}_n = \beta\widetilde{e}_n$. Then $S^2\widetilde{e}_n = \beta^2\widetilde{e}_n$. But $S^2\widetilde{e}_n = \alpha\widetilde{e}_n$ and $\alpha < 0$, that is, we have a contradiction. Hence $\alpha > 0$ and there are two complementary subspaces T_1 and T_2 of T_xM such that $S_{x|T_1} = \sqrt{\alpha}\, I_x$ and $S_{x|T_2} = -\sqrt{\alpha}\, I_x$. In particular S_x is diagonalizable, which is again a contradiction.

Summing up, for each $x \in M$ we have $S_x = \varrho I_x$. By the Codazzi equation ϱ is constant on M. Since $R^* \neq 0$, ϱ is not zero. Hence

$$R^*(X, Y)Z = \varrho^2(h(Y, Z)X - h(X, Z)Y)$$

and consequently

$$(\nabla_V R^*)(X, Y)Z = \varrho^2(\nabla h(V, Y, Z)X - \nabla h(V, X, Z)Y).$$

Therefore $\nabla R^* = 0$ implies that $\nabla h = 0$ and, by a theorem of Berwald, $f : M \to \mathbb{R}^{n+1}$ is a quadratic hypersurface. The proof is complete.

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