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Oscillation criteria for second order self-adjoint matrix differential equations

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Abstract. Some results concerning oscillation of second order self-adjoint matrix differential equations are obtained. These may be regarded as a generalization of results for the corresponding scalar equations.

1. Consider the self-adjoint second order linear differential equation

$$[\sigma(t)y']' + c(t)y = 0, \tag{1.1}$$

where $\sigma, c \in C([0,\infty),\mathbb{R})$ and $\sigma(t) > 0$. A solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*. Equation (1.1) is said to be *oscillatory* if all its solutions are oscillatory.

Leighton's criterion (see [7, 10]) states that (1.1) is oscillatory if

$$\int_0^\infty \frac{dt}{\sigma(t)} = \infty \quad \text{and} \quad \int_0^\infty c(t)\,dt = \infty.$$

In 1949, Wintner [12] showed that

$$y'' + c(t)y = 0 \tag{1.2}$$

is oscillatory if $\lim_{t\to\infty} C(t) = \infty$, where

$$C(t) = \frac{1}{t}\int_0^t \Big(\int_0^s c(u)\,du\Big)\,ds. \tag{1.3}$$

On the other hand, Hartman [5] has proved that nonoscillation of (1.2) implies that either $C(t)$ tends to a finite limit or $\liminf_{t\to\infty} C(t) = -\infty$. Thus the following oscillation results follow:

Lemma 1.1. *If* $\lim_{t\to\infty} C(t) = \infty$, *then* (1.2) *is oscillatory.*

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Lemma 1.2. *If* $-\infty < \liminf_{t\to\infty} C(t) < \limsup_{t\to\infty} C(t) \le \infty$, *then* (1.2) *is oscillatory.*

In [4], Coles developed the idea of weighted averages of $\int_0^x c(s)\,ds$ in order to obtain additional information about oscillation of (1.2). He proved the following results:

Theorem 1.3. *Let f be a nonnegative, locally integrable function on* $[0,\infty)$ *such that* $\int_0^x f(t)\,dt \not\equiv 0$. *If*

$$\int_a^\infty f(t)\Big(\int_0^t f^2(s)\,ds\Big)^{-1}\Big(\int_0^t f(s)\,ds\Big)^k dt = \infty \tag{1.4}$$

for some k, $0 \le k < 1$, *and for* $a > 0$, *and*

$$\lim_{x\to\infty}\Big(\int_0^x f(t)\,dt\Big)^{-1}\int_0^x f(t)\Big(\int_0^t c(s)\,ds\Big)\,dt = \infty,$$

then (1.2) *is oscillatory.*

Theorem 1.4. *If* $C(t)$, *given by* (1.3), *does not approach a finite limit as* $t\to\infty$ *and if there is a nonnegative, locally integrable function f on* $[0,\infty)$ *satisfying* $\int_0^x f(t)\,dt \not\equiv 0$ *and* (1.4) *and*

$$\liminf_{x\to\infty}\Big(\int_0^x f(t)\,dt\Big)^{-1}\int_0^x f(t)\Big(\int_0^t c(s)\,ds\Big)\,dt > -\infty,$$

then (1.2) *is oscillatory.*

Clearly, Theorems 1.3 and 1.4 generalize Lemmas 1.1 and 1.2 respectively.

In this paper, we generalize Theorems 1.3 and 1.4 to self-adjoint second order matrix differential equations of the form

$$(P(t)Y')' + Q(t)Y = 0 \tag{E}$$

on $[0,\infty)$, where $Y(t)$, $P(t)$ and $Q(t)$ are $n\times n$ real, continuous matrix functions on $[0,\infty)$ such that $Q(t)$ is symmetric and $P(t)$ is symmetric and positive definite. A solution $Y(t)$ of (E) is said to be *nontrivial* if $\det Y(t) \ne 0$ for at least one $t\in[0,\infty)$. A solution $Y(t)$ of (E) is said to be *prepared* or *self-conjugate* if

$$Y^*(t)(P(t)Y'(t)) = (P(t)Y'(t))^*Y(t) \tag{1.5}$$

for $t\in[0,\infty)$, where, for any matrix A, the transpose of A is denoted by A^*. It is easy to see that for any solution $Y(t)$ of (E),

$$Y^*(t)(P(t)Y'(t)) - (P(t)Y'(t))^*Y(t) = \text{a constant}.$$

In most of the literature dealing with oscillation of matrix differential equations, it is tacitly assumed that the constant in the above identity is a zero matrix. However, Howard (see [6, pp. 185, 188]) explicitly assumed the condition (1.5). A nontrivial prepared solution $Y(t)$ of (E) is said to be *oscillatory* if for every $t_0 \geq 0$ it is possible to find a $t_1 \geq t_0$ such that $\det Y(t_1) = 0$; otherwise, $Y(t)$ is called *nonoscillatory*. Equation (E) is said to be *oscillatory* if every nontrivial prepared solution of the equation is oscillatory. The oscillation of (E) is defined through its nontrivial prepared solutions because it is possible (see [9]) that (E) admits a nontrivial nonprepared nonoscillatory solution.

For any $n \times n$ real symmetric matrix A, the eigenvalues $\lambda_k(A)$ of A, $1 \leq k \leq n$, are real and hence may be arranged as $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$. For any $n \times n$ real symmetric matrices A and B, we write $A \geq B$ to mean that $A - B \geq 0$, that is, $A - B$ is positive semi-definite, and $A > B$ to mean that $A - B > 0$, that is, $A - B$ is positive definite. It is well known that $A \geq B$ and $B \geq 0$ imply that $A \geq 0$.

If S is the real linear space of all real symmetric $n \times n$ matrices, then $\operatorname{tr} : S \to \mathbb{R}$ is a linear functional and $(\operatorname{tr} A)^2 \leq n \operatorname{tr}(A^2)$ for every $A \in S$. Further, for $A, B \in S$, (i) $A \geq B$ implies that $\operatorname{tr} A \geq \operatorname{tr} B$, (ii) $\lambda_n(A) \leq \operatorname{tr} A/n \leq \lambda_1(A)$. If $A \geq 0$, then $\lambda_1(A) \leq \operatorname{tr} A \leq n\lambda_1(A)$. We recall that

$$\operatorname{tr} A = \sum_{k=1}^{n} \lambda_k(A) = \sum_{k=1}^{n} a_{kk} \quad \text{if } A = (a_{ij})_{n \times n}.$$

Moreover,

$$\operatorname{tr} \int_0^t Q(s)\, ds = \int_0^t \operatorname{tr} Q(s)\, ds.$$

One may see [8] for these properties.

If $P(t) \equiv I$, the identity matrix, then (E) takes the form

$$(\mathrm{E}_1) \qquad Y'' + Q(t)Y = 0.$$

Oscillation of (E) must be studied separately from (E_1) since, like in the scalar case, there is no oscillation-preserving transformation of the independent variable that allows the passage between the two forms. In most of the literature (see [2, 3, 8] and the references therein), oscillation criteria for (E) or (E_1) are given in terms of $\operatorname{tr}(\int_0^t Q(s)\, ds)$ or $\lambda_1(\int_0^t Q(s)\, ds)$, the first eigenvalue of $\int_0^t Q(s)\, ds$. However, these results are not always comparable. In this paper, we obtain sufficient conditions for oscillation of (E) in terms of $\int_0^t \operatorname{tr} Q(s)\, ds$. Some results are stated in terms of $\lambda_1(\int_0^t Q(s)\, ds)$. Examples are given to illustrate usefulness of each of these results.

The motivation for this work came from the above observation and from the observation that a very extensive literature exists (see [1, 11, 13] and the references therein) for the oscillation theory of (1.1) or (1.2), whereas the corresponding theory for (E) or (E_1) is less developed.

2. In this section we obtain sufficient conditions for oscillation of (E). The following conditions are needed for our results in the sequel:

(H_1) $P^{-1}(t) \geq I$,

(H_2) $\lim_{t\to\infty} A(t) = \infty$, where $A(t) = A(t, 0)$ and

$$A(t,\sigma) = \Big(\int_\sigma^t f(s)\,ds\Big)^{-1} \int_\sigma^t f(s)\Big(\int_\sigma^s \operatorname{tr} Q(u)\,du\Big)\,ds, \qquad t \geq \sigma \geq 0,$$

(H_3) $\displaystyle\int_a^\infty f(t)\Big(\int_0^t f^2(s)\,ds\Big)^{-1}\Big(\int_0^t f(s)\,ds\Big)^k\,dt = \infty$

for some k, $0 \leq k < 1$, and some $a > 0$, where f is a nonnegative, locally integrable function on $[0,\infty)$ such that $\int_0^t f(s)\,ds \not\equiv 0$,

(H_4) $\displaystyle\limsup_{t\to\infty} \frac{1}{t}\int_0^t \Big(\int_0^s \operatorname{tr} Q(u)\,du\Big)\,ds = \infty$,

(H_5) $\liminf_{t\to\infty} A(t) > -\infty$, where $A(t)$ is given by (H_2),

(H_6) the eigenvalues $\lambda_i(C(t))$, $1 \leq i \leq n$, of any real symmetric matrix $C(t)$ may be arranged in the form $\lambda_1(C(t)) \geq \ldots \geq \lambda_n(C(t))$.

(H_7) $\lim_{t\to\infty} B(t) = \infty$, where $B(t) = B(t, 0)$ and

$$B(t,\sigma) = \Big(\int_\sigma^t f(s)\,ds\Big)^{-1} \int_\sigma^t f(s)\lambda_1\Big(\int_\sigma^s Q(u)\,du\Big)\,ds, \qquad t \geq \sigma \geq 0,$$

(H_8) $\liminf_{t\to\infty} B(t) > -\infty$, where $B(t)$ is given by (H_7),

(H_9) $\displaystyle\limsup_{t\to\infty} \frac{1}{t}\int_0^t \lambda_1\Big(\int_0^s Q(u)\,du\Big)\,ds = \infty$.

Remark 1. (i) (H_3) implies that

$$\int_{t_1}^\infty f(t)\Big(\int_{t_0}^t f^2(s)\,ds\Big)^{-1}\Big(\int_{t_1}^t f(s)\,ds\Big)^k\,dt = \infty, \qquad t_1 > t_0 > 0.$$

(ii) (H_3) implies that $\int_0^\infty f(t)\,dt = \infty$.

(iii) (H_2) and $\int_0^\infty f(t)\,dt = \infty$ imply that $\lim_{t\to\infty} A(t, t_0) = \infty$ for every $t_0 > 0$.

(iv) (H_4) implies that

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^t \Big(\int_{t_0}^s \operatorname{tr} Q(u)\,du\Big)\,ds = \infty \qquad \text{for every } t_0 > 0.$$

(v) (H_5) and $\int_0^\infty f(t)\,dt = \infty$ imply that $\liminf_{t\to\infty} A(t, t_0) > -\infty$ for every $t_0 > 0$.

(vi) If $\int_0^\infty f(t)\,dt = \infty$ and $g(t)$ is nondecreasing, then the function $(\int_0^t f(s)\,ds)^{-1}(\int_0^t f(s)g(s)\,ds)$ is nondecreasing.

(vii) If $\int_0^\infty f(t)\,dt = \infty$, $g(t)$ is nondecreasing and the function $(\int_0^t f(s)\,ds)^{-1}(\int_0^t f(s)g(s)\,ds)$ is bounded, then $g(t)$ is bounded.

THEOREM 2.1. *If* (H_1)–(H_3) *hold, then* (E) *is oscillatory.*

Proof. If possible, suppose that (E) is not oscillatory. Hence there exists a nontrivial prepared solution $Y(t)$ of (E) such that $\det Y(t) \neq 0$ for $t \geq t_0 > a$. Setting

$$R(t) = P(t)Y'(t)Y^{-1}(t) \tag{2.1}$$

for $t \geq t_0$, we observe that $R^*(t) = R(t)$ due to (1.5) and

$$R'(t) + R(t)P^{-1}(t)R(t) + Q(t) = 0. \tag{2.2}$$

Integrating (2.2) from t_0 to t yields

$$R(t) + \int_{t_0}^{t} R(s)P^{-1}(s)R(s)\,ds + \int_{t_0}^{t} Q(s)\,ds = R(t_0).$$

Hence

$$\operatorname{tr} R(t) + \int_{t_0}^{t} \operatorname{tr}(R(s)P^{-1}(s)R(s))\,ds + \int_{t_0}^{t} \operatorname{tr} Q(s)\,ds = \operatorname{tr} R(t_0).$$

Multiplying the above identity through by $f(t)$ and then integrating from t_0 to t, we obtain

$$\begin{aligned}\int_{t_0}^{t} f(s)\operatorname{tr} R(s)\,ds + \int_{t_0}^{t} f(s)\Big(\int_{t_0}^{s} \operatorname{tr}(R(u)P^{-1}(u)R(u))\,du\Big)\,ds \\ = (\operatorname{tr} R(t_0) - A(t,t_0))\int_{t_0}^{t} f(s)\,ds < 0\end{aligned} \tag{2.3}$$

for large t due to (H_2) and (H_3) (see Remark 1(ii), (iii)). From (H_1) it follows that

$$R(t)P^{-1}(t)R(t) \geq R^2(t) \geq 0$$

and hence (2.3) yields

$$\int_{t_0}^{t} f(s)\operatorname{tr} R(s)\,ds < 0 \tag{2.4}$$

for large t. Consequently, from (2.3) we get

$$\left[\int_{t_0}^{t} f(s)\Big(\int_{t_0}^{s}\operatorname{tr}(R(u)P^{-1}(u)R(u))\,du\Big)\,ds\right]^2 \tag{2.5}$$
$$\le \left[\int_{t_0}^{t} f(s)\operatorname{tr}R(s)\,ds\right]^2 \le \Big(\int_{t_0}^{t} f^2(s)\,ds\Big)\Big(\int_{t_0}^{t}(\operatorname{tr}R(s))^2\,ds\Big)$$
$$\le n\Big(\int_{t_0}^{t} f^2(s)\,ds\Big)\Big(\int_{t_0}^{t}\operatorname{tr}R^2(s)\,ds\Big)$$
$$\le n\Big(\int_{t_0}^{t} f^2(s)\,ds\Big)\Big(\int_{t_0}^{t}\operatorname{tr}(R(s)P^{-1}(s)R(s))\,ds\Big),$$

where the Cauchy–Schwarz inequality is used. If

$$r(t)=\int_{t_0}^{t} f(s)\Big(\int_{t_0}^{s}\operatorname{tr}(R(u)P^{-1}(u)R(u))\,du\Big)\,ds, \tag{2.6}$$

then, for $t \ge t_1 > t_0$, we have

$$r(t) \ge \int_{t_1}^{t} f(s)\Big(\int_{t_0}^{s}\operatorname{tr}(R(u)P^{-1}(u)R(u))\,du\Big)\,ds$$
$$\ge \Big(\int_{t_1}^{t} f(s)\,ds\Big)\Big(\int_{t_0}^{t_1}\operatorname{tr}(R(u)P^{-1}(u)R(u))\,du\Big).$$

Hence, using (2.5), we obtain

$$\Big(\int_{t_1}^{t} f(s)\,ds\Big)^k\Big(\int_{t_0}^{t_1}\operatorname{tr}(R(u)P^{-1}(u)R(u))\,du\Big)^k$$
$$\le r^k(t) = r^{k-2}(t)r^2(t)$$
$$\le nr^{k-2}(t)\Big(\int_{t_0}^{t} f^2(s)\,ds\Big)\Big(\int_{t_0}^{t}\operatorname{tr}(R(s)P^{-1}(s)R(s))\,ds\Big),$$

that is,

$$f(t)\Big(\int_{t_1}^{t} f(s)\,ds\Big)^k\Big(\int_{t_0}^{t_1}\operatorname{tr}(R(u)P^{-1}(u)R(u))\,du\Big)^k$$
$$\le nr^{k-2}(t)r'(t)\int_{t_0}^{t} f^2(s)\,ds.$$

Integrating from t_1 to t, we get

$$\Big(\int_{t_0}^{t_1} \operatorname{tr}(R(u)P^{-1}(u)R(u))\,du\Big)^k \times \int_{t_1}^{t} f(s)\Big(\int_{t_0}^{s} f^2(u)\,du\Big)^{-1}\Big(\int_{t_1}^{s} f(u)\,du\Big)^k\,ds$$
$$\le n\int_{t_1}^{t} r^{k-2}(s)r'(s)\,ds \le \frac{n}{1-k}\cdot\frac{1}{r^{1-k}(t_1)} < \infty,$$

which contradicts (H_3) (see Remark 1(i)). Hence the theorem is proved.

Remark 2. Theorem 2.1 may be viewed as a generalization of the following theorem which is an extension of Theorem 1.3.

Theorem 2.2. *Suppose that the conditions of Theorem* 1.3 *hold. If* $\int_0^\infty \sigma^{-1}(t)\,dt = \infty$, *then* (1.1) *is oscillatory.*

Theorem 2.3. *Let* (H_1) *and* (H_3)–(H_5) *hold. Then* (E) *is oscillatory.*

Proof. If (E) is not oscillatory, then it admits a nontrivial prepared solution $Y(t)$ such that $\det Y(t) \neq 0$ for $t \ge t_0 > a$. Setting $R(t)$ as in (2.1) for $t \ge t_0$, we obtain (2.2) and $R^*(t) = R(t)$. Proceeding as in the proof of Theorem 2.1, we obtain (2.3). Thus

$$\Big(\int_{t_0}^{t} f(s)\,ds\Big)^{-1}\Big[\int_{t_0}^{t} f(s)\operatorname{tr} R(s)\,ds + \int_{t_0}^{t} f(s)\Big(\int_{t_0}^{s}\operatorname{tr}(R(u)P^{-1}(u)R(u))\,du\Big)\,ds\Big] \le \operatorname{tr} R(t_0) - A(t,t_0) \le L \tag{2.7}$$

for $t \ge t_1 > t_0$, by (H_3) and (H_5), where L is a constant (see Remark 1(v)).

We claim that $r(t)(\int_{t_0}^t f(s)\,ds)^{-1}$ is bounded, where $r(t)$ is given by (2.6). Suppose it is unbounded. Since $\int_{t_0}^t \operatorname{tr}(R(u)P^{-1}(u)R(u))\,du$ is nondecreasing and (H_3) holds, it follows that $r(t)(\int_{t_0}^t f(s)\,ds)^{-1}$ is nondecreasing (see Remark 1(vi)). Hence

$$\lim_{t\to\infty} r(t)\Big(\int_{t_0}^{t} f(s)\,ds\Big)^{-1} = \infty. \tag{2.8}$$

From (2.7) it follows that, for $t \geq t_1$,

$$\int_{t_0}^{t} f(s) \operatorname{tr} R(s)\, ds + r(t) \leq L \int_{t_0}^{t} f(s)\, ds,$$

that is,

$$\int_{t_0}^{t} f(s) \operatorname{tr} R(s)\, ds + \frac{1}{2} r(t) \leq \left[L - \frac{1}{2} r(t) \Big(\int_{t_0}^{t} f(s)\, ds \Big)^{-1} \right] \int_{t_0}^{t} f(s)\, ds.$$

Thus the left hand side is negative for large t, due to (2.8). Consequently, we obtain (2.4). Proceeding as in the proof of Theorem 2.1, we arrive at a contradiction to (H_3). Hence our claim holds.

From Remark 1(vii) it now follows that $\int_{t_0}^{t} \operatorname{tr}(R(u)P^{-1}(u)R(u))\, du$ is bounded. If

$$\Big| \operatorname{tr} R(t_0) - \int_{t_0}^{t} \operatorname{tr}(R(u)P^{-1}(u)R(u))\, du \Big| \leq M,$$

where M is a constant, then from (2.2) we obtain

$$\int_{t_0}^{t} \operatorname{tr} Q(s)\, ds \leq M - \operatorname{tr} R(t).$$

Hence

$$\frac{1}{t} \int_{t_0}^{t} \Big(\int_{t_0}^{s} \operatorname{tr} Q(u)\, du \Big)\, ds \leq \frac{M(t-t_0)}{t} - \frac{1}{t} \int_{t_0}^{t} \operatorname{tr} R(s)\, ds. \tag{2.9}$$

By the Cauchy–Schwarz inequality

$$\begin{aligned} \left[\frac{1}{t} \int_{t_0}^{t} \operatorname{tr} R(s)\, ds \right]^2 &\leq (t-t_0) \frac{1}{t^2} \int_{t_0}^{t} (\operatorname{tr} R(s))^2\, ds \leq (t-t_0) \frac{n}{t^2} \int_{t_0}^{t} \operatorname{tr}(R(s))^2\, ds \\ &\leq n \Big(1 - \frac{t_0}{t} \Big) \frac{1}{t} \int_{t_0}^{t} \operatorname{tr}(R(s)P^{-1}(s)R(s))\, ds. \end{aligned}$$

Hence

$$\lim_{t\to\infty} \left[\frac{1}{t} \int_{t_0}^{t} \operatorname{tr} R(s)\, ds \right]^2 = 0,$$

since $\int_{t_0}^{t} \operatorname{tr}(R(s)P^{-1}(s)R(s))\, ds$ is bounded. Thus

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{tr} R(s)\, ds = 0.$$

From (2.9) it follows that

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \Big(\int_{t_0}^{s} \operatorname{tr} Q(u)\, du \Big)\, ds \le M,$$

a contradiction to (H_4) due to Remark 1(iv). Hence the proof of the theorem is complete.

Remark 3. Theorem 2.3 is a generalization of Theorem 1.4.

The following examples illustrate the above results.

Example 1. Consider

$$(P(t)Y')' + Q(t)Y = 0, \qquad t \ge 0, \tag{2.10}$$

where

(2.11)

$$P(t) = \frac{1}{2(t+2)} \begin{bmatrix} 2 & 0 \\ 0 & t+2 \end{bmatrix}, \quad Q(t) = \begin{bmatrix} (1/2) - \cos t & 0 \\ 0 & (1/2) + \cos t \end{bmatrix}.$$

Hence

$$P^{-1}(t) - I = \begin{bmatrix} t+1 & 0 \\ 0 & 1 \end{bmatrix} > 0 \quad \text{and} \quad \operatorname{tr} Q(t) = 1.$$

Taking $f(t)=t$ and $k=2/3$, we observe that the assumptions (H_1)–(H_3) are satisfied. From Theorem 2.1 it follows that (2.10) is oscillatory. In particular,

$$Y(t) = \begin{bmatrix} y_1(t) & 0 \\ 0 & y_2(t) \end{bmatrix}$$

is an oscillatory solution of (2.10), where $y_i(t)$ is a solution of

$$(p_i(t)y')' + q_i(t)y = 0, \qquad t \ge 0, \tag{2.12_i}$$

$i = 1, 2$, with $p_1(t) = 1/(t+2)$, $p_2(t) = 1/2$, $q_1(t) = 1/2 - \cos t$, and $q_2(t) = 1/2 + \cos t$. Leighton's criterion implies that $(2.12)_i$ is oscillatory, $i = 1, 2$, and hence $y_1(t)$ and $y_2(t)$ are oscillatory functions. Clearly, $Y(t)$ is nontrivial and prepared.

Example 2. Let $P(t)$ be as in Example 1 and

$$Q(t) = \begin{bmatrix} q(t) - \cos t & 0 \\ 0 & q(t) + \cos t \end{bmatrix} \tag{2.13}$$

where $q(t) = q_n(t)$, $t \in [2n, 2n+2]$, $n = 0, 1, 2, \ldots$, and

$$q_n(t) = \begin{cases} 0, & t \in [2n, 2n+1], \\ 2(t - 2n - 1), & t \in [2n+1, 2n+3/2], \\ -2t + 4(n+1), & t \in [2n+3/2, 2n+2]. \end{cases}$$

Clearly, $q(t)$ is a nonnegative continuous function on $[0, \infty)$ and hence $Q(t)$ is a continuous matrix function on $[0, \infty)$. Clearly, $\operatorname{tr} Q(t) = 2q(t)$. Since

$\int_{2n}^{2n+2} q_n(t)\,dt = 1/2$ for $n = 0, 1, 2, \dots$, we have, for $t \in (2n+2, 2n+4]$,

$$\begin{aligned}
\frac{1}{t}\int_0^t\Big(\int_0^s \operatorname{tr} Q(u)\,du\Big)\,ds &= \frac{2}{t}\int_0^t\Big(\int_0^s q(u)\,du\Big)\,ds \\
&= \frac{2}{t}\Big[\sum_{i=1}^{n+1}\int_{2(i-1)}^{2i}\Big(\int_0^s q(u)\,du\Big)\,ds + \int_{2n+2}^{t}\Big(\int_0^s q(u)\,du\Big)\,ds\Big] \\
&\geq \frac{1}{t}\sum_{i=1}^{n+1}\int_{2(i-1)}^{2i}\Big(\int_0^s q(u)\,du\Big)\,ds \\
&\geq \frac{1}{t}\sum_{i=1}^{n+1}\int_{2(i-1)}^{2i}\Big[\int_0^2 q_0(u)\,du + \int_2^4 q_1(u)\,du + \dots + \int_{2(i-2)}^{2(i-1)} q_{i-2}(u)\,du\Big]\,ds \\
&\geq \frac{2}{t}\sum_{i=1}^{n+1}\frac{i-1}{2} = \frac{1}{t}\cdot\frac{n(n+1)}{2} \geq \frac{n(n+1)}{4(n+2)}.
\end{aligned}$$

Thus

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\Big(\int_0^s \operatorname{tr} Q(u)\,du\Big)\,ds = \infty.$$

If

$$f(t) = \begin{cases} 1, & t \in [2n, 2n+1], \\ 0, & t \in (2n+1, 2n+2], \end{cases}$$

for $n = 0, 1, 2, \dots$, then $f(t)$ is a nonnegative, locally integrable function on $[0,\infty)$ such that $\int_0^t f(s)\,ds \not\equiv 0$, and

$$\begin{aligned}
A(t) &= \Big(\int_0^t f(s)\,ds\Big)^{-1}\int_0^t f(s)\Big(\int_0^s \operatorname{tr} Q(u)\,du\Big)\,ds \\
&= \Big(\int_0^t f(s)\,ds\Big)^{-1}\int_0^t f(s)\Big(\int_0^s q(u)\,du\Big)\,ds = 0
\end{aligned}$$

for $t \in (0,\infty)$, due to the definitions of f and q. Further, for $0 < k < 1$, $a > 0$ and $2n < t \leq 2n+2$,

$$\begin{aligned}
\int_a^t f(s)\Big(\int_0^s f^2(u)\,du\Big)^{-1}\Big(\int_0^s f(u)\,du\Big)^k\,ds &= \int_a^1 + \int_2^3 + \dots + \int_{2n-2}^{2n-1} + \int_{2n}^{t} \\
&> \Big(\frac{1}{k} - a^k\Big) + \Big(\frac{2^k}{k} - \frac{1}{k}\Big) + \Big(\frac{3^k}{k} - \frac{2^k}{k}\Big) + \dots + \Big(\frac{n^k}{k} - \frac{(n-1)^k}{k}\Big) \\
&= \frac{n^k}{k} - a^k.
\end{aligned}$$

Hence

$$\int_a^\infty f(t)\Big(\int_0^t f^2(s)\,ds\Big)^{-1}\Big(\int_0^t f(s)\,ds\Big)^k\,dt=\infty.$$

As all the assumptions of Theorem 2.3 are satisfied, the matrix equation

$$(P(t)Y')'+Q(t)Y=0 \tag{2.14}$$

is oscillatory, where $P(t)$ and $Q(t)$ are given by (2.11) and (2.13) respectively. In particular,

$$Y(t)=\begin{bmatrix} y_1(t) & 0\\ 0 & y_2(t)\end{bmatrix}$$

is a nontrivial, prepared, oscillatory solution of (2.14) where $y_1(t)$ and $y_2(t)$ are solutions of

$$\Big(\frac{1}{t+2}y'\Big)'+(q(t)-\cos t)y=0 \tag{2.15}$$

and

$$\Big(\frac{1}{2}y'\Big)'+(q(t)+\cos t)y=0 \tag{2.16}$$

respectively. From Theorem 2.2 it follows that equations (2.15) and (2.16) are oscillatory.

3. Discussion. Since $Q(t)$ is a real symmetric matrix function and $\operatorname{tr}\int_\sigma^t Q(s)\,ds\le n\lambda_1(\int_\sigma^t Q(s)\,ds)$, (H_2) implies (H_7), (H_4) implies (H_9), and (H_5) implies (H_8). These implications hold provided it is possible to determine the largest eigenvalue of $\int_\sigma^t Q(s)\,ds$. Thus it would be interesting to establish the following theorems:

THEOREM 3.1. *If* (H_1), (H_3), (H_6) *and* (H_7) *hold*, *then* (E) *is oscillatory.*

THEOREM 3.2. *If* (H_1), (H_3), (H_6), (H_8) *and* (H_9) *are satisfied*, *then* (E) *is oscillatory.*

Although it appears that Theorems 3.1 and 3.2 are generalizations of Theorems 2.1 and 2.3 respectively, it is really not true in view of the assumption (H_6) which is not required for the proof of the latter theorems. If $C(t)=C$, a real symmetric matrix with constant entries, then (H_6) follows immediately from the natural ordering of real numbers. In fact, (H_6) makes Theorems 2.1 and 2.3 independent of Theorems 3.1 and 3.2. In the following we give some examples to make this point clear. We may note that Theorem 3.1 cannot be applied to Example 1 since (H_6) fails to hold. Indeed, the eigenvalues of $\int_0^t Q(s)\,ds$ are given by $t/2-\sin t$ and $t/2+\sin t$ and these are not comparable. For a similar reason, Theorem 3.2 cannot be applied to Example 2. On the other hand, there are examples to which Theorems 3.1 and 3.2 can be applied but Theorems 2.1 or 2.3 cannot.

EXAMPLE 3. Consider

$$(P(t)Y')' + Q(t)Y = 0, \qquad t \geq 0, \tag{3.1}$$

with

$$P(t) = \frac{1}{2(t^2+1)}\begin{bmatrix} 2 & 0 \\ 0 & t^2+1 \end{bmatrix} \quad \text{and} \quad Q(t) = \begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}.$$

Since $\operatorname{tr} Q(t) = 0$, ($H_2$) fails to hold and hence Theorem 2.1 cannot be applied to (3.1). However, Theorem 3.1 holds for (3.1). Indeed, here

$$P^{-1}(t) - I = \begin{bmatrix} t^2 & 0 \\ 0 & 1 \end{bmatrix} > 0$$

and $\lambda_1(\int_0^t Q(s)\,ds) = t^2/2$ implies, by taking $f(t) = t$, that

$$\Big(\int_0^t f(s)\,ds\Big)^{-1}\int_0^t f(s)\lambda_1\Big(\int_0^s Q(u)\,du\Big)\,ds = \frac{2}{t^2}\int_0^t \frac{s^3}{2}\,ds = \frac{t^2}{4} \to \infty$$

as $t \to \infty$. Moreover, for $a > 0$ and $k = 4/5$, we obtain

$$\int_a^t f(s)\Big(\int_0^s f^2(u)\,du\Big)^{-1}\Big(\int_0^s f(u)\,du\Big)^k\,ds$$
$$= \int_a^t \frac{3}{s^2}\cdot\frac{s^{2k}}{2^k}\,ds = \frac{3}{2^k}\cdot\frac{1}{2k-1}[t^{2k-1} - a^{2k-1}] \to \infty$$

as $t \to \infty$. Thus (H_1), (H_3), (H_6) and (H_7) are satisfied.

Clearly,

$$Y(t) = \begin{bmatrix} y_1(t) & 0 \\ 0 & y_2(t) \end{bmatrix}$$

is a nontrivial, prepared, oscillatory solution of (3.1), where $y_1(t)$ is an oscillatory solution of

$$\Big(\frac{1}{t^2+1}y'\Big)' + ty = 0$$

and $y_2(t)$ is a nonoscillatory solution of

$$\Big(\frac{1}{2}y'\Big)' - ty = 0.$$

EXAMPLE 4. Clearly, all the conditions of Theorem 3.2 are satisfied for the matrix equation

$$(P(t)Y')' + Q(t)Y = 0, \qquad t \geq 0, \tag{3.2}$$

with

$$P(t) = \frac{1}{2(t+2)} \begin{bmatrix} 2 & 0 \\ 0 & t+2 \end{bmatrix} \quad \text{and} \quad Q(t) = \begin{bmatrix} q(t) & 0 \\ 0 & -q(t) \end{bmatrix}$$

where $q(t)$ and $f(t)$ are as in Example 2 and $0 < k < 1$, since $\lambda_1(\int_0^t Q(s)\,ds) = \int_0^t q(s)\,ds$. However, Theorem 2.3 fails to hold for (3.2) as $\operatorname{tr} Q(t) = 0$.

We may note that

$$Y(t) = \begin{bmatrix} y_1(t) & 0 \\ 0 & y_2(t) \end{bmatrix}$$

is a nontrivial, prepared, oscillatory solution of (3.2), where $y_1(t)$ is an oscillatory solution of

$$\left(\frac{1}{t^2+1} y'\right)' + q(t)y = 0,$$

which is oscillatory by Theorem 2.2, and $y_2(t)$ is a nonoscillatory solution of

$$\left(\frac{1}{2} y'\right)' - q(t)y = 0.$$

The proof of Theorems 3.1 and 3.2 will be given elsewhere.

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