

**Existence of solutions and monotone iterative method
for infinite systems of
parabolic differential-functional equations**

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Abstract. We consider the Fourier first boundary value problem for an infinite system of weakly coupled nonlinear differential-functional equations. To prove the existence and uniqueness of solution, we apply a monotone iterative method using J. Szarski's results on differential-functional inequalities and a comparison theorem for infinite systems.

1. Introduction. We consider an infinite system of weakly coupled differential-functional equations of the form

$$(1) \quad \mathcal{F}^i[z^i](t, x) = f^i(t, x, z(t, \cdot)), \quad i \in S,$$

where

$$\mathcal{F}^i := \frac{\partial}{\partial t} - A^i, \quad A^i := \sum_{j,k=1}^m a_{jk}^i(t, x) \frac{\partial^2}{\partial x_j \partial x_k},$$

$x = (x_1, \dots, x_m)$, $(t, x) \in (0, T) \times G := D$, $T < \infty$, $G \subset \mathbb{R}^m$ and G is an open bounded domain with $C^{2+\alpha}$ ($0 < \alpha \leq 1$) boundary.

Let $B(S)$ be the Banach space of mappings

$$v : S \ni i \rightarrow v^i \in \mathbb{R},$$

with the finite norm

$$\|v\|_{B(S)} := \sup\{|v^i| : i \in S\},$$

where S is a denumerable set of indices (finite or infinite). The case of finite systems ($B(S) = \mathbb{R}^r$) was treated in [3, 4]. For infinite countable S we have $B(S) = l^\infty$ and we now focus on such infinite systems. Thus,

$$\|v\|_{B(S)} = \|v\|_{l^\infty}.$$

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Denote by $C_S(\overline{G})$ the real Banach space of mappings

$$w : \overline{G} \ni x \rightarrow (w(x) : S \ni i \rightarrow w^i(x) \in \mathbb{R}) \in l^\infty,$$

where w^i are continuous in \overline{G} , with the finite norm

$$\|w\| := \sup\{|w^i(x)| : x \in \overline{G}, i \in S\}.$$

For any fixed $t \in [0, T)$, we denote by $z(t, \cdot) = (z^1(t, \cdot), z^2(t, \cdot), \dots)$ the function

$$z(t, \cdot) : \overline{G} \ni x \rightarrow z(t, x) \in l^\infty$$

which is an element of the space $C_S(\overline{G})$. We denote the space of these functions by $C_S(\overline{D})$.

For system (1) we consider the following *Fourier first boundary value problem*:

Find a regular solution $z = z(t, x)$ of (1) in \overline{D} satisfying the boundary condition

$$(2) \quad z(t, x) = g(t, x) \quad \text{for } (t, x) \in \Sigma,$$

where $\sigma := (0, T) \times \partial G$, $D_0 := \{(t, x) : t = 0, x \in \overline{G}\}$, $\Sigma := D_0 \cup \sigma$, $\overline{D} := D \cup \Sigma$ and $g = (g^1, g^2, \dots)$.

To prove the existence and uniqueness of the solution, we apply an iterative successive approximations method (see [3, 4]). We use J. Szarski's results [9, 10] on differential-functional inequalities and a comparison theorem for infinite systems and parabolic differential inequalities [8].

Infinite systems of parabolic differential and differential-integral equations are used to describe polymerization-type chemical reaction phenomena (coagulation and fragmentation of clusters) [2], [6]. An infinite system of ordinary differential equations was introduced by M. Smoluchowski ([7], 1917) as a model for coagulation of colloids moving according to a Brownian motion.

2. Notations, definitions and assumptions. A mapping $z \in C_S(\overline{D})$ will be called *regular* if the functions z^i ($i \in S$) are continuous in \overline{D} and have continuous derivatives $\partial z^i / \partial t$, $\partial^2 z^i / \partial x_i \partial x_k$ in D for $j, k = 1, \dots, m$. We write briefly $z \in C_S^{\text{reg}}(\overline{D})$.

A regular mapping will be called a *regular solution* of problem (1), (2) in \overline{D} if the above equations are satisfied in D and the boundary condition (2) is satisfied.

The Hölder space $C^{l+\alpha}(\overline{D}) := C^{(l+\alpha)/2, l+\alpha}(\overline{D})$ ($l = 0, 1, 2, \dots$; $0 < \alpha \leq 1$) is the space of continuous functions f whose derivatives $\partial^{r+s} f / \partial t^r \partial x^s := D_t^r D_x^s f(t, x)$ ($0 \leq 2r + s \leq l$) all exist and are Hölder continuous with

exponent α ($0 < \alpha \leq 1$) in D , with the finite norm

$$|f|^{l+\alpha} := \sup_{\substack{(t,x) \in D \\ 0 \leq 2r+s \leq l}} |D_t^r D_x^s f(t,x)| + \sup_{\substack{P, P' \in D \\ 2r+s=l \\ P \neq P'}} \frac{|D_t^r D_x^s f(t,x) - D_t^r D_x^s f(t',x')|}{[d(P, P')]^\alpha},$$

where $P = (t, x)$, $P' = (t', x')$ and $d(P, P')$ is the parabolic distance defined by

$$d(P, P') := (|t - t'| + \|x - x'\|_{\mathbb{R}^m}^2)^{1/2},$$

and $\|x\|_{\mathbb{R}^m} := (\sum_{j=1}^m x_j^2)^{1/2}$.

We denote by $C_S^{l+\alpha}(\bar{D})$ the Banach space of mappings z such that $z^i \in C^{l+\alpha}(\bar{D})$ for all $i \in S$.

In the space $C_S(\bar{D})$ the following order is introduced: for $z, \tilde{z} \in C_S(\bar{D})$ the inequality $z \leq \tilde{z}$ means that $z^i(t, x) \leq \tilde{z}^i(t, x)$ for all $(t, x) \in \bar{D}$, $i \in S$.

We assume that the operators \mathcal{A}^i ($i \in S$) are *uniformly elliptic* in \bar{D} , i.e., there exists a constant $\mu > 0$ such that

$$\sum_{j,k=1}^m a_{jk}^i(t, x) \xi_j \xi_k \geq \mu \sum_{j=1}^m \xi_j^2$$

for all $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, $(t, x) \in \bar{D}$, $i \in S$.

We say that the operators $\mathcal{F}^i = \partial/\partial t - \mathcal{A}^i$ ($i \in S$) are *uniformly parabolic* in \bar{D} when the operators \mathcal{A}^i are uniformly elliptic in \bar{D} .

Functions $u = u(t, x)$ and $v = v(t, x) \in C_S^{reg}(\bar{D})$ satisfying the systems of inequalities

$$(3) \quad \begin{cases} \mathcal{F}^i[u^i](t, x) \leq f^i(t, x, u(t, \cdot)) & \text{for } (t, x) \in D, \ i \in S, \\ u(t, x) \leq g(t, x) & \text{for } (t, x) \in \Sigma, \end{cases}$$

$$(4) \quad \begin{cases} \mathcal{F}^i[v^i](t, x) \geq f^i(t, x, v(t, \cdot)) & \text{for } (t, x) \in D, \ i \in S, \\ v(t, x) \geq g(t, x) & \text{for } (t, x) \in \Sigma \end{cases}$$

are called, respectively, a *lower* and an *upper function* for problem (1), (2) in \bar{D} .

We assume that the functions

$$f^i : D \times C_S(\bar{G}) \ni (t, x, s) \rightarrow f^i(t, x, s) \in \mathbb{R}, \quad i \in S,$$

satisfy the following assumptions:

(H_f) $f^i(\cdot, \cdot, s) \in C^{0+\alpha}(\bar{D})$ ($i \in S$);

(L) f^i ($i \in S$) satisfy the Lipschitz condition with respect to s , i.e., for all s, \tilde{s} we have

$$|f^i(t, x, s) - f^i(t, x, \tilde{s})| \leq L \|s - \tilde{s}\| \quad \text{for } (t, x) \in D,$$

where $L > 0$ is constant;

(W) f^i ($i \in S$) are increasing with respect to s .

(H_a) The coefficients $a_{jk}^i = a_{jk}^i(t, x)$, $a_{jk}^i = a_{kj}^i$ ($j, k = 1, \dots, m$, $i \in S$) in (1) are Hölder continuous with respect to t and x in \bar{D} , i.e., $a_{jk}^i \in C^{0+\alpha}(\bar{D})$.

(H_g) $g^i \in C^{2+\alpha}(\Sigma)$ for $i \in S$.

We remark that if $g^i \in C^{2+\alpha}(\Sigma)$ and $\partial G \in C^{2+\alpha}$ then, without loss of generality, we can consider the homogeneous boundary condition

$$(5) \quad z(t, x) = 0 \quad \text{for } (t, x) \in \Sigma.$$

Accordingly, in what follows we confine ourselves to considering the homogeneous problem (1), (5) in \bar{D} only.

ASSUMPTION (A). There exists at least one pair $u_0 = u_0(t, x)$, $v_0 = v_0(t, x)$ of a lower and an upper function for problem (1), (5) in \bar{D} .

3. Existence and uniqueness theorem

THEOREM. *Let all the above assumptions hold. Consider the following infinite systems of linear equations:*

$$(6) \quad \mathcal{F}^i[u_n^i](t, x) = f^i(t, x, u_{n-1}(t, \cdot)),$$

$$(7) \quad \mathcal{F}^i[v_n^i](t, x) = f^i(t, x, v_{n-1}(t, \cdot)), \quad i \in S,$$

for $n = 1, 2, \dots$ with boundary condition (5). Then

(i) *there exist unique regular solutions u_n and v_n ($n = 1, 2, \dots$) of systems (6) and (7) with boundary condition (5) in \bar{D} ;*

(ii) *the inequalities*

$$(8) \quad u_{n-1}(t, x) \leq u_n(t, x), \quad v_n(t, x) \leq v_{n-1}(t, x)$$

hold for $(t, x) \in \bar{D}$ ($n = 1, 2, \dots$);

(iii) *the functions u_n and v_n ($n = 1, 2, \dots$) are lower and upper functions for problem (1), (5) in \bar{D} , respectively;*

(iv) $\lim_{n \rightarrow \infty} [v_n(t, x) - u_n(t, x)] = 0$ *uniformly in \bar{D} ;*

(v) *the function*

$$z(t, x) = \lim_{n \rightarrow \infty} u_n(t, x)$$

is a unique regular solution of problem (1), (5) in \bar{D} and $z \in C_S^{2+\alpha}(\bar{D})$.

Before going into the proof we introduce the Nemytskiĭ operator and prove some lemmas. Nemytskiĭ operators play an important role in the theory of nonlinear equations. For more information see [1].

REMARK 1. If u and v are lower and upper functions for problem (1), (5) in \bar{D} , respectively, and z is a regular solution of this problem, then by the

Szarski theorem on differential-functional inequalities for infinite systems of parabolic type [10] we have

$$(9) \quad u(t, x) \leq z(t, x) \leq v(t, x) \quad \text{for } (t, x) \in \bar{D}.$$

In particular we have

$$(10) \quad u_0(t, x) \leq z(t, x) \leq v_0(t, x) \quad \text{for } (t, x) \in \bar{D}.$$

Let $\beta \in C_S(\bar{D})$ be a sufficiently regular function. Denote by \mathcal{P} the operator

$$\mathcal{P} : \beta \rightarrow \gamma = \mathcal{P}\beta,$$

where γ is the (supposedly unique) solution of the boundary value problem

$$(11) \quad \begin{cases} \mathcal{F}^i[\gamma^i](t, x) = f^i(t, x, \beta(t, \cdot)) & \text{for } (t, x) \in D, \quad i \in S, \\ \gamma(t, x) = 0 & \text{for } (t, x) \in \Sigma. \end{cases}$$

The operator \mathcal{P} is the composition of the nonlinear Nemytskiĭ operator $\mathbf{F} = (\mathbf{F}^1, \mathbf{F}^2, \dots)$,

$$\mathbf{F} : \beta \rightarrow \delta = \mathbf{F}\beta,$$

where

$$\mathbf{F}^i \beta(t, x) := f^i(t, x, \beta^1(t, \cdot), \beta^2(t, \cdot), \dots) = \delta^i(t, x), \quad i \in S,$$

and $\delta = (\delta^1, \delta^2, \dots)$, and the linear operator

$$\mathcal{G} : \delta \rightarrow \gamma,$$

where γ is the (supposedly unique) solution of the linear problem

$$(12) \quad \begin{cases} \mathcal{F}^i[\gamma^i](t, x) = \delta^i(t, x) & \text{in } D, \quad i \in S, \\ \gamma(t, x) = 0 & \text{on } \Sigma. \end{cases}$$

Hence $\mathcal{P} = \mathcal{G} \circ \mathbf{F}$.

LEMMA 1. *If $\beta \in C_S^{0+\alpha}(\bar{D})$ and the function $f = (f^1, f^2, \dots)$ generating the Nemytskiĭ operator \mathbf{F} satisfies conditions (H_f) and (L) , then*

$$\delta = \mathbf{F}\beta \in C_S^{0+\alpha}(\bar{D}).$$

Proof. Since $\beta \in C_S^{0+\alpha}(\bar{D})$, we have

$$|\beta^i(t, x) - \beta^i(t', x')| \leq K(|t - t'| + \|x - x'\|_{\mathbb{R}^m}^2)^{\alpha/2}$$

for all $(t, x), (t', x') \in \bar{D}$ and $i \in S$, where K is some nonnegative constant. Hence

$$\|\beta(t, \cdot) - \beta(t', \cdot)\| = \sup_{\substack{i \in S \\ x \in \bar{G}}} \{|\beta^i(t, x) - \beta^i(t', x)|\} \leq K|t - t'|^{\alpha/2}.$$

From (H_f) and (L) it follows that

$$\begin{aligned}
& |\delta^i(t, x) - \delta^i(t', x')| \\
&= |\mathbf{F}^i \beta(t, x) - \mathbf{F}^i \beta(t', x')| \\
&= |f^i(t, x, \beta(t, \cdot)) - f^i(t', x', \beta(t', \cdot))| \\
&\leq |f^i(t, x, \beta(t, \cdot)) - f^i(t', x', \beta(t, \cdot))| + |f^i(t', x', \beta(t, \cdot)) - f^i(t', x', \beta(t', \cdot))| \\
&\leq K_1(|t - t'| + \|x - x'\|_{\mathbb{R}^m}^2)^{\alpha/2} + L\|\beta(t, \cdot) - \beta(t', \cdot)\| \\
&\leq K_1(|t - t'| + \|x - x'\|_{\mathbb{R}^m}^2)^{\alpha/2} + LK|t - t'|^{\alpha/2} \\
&\leq K^*(|t - t'| + \|x - x'\|_{\mathbb{R}^m}^2)^{\alpha/2},
\end{aligned}$$

where $K^* = K_1 + LK$ for all $(t, x), (t', x') \in \bar{D}$, $i \in S$. Therefore $\delta \in C_S^{0+\alpha}(\bar{D})$. ■

LEMMA 2. If $\delta \in C_S^{0+\alpha}(\bar{D})$ and the coefficients satisfy assumption (H_α) , then problem (1), (5) has a unique regular solution $\gamma \in C_S^{2+\alpha}(\bar{D})$.

PROOF. Observe that system (12) has the following property: the i th equation depends on the i th unknown function only. Therefore, applying the theorem on the existence and uniqueness of solution of Fourier's first problem for a linear parabolic equation (see A. Friedman [5], Theorems 6 and 7, p. 65), the statement of the lemma follows immediately. ■

Lemmas 1 and 2 yield

COROLLARY. $\mathcal{P} = \mathcal{G} \circ \mathbf{F} : C_S^{0+\alpha}(\bar{D}) \ni \beta \rightarrow \gamma = \mathcal{P}\beta \in C_S^{2+\alpha}(\bar{D})$. ■

LEMMA 3. If β is an upper function (resp. a lower function) for problem (1), (5) in \bar{D} , then $\mathcal{P}\beta(t, x) \leq \beta(t, x)$ (resp. $\mathcal{P}\beta(t, x) \geq \beta(t, x)$) in \bar{D} .

PROOF. If β is an upper function, then by (4) we have

$$\mathcal{F}^i[\beta^i](t, x) \geq f^i(t, x, \beta(t, \cdot)) \quad \text{in } D, \quad i \in S.$$

From the definition of the operator \mathcal{P} (see (12)) it follows that

$$\mathcal{F}^i[\gamma^i](t, x) = f^i(t, x, \beta(t, \cdot)) \quad \text{in } D, \quad i \in S.$$

Therefore

$$\mathcal{F}^i[\gamma^i - \beta^i](t, x) \leq 0 \quad \text{in } D, \quad i \in S.$$

and

$$\gamma(t, x) - \beta(t, x) \leq 0 \quad \text{on } \Sigma.$$

Hence, by the Szarski theorem [10], we have $\gamma(t, x) - \beta(t, x) \leq 0$ in \bar{D} so

$$(13) \quad \mathcal{P}\beta(t, x) = \gamma(t, x) \leq \beta(t, x) \quad \text{in } \bar{D}. \quad \blacksquare$$

LEMMA 4. If β is an upper (resp. a lower) function for problem (1), (5) in \bar{D} , then $\gamma = \mathcal{P}\beta$ is also an upper (resp. a lower) function for problem (1), (5) in \bar{D} .

Proof. From (11), (13) and condition (W),

$$\mathcal{F}^i[\gamma^i](t, x) - f^i(t, x, \gamma(t, \cdot)) = f^i(t, x, \beta(t, \cdot)) - f^i(t, x, \gamma(t, \cdot)) \geq 0$$

in D , $i \in S$, and $\gamma(t, x) = 0$ on Σ . From the Corollary it follows that γ is a regular function, so it is an upper function for problem (1), (5) in \bar{D} .

Proof of Theorem. Starting from a lower function u_0 and an upper function v_0 , we define by induction

$$\begin{aligned} u_1 &= \mathcal{P}u_0, & u_n &= \mathcal{P}u_{n-1}, \\ v_1 &= \mathcal{P}v_0, & v_n &= \mathcal{P}v_{n-1}, \quad n = 1, 2, \dots \end{aligned}$$

From Lemmas 1, 2 and 4 it follows that u_n and v_n ($n = 1, 2, \dots$) are respectively a lower and an upper function for problem (1), (5) in \bar{D} .

By induction, from Lemma 3 we have

$$\begin{aligned} u_{n-1}(t, x) &\leq \mathcal{P}u_{n-1}(t, x) = u_n(t, x), \\ v_n(t, x) &= \mathcal{P}v_{n-1}(t, x) \leq v_{n-1}(t, x) \quad (n = 1, 2, \dots) \text{ for } (t, x) \in \bar{D}. \end{aligned}$$

Therefore

$$\begin{aligned} u_0(t, x) &\leq u_1(t, x) \leq \dots \leq u_n(t, x) \\ &\leq \dots \leq v_n(t, x) \leq \dots \leq v_1(t, x) \leq v_0(t, x) \end{aligned}$$

for $(t, x) \in \bar{D}$.

We now show by induction that

$$(14) \quad w_n^i(t, x) \leq N_0 \frac{(Lt)^n}{n!}, \quad n = 0, 1, 2, \dots, \text{ for } (t, x) \in \bar{D}, \quad i \in S,$$

where by (9) and (10),

$$(15) \quad w_n^i(t, x) = v_n^i(t, x) - u_n^i(t, x) \geq 0 \quad \text{in } \bar{D}$$

and

$$N_0 = \max_{i \in S} \max_{(t, x) \in \bar{D}} [v_0^i(t, x) - u_0^i(t, x)] \geq 0;$$

owing to the regularity of u_0 and v_0 we have $N_0 < \infty$.

It is obvious that (14) holds for w_0 . Suppose it holds for w_n . Since the functions f^i ($i \in S$) satisfy the Lipschitz condition (L), by (6), (7), (8), (14) and (15), we get

$$\mathcal{F}^i[w_{n+1}^i](t, x) = f^i(t, x, v_n(t, x)) - f^i(t, x, u_n(t, \cdot)) \leq L \|w_n(t, \cdot)\|.$$

By the definition of the norm in $C_S(\bar{D})$ and by (14) we get

$$\|w_n(t, \cdot)\| \leq \frac{(Lt)^n}{n!},$$

so we finally obtain

$$(16) \quad \mathcal{F}^i[w_{n+1}^i](t, x) \leq N_0 \frac{L^{n+1} t^n}{n!} \quad \text{for } (t, x) \in D, \quad i \in S,$$

and

$$w_{n+1}(t, x) = 0 \quad \text{for } (t, x) \in \Sigma.$$

Consider the comparison system

$$(17) \quad \mathcal{F}^i[M_{n+1}^i](t, x) = N_0 \frac{L^{n+1}t^n}{n!} \quad \text{for } (t, x) \in D, \quad i \in S,$$

with the boundary condition

$$(18) \quad M_{n+1}(t, x) \geq 0 \quad \text{on } \Sigma.$$

It is obvious that the functions

$$M_{n+1}^i(t, x) = N_0 \frac{(Lt)^{n+1}}{(n+1)!}, \quad i \in S.$$

are regular solutions of (17), (18) in \bar{D} .

Applying a theorem on differential inequalities of parabolic type ([8], Theorem 64.1, p. 195) to systems (16) and (17) we get

$$w_{n+1}^i(t, x) \leq M_{n+1}^i(t, x) = N_0 \frac{(Lt)^{n+1}}{(n+1)!} \quad \text{for } (t, x) \in \bar{D}, \quad i \in S,$$

so the induction step is proved and so is inequality (14).

As a direct consequence of (14) we obtain

$$(19) \quad \lim_{n \rightarrow \infty} [v_n(t, x) - u_n(t, x)] = 0 \quad \text{uniformly in } \bar{D}.$$

The functional sequences $\{u_n\}$ and $\{v_n\}$ are monotone and bounded, and (19) holds, so there exists a continuous function $U = U(t, x)$ in \bar{D} such that

$$(20) \quad \lim_{n \rightarrow \infty} u_n(t, x) = U(t, x), \quad \lim_{n \rightarrow \infty} v_n(t, x) = U(t, x) \quad \text{uniformly in } \bar{D}.$$

Since the functions f^i ($i \in S$) are monotone (condition (W)), from (8) it follows that the functions $f^i(t, x, u_{n-1}(t, \cdot))$ ($i \in S$) are uniformly bounded in D with respect to n . Hence we conclude by Lemma 2 that all the functions $u_n \in C_S^{2+\alpha}(\bar{D})$ for $n = 1, 2, \dots$ satisfy the Hölder condition with a constant independent of n . Hence $U \in C_S^{0+\alpha}(\bar{D})$.

If we now consider the system of equations

$$(21) \quad \mathcal{F}^i[z^i](t, x) = f^i(t, x, U(t, \cdot)) = \mathbf{F}^i U(t, x) \quad \text{for } (t, x) \in D, \quad i \in S$$

with boundary condition (5), then by Lemma 1 we have $\mathbf{F}^i U \in C_S^{0+\alpha}(\bar{D})$. Therefore by Lemma 2 this problem has a unique regular solution $z \in C_S^{2+\alpha}(\bar{D})$.

Let us now consider systems (6) and (21) together, and apply Szarski's theorem ([8], Theorem 51.1, p. 147) on the continuous dependence of solution of the first problem on initial and boundary values and on the right-hand sides of the systems.

Since the f^i ($i \in S$) satisfy the Lipschitz condition (L), by (20) we have

$$\lim_{n \rightarrow \infty} f^i(t, x, u_n(t, \cdot)) = f^i(t, x, U(t, \cdot)) \quad \text{uniformly in } \bar{D}.$$

Hence

$$(22) \quad \lim_{n \rightarrow \infty} u_n(t, x) = z(t, x).$$

By (20) and (22),

$$z = z(t, x) = U(t, x) \quad \text{for } (t, x) \in \bar{D}$$

is a regular solution of problem (1), (5) in \bar{D} and $z \in C_S^{2+\alpha}(\bar{D})$.

The uniqueness of the solution follows from Szarski's uniqueness criterion [9]. It also follows directly from inequality (14). ■

REMARK 2. Instead of Assumption (A), one may use the following stronger assumption (see [4]):

ASSUMPTION (A*). There exists at least one pair $u_0 = u_0(t, x)$, $v_0 = v_0(t, x)$ of a lower and an upper function for problem (1), (5) in \bar{D} such that

$$u_0(t, x) \leq v_0(t, x) \quad \text{for } (t, x) \in \bar{D}.$$

Under Assumption (A*), the other assumptions on the functions f^i may be weakened in such a way that conditions (H_f), (L) and (W) hold only locally in the set \mathcal{K} , where $\mathcal{K} := \{(t, x, s) : (t, x) \in \bar{D}, s \in \langle u_0, v_0 \rangle\}$ and $\langle u_0, v_0 \rangle$ is the segment defined by

$$\langle u_0, v_0 \rangle := \{s \in C_S(\bar{D}) : u_0(t, x) \leq s(t, x) \leq v_0(t, x) \text{ for } (t, x) \in \bar{D}\}.$$

The existence and uniqueness of the solution will then be guaranteed in the set $\langle u_0, v_0 \rangle$, though. ■

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