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Existence of solutions for a multivalued boundary value problem with non-convex and unbounded right-hand side

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Abstract. Let $F : [a,b]\times\mathbb{R}^n\times\mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a multifunction with possibly non-convex and unbounded values. The main result of this paper (Theorem 1) asserts that, given the multivalued boundary value problem

$$(\mathrm{P}_F)\qquad \begin{cases} u'' \in F(t,u,u'),\\ u(a) = u(b) = \vartheta_{\mathbb{R}^n}, \end{cases}$$

if an appropriate restriction of the multifunction F has non-empty and closed values and satisfies the lower Scorza Dragoni property and a weak integrable boundedness type condition, then we can substitute the problem (P_F) with another one (P_G), with a suitable convex right-hand side G, such that every generalized solution of (P_G) is also a generalized solution of (P_F) (see also Remark 1 and Corollary 1).

As a consequence of our results, in conjunction with those in [13] and [18], some existence theorems for multivalued boundary value problems are then presented (see Theorem 2, Corollary 2 and Theorem 3).

Finally, some applications are given to the existence of generalized solutions for two implicit boundary value problems (Theorems 4–6).

1. Introduction. Let $([a,b],\mathcal{L},\mu)$ be the Lebesgue measure space on the compact real interval $[a,b]$; $\mathbb{R}^n$ the euclidean n-space, whose zero element is denoted by $\vartheta_{\mathbb{R}^n}$; $s\in[1,\infty]$; $W^{2,s}([a,b],\mathbb{R}^n) := \{u : [a,b]\to\mathbb{R}^n \mid u\in C^1([a,b],\mathbb{R}^n),\ u'\in AC([a,b],\mathbb{R}^n),\ u''\in L^s([a,b],\mathbb{R}^n)\}$; $F : [a,b]\times\mathbb{R}^n\times\mathbb{R}^n\to 2^{\mathbb{R}^n}$ a multifunction.

Consider the problem

$$(\mathrm{P}_F)\qquad \begin{cases} u'' \in F(t,u,u'),\\ u(a) = u(b) = \vartheta_{\mathbb{R}^n}. \end{cases}$$

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A function $u : [a,b] \to \mathbb{R}^n$ is said to be a *generalized solution* of the problem (P_F) in $W^{2,s}([a,b],\mathbb{R}^n)$ if $u \in W^{2,s}([a,b],\mathbb{R}^n)$, $u(a) = u(b) = \vartheta_{\mathbb{R}^n}$, and $u''(t) \in F(t,u(t),u'(t))$ a.e. in $[a,b]$.

This paper is arranged as follows. After some notations and preliminary results given in Section 2, in Section 3 we prove our main result (Theorem 1) which states that, if $F(t,x,z)$ is a multifunction, with possibly non-convex and unbounded values, such that an appropriate restriction of F satisfies the lower Scorza Dragoni property and a weak integrable boundedness type condition with a function $m \in L^s([a,b],\mathbb{R}_0^+)$, then there exists another multifunction $G : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$, with non-empty, closed and convex values, such that $G(\cdot,x,z)$ is measurable, $G(t,\cdot,\cdot)$ has closed graph, G is integrably bounded by m, and every generalized solution of the problem

$$(\mathrm{P}_G) \qquad \begin{cases} u'' \in G(t,u,u'), \\ u(a) = u(b) = \vartheta_{\mathbb{R}^n}, \end{cases}$$

in $W^{2,s}([a,b],\mathbb{R}^n)$ is also a generalized solution of (P_F) in $W^{2,s}([a,b],\mathbb{R}^n)$ (see also Remark 1 and Corollary 1).

The technical approach consists in the substitution of the multifunction F with another one H, which is integrably bounded by m and has the lower Scorza Dragoni property, and in the use of Bressan's directional continuous selections ([6]) in order to obtain G by means of a convexification.

In Section 4, some existence theorems for problem (P_F) follow as a simple consequence of our theorems and Theorem 2.1 of [13] (see Theorem 2 and Corollary 2). They both improve Theorem 3 of [8]. Moreover, by using a result of [18] and our Theorem 2, an existence theorem for the problem

$$(\mathrm{P}_{F\circ G}) \qquad \begin{cases} u'' \in F(G(t,u,u')), \\ u(a) = u(b) = \vartheta_{\mathbb{R}^n}, \end{cases}$$

is given (Theorem 3), where the multifunction $F \circ G$ is not required to be lower or upper semicontinuous, and its values can be non-convex, non-closed and unbounded (see also Remark 4).

In Section 5, some applications are given of our results to the existence of generalized solutions in $W^{2,s}([a,b],\mathbb{R}^n)$ for a boundary value problem for second-order implicit equations $f(t,u,u',u'') = 0$. Usually, in the literature, very strong conditions are required for $f(t,u,u',\cdot)$ to assure existence of solutions for such a problem (such as lipschitzianity, with Lipschitz constant strictly less than 1). The first attempt to obtain existence theorems where rather general conditions on the function f with respect to the last variable are required seems to be [14], to which we refer for other bibliographical references.

We give three theorems.

The first one (Theorem 4) is an existence theorem for the boundary value problem

$$(\mathrm{P}^i_f) \qquad \begin{cases} f(t,u,u',u'') = 0, \\ u(a) = u(b) = \vartheta_{\mathbb{R}^n}, \end{cases}$$

where, given a non-empty, connected, locally connected, but possibly non-closed and unbounded subset Y of $\mathbb{R}^n$, $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times Y \to \mathbb{R}$ is a function which, besides other conditions, is continuous in its last variable (for suitable values of (t,u,u')) and satisfies with respect to the other variables a condition weaker than the Scorza Dragoni property.

The second one (Theorem 5) is another existence theorem for the boundary value problem (P^i_f), where Y is a non-empty, bounded, connected and locally connected, but possibly non-closed subset of $\mathbb{R}^n$, and f is again continuous in u''. This theorem, just as Theorem 2.1 of [14], in which Y is also closed, gives existence of solutions in $W^{2,\infty}([a,b],\mathbb{R}^n)$.

The last one (Theorem 6) is an existence theorem for the boundary value problem

$$(\mathrm{P}^i_{f,g}) \qquad \begin{cases} f(u'') = g(t,u,u'), \\ u(a) = u(b) = \vartheta_{\mathbb{R}^n}, \end{cases}$$

where, given a non-empty subset Y of $\mathbb{R}^n$, $f : Y \to \mathbb{R}$ is not required to be continuous, and a suitable restriction of $g : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ has the Scorza Dragoni property. Theorem 6 improves Theorem 2.2 of [14], in which the continuity of f and g is required, Y is a non-empty, compact, connected and locally connected subset of $\mathbb{R}^n$, and only generalized solutions in $W^{2,\infty}([a,b],\mathbb{R}^n)$ can be obtained.

Finally, we give an example which shows that our Theorems 4 and 6 can be used to obtain existence of solutions also for boundary value problems with no solutions in $W^{2,\infty}([a,b],\mathbb{R}^n)$.

2. Notations and preliminaries. Let A, B be two non-empty sets. A *multifunction* $\Phi : A \to 2^B$ is a function from A into the family of all subsets of B. The *graph* of Φ is the set $\mathrm{gr}(\Phi) := \{(a,b) \in A \times B : b \in \Phi(a)\}$. If Ω is a subset of B, we put $\Phi^-(\Omega) := \{a \in A : \Phi(a) \cap \Omega \neq \emptyset\}$ and $\Phi^+(\Omega) := \{a \in A : \Phi(a) \subset \Omega\}$. If C is a non-empty subset of A, we put $\Phi(C) := \bigcup_{c \in C} \Phi(c)$, and we denote by $\Phi_{|C}$ the restriction of Φ to C.

If (A, τ_A) is a topological space and $E \subset A$, then $\mathrm{int}(E)$ and $\overline{E}$ denote, as usual, the interior and the closure of the set E respectively; $\mathcal{B}(A)$ denotes the σ-algebra generated by τ_A.

If (B, τ_B) is a topological space, then $\overline{\Phi}$ denotes the multifunction from A into 2^B defined by $\overline{\Phi}(a) = \overline{\Phi(a)}$.

If $(A, \mathcal{F}_A)$ is a measurable space and (B, τ_B) a topological space, we say that Φ is *measurable* (or $\mathcal{F}_A$-*measurable*) if $\Phi^-(\Omega) \in \mathcal{F}_A$ for every $\Omega \in \tau_B$.

If (A, τ_A) and (B, τ_B) are two topological spaces, we say that Φ is *lower* (resp. *upper*) *semicontinuous* if $\Phi^-(\Omega) \in \tau_A$ (resp. $\Phi^+(\Omega) \in \tau_A$) for every $\Omega \in \tau_B$; Φ is said to be *continuous* if it is simultaneously lower and upper semicontinuous. We say that a multifunction $\Psi : [a, b] \times A \to 2^B$ has the *lower Scorza Dragoni property* if for every $\varepsilon > 0$ there exists a compact set $T_\varepsilon \subset [a, b]$, with $\mu([a, b] \setminus T_\varepsilon) < \varepsilon$, such that $\Psi_{|T_\varepsilon \times A}$ is lower semicontinuous; we say that a function $f : [a, b] \times A \to B$ has the *Scorza Dragoni property* if for every $\varepsilon > 0$ there exists a compact set $T_\varepsilon \subset [a, b]$, with $\mu([a, b] \setminus T_\varepsilon) < \varepsilon$, such that $f_{|T_\varepsilon \times A}$ is continuous.

Let (A, ϱ) be a metric space. For every $a \in A$ and every $r \geq 0$, we denote by $B_\varrho(a, r) := \{a' \in A : \varrho(a, a') \leq r\}$ the closed ball of center a and radius r and by $B^\circ_\varrho(a, r) := \{a' \in A : \varrho(a, a') < r\}$ the corresponding open ball. If $x \in A$ and C is a non-empty subset of A, we put $\varrho(x, A) := \inf\{\varrho(x, c) : c \in C\}$. As usual, when the metric is clear from the context, we use the notations $B(a, r)$ and $B^\circ(a, r)$ respectively.

For all $(t, \sigma) \in [a, b] \times [a, b]$, put

$$K(t,\sigma) := \begin{cases} \dfrac{(b-t)(\sigma-a)}{b-a} & \text{if } a \leq \sigma \leq t \leq b, \\ \dfrac{(b-\sigma)(t-a)}{b-a} & \text{if } a \leq t \leq \sigma \leq b. \end{cases}$$

Lemma 1 (cf. [13]). *If $u \in W^{2,p}([a, b], \mathbb{R}^n)$, $p \in [1, \infty]$, and $u(a) = u(b) = \vartheta_{\mathbb{R}^n}$, then*

$$u(t) = -\int_a^b K(t,\sigma) u''(\sigma)\, d\sigma, \tag{1}$$

$$u'(t) = -\int_a^b \frac{\partial K(t,\sigma)}{\partial t} u''(\sigma)\, d\sigma. \tag{2}$$

To simplify the notations, in the following Lemmas 2 and 3 we assume the indeterminate expressions, when $p = 1$ or $p = \infty$, to be read as $\lim_{p\to 1^+}$ or $\lim_{p\to\infty}$ respectively.

Lemma 2 (cf. [13], Lemma 1.1). *Let $p \in [1, \infty]$. Then, for every $t \in [a, b]$, we have*

$$\|K(t,\cdot)\|_{L^p([a,b],\mathbb{R})} \leq \frac{(b-a)^{1+1/p}}{4(p+1)^{1/p}}, \tag{3}$$

$$\left\| \frac{\partial K(t,\sigma)}{\partial t} \right\|_{L^p([a,b],\mathbb{R})} \leq \frac{(b-a)^{1/p}}{(p+1)^{1/p}}. \tag{4}$$

In the following, $\|\cdot\|$ denotes a fixed norm on $\mathbb{R}^n$ and d the metric induced by $\|\cdot\|$.

LEMMA 3. *If* $u \in W^{2,p}([a,b],\mathbb{R}^n)$, $p \in [1,\infty]$, *and* $u(a) = u(b) = \vartheta_{\mathbb{R}^n}$, *then*, *for every* $t \in [a,b]$, *we have*

$$\|u(t)\| \le \frac{b-a}{4}\left[\frac{(b-a)(p-1)}{2p-1}\right]^{1-1/p}\|u''\|_{L^p([a,b],\mathbb{R}^n)}, \tag{5}$$

$$\|u'(t)\| \le \left[\frac{(b-a)(p-1)}{2p-1}\right]^{1-1/p}\|u''\|_{L^p([a,b],\mathbb{R}^n)}. \tag{6}$$

Moreover, *for every* $t, t^* \in [a,b]$ *with* $a \le t < t^* \le b$, *we have*

$$\|u(t^*) - u(t)\| \le \left[\frac{(b-a)(p-1)}{2p-1}\right]^{1-1/p}\|u''\|_{L^p([a,b],\mathbb{R}^n)}(t^* - t). \tag{7}$$

Proof. By using (1), Hölder's inequality and (3), we obtain

$$\begin{aligned}\|u(t)\| &= \Big\|\int_a^b K(t,\sigma)u''(\sigma)\,d\sigma\Big\| \le \int_a^b |K(t,\sigma)|\cdot\|u''(\sigma)\|\,d\sigma\\ &\le \|K(t,\cdot)\|_{L^{p/(p-1)}([a,b],\mathbb{R})}\|u''\|_{L^p([a,b],\mathbb{R}^n)}\\ &\le \frac{b-a}{4}\left[\frac{(b-a)(p-1)}{2p-1}\right]^{1-1/p}\|u''\|_{L^p([a,b],\mathbb{R}^n)}.\end{aligned}$$

Similarly, by using (2), Hölder's inequality and (4), we obtain

$$\begin{aligned}\|u'(t)\| &= \Big\|\int_a^b \frac{\partial K(t,\sigma)}{\partial t}u''(\sigma)\,d\sigma\Big\| \le \int_a^b \Big|\frac{\partial K(t,\sigma)}{\partial t}\Big|\|u''(\sigma)\|\,d\sigma\\ &\le \Big\|\frac{\partial K(t,\cdot)}{\partial t}\Big\|_{L^{p/(p-1)}([a,b],\mathbb{R})}\|u''\|_{L^p([a,b],\mathbb{R}^n)}\\ &\le \left[\frac{(b-a)(p-1)}{2p-1}\right]^{1-1/p}\|u''\|_{L^p([a,b],\mathbb{R}^n)}.\end{aligned}$$

Finally, (7) is an immediate consequence of (6) and the weak form of the mean value theorem. ■

We recall that, given a set $L \in \mathcal{L}$, a point t is a *point of density* for L if

$$\lim_{\eta\to 0^+}\frac{\mu(L\cap[t-\eta,t+\eta])}{2\eta} = 1.$$

The "density theorem" (cf., for instance, [16], p. 17) asserts that almost every point of a set $L \in \mathcal{L}$ is a point of density for L.

LEMMA 4. *Let* $G : [a,b]\times\mathbb{R}^n\times\mathbb{R}^n \to 2^{\mathbb{R}^n}$. *Let* $E \in \mathcal{L}$, $s \in [1,\infty]$, *and* $u \in W^{2,s}([a,b],\mathbb{R}^n)$ *be such that* $u''(t) \in G(t,u(t),u'(t))$ *a.e. in* E. *Let* T *be the set of all* $t \in E$ *such that*:

1) $u''(t) \in G(t, u(t), u'(t))$;
2) *there exists a strictly decreasing sequence* $(t_j)_j$ *in* E *such that*

$$t_j \xrightarrow{j} t, \qquad u''(t_j) \xrightarrow{j} u''(t), \qquad u''(t_j) \in G(t_j, u(t_j), u'(t_j)).$$

Then $\mu(T) = \mu(E)$.

Proof. Let $T_1 := \{t \in E : u''(t) \in G(t, u(t), u'(t))\}$. By hypothesis, $\mu(T_1) = \mu(E)$.

Since $u'' \in L^s([a,b], \mathbb{R}^n)$, in particular it satisfies the assumption of Lusin's theorem. Thus, for every $\varepsilon > 0$ there exists $T_\varepsilon \subset [a,b]$ such that $\mu(T_\varepsilon) > b - a - \varepsilon$ and $u''_{|T_\varepsilon}$ is continuous.

Put $T_2 := T_1 \cap T_\varepsilon$. Then $\mu(T_2) = \mu(E \cap T_\varepsilon) > \mu(E) - \varepsilon$, $u''_{|T_2}$ is continuous, and $u''(t) \in G(t, u(t), u'(t))$ for every $t \in T_2$.

Let T_3 be the set of all points of T_2 which are points of density for T_2. By the density theorem and the definition of point of density, we obtain $\mu(T_3) = \mu(T_2) > \mu(E) - \varepsilon$, and for every $t \in T_3$ there exists a strictly decreasing sequence $(t_j)_j$ in T_2, such that $t_j \xrightarrow{j} t$. Thus, $T_3 \subset T$, so that $\mu(T) \geq \mu(T_3) > \mu(E) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the conclusion follows. ■

Lemma 5. *Let* (A, τ_A) *be a topological space and* (Y, ϱ) *a metric space. Let* $F : A \to 2^Y$ *be a lower semicontinuous multifunction*, $m : A \to \mathbb{R}_0^+$ *a lower semicontinuous function, and* $y \in Y$. *Then the multifunction* $I_y : A \to 2^Y$ *defined by* $I_y(t) := F(t) \cap B^\circ(y, m(t))$ *is lower semicontinuous.*

Proof. Let Ω be an open subset of Y and $t_0 \in I_y^-(\Omega)$. Then there is $y_0 \in F(t_0) \cap B^\circ(y, m(t_0)) \cap \Omega$. In particular, $\varrho(y_0, y) < m(t_0)$. Let $\delta > 0$ be such that $\varrho(y_0, y) + \delta < m(t_0)$. Obviously, $y_0 \in F(t_0) \cap B^\circ(y_0, \delta) \cap \Omega$. By the hypotheses on F and m, there exists an open neighborhood U of t_0 such that $F(t) \cap B^\circ(y_0, \delta) \cap \Omega \neq \emptyset$ and $\varrho(y_0, y) + \delta < m(t)$ for every $t \in U$. Then, for every $t \in U$, since $B^\circ(y_0, \delta) \subset B^\circ(y, m(t))$, we have $F(t) \cap B^\circ(y, m(t)) \cap \Omega \neq \emptyset$. ■

Lemma 6. *Let* $(A, \mathcal{F}_A)$ *be a measurable space*, (X, τ_X) *a second-countable topological space and* (Y, ϱ) *a metric space in which bounded sets are relatively compact. Let* $G : A \times X \to 2^Y$ *be a multifuction, with non-empty values, such that*:

(i_1) $\overline{G}(t, \cdot)$ *has closed graph for every* $t \in A$;
(i_2) $\{x \in X : G(\cdot, x)$ *is* $\mathcal{F}_A$*-measurable*$\}$ *is dense in* X.

Then, for each $y \in Y$ *and for each* $B \subset X$ *such that* $\overline{B} = \overline{\mathrm{int}(B)} \neq \emptyset$, *the extended real function* $t \mapsto \sup_{x \in B} \varrho(y, G(t, x))$ *is* $\mathcal{F}_A$*-measurable.*

Proof. Let $\{B_i : i \in \mathbb{N}\}$ be a countable base for τ_X. Put $\mathbb{N}_B := \{i \in \mathbb{N} : B_i \cap B \neq \emptyset\}$. By ($i_2$), for each $i \in \mathbb{N}_B$ there exists $x_i \in B_i \cap \mathrm{int}(B)$ such that $\varrho(y, G(\cdot, x_i))$ is $\mathcal{F}_A$-measurable. The countable set $D := \{x_i : i \in \mathbb{N}_B\}$ is dense in B.

The extended real function $t \mapsto \sup_{i\in\mathbb{N}_B} \varrho(y, G(., x_i))$ is $\mathcal{F}_A$-measurable; thus the conclusion follows if we prove that

$$\sup_{x\in B} \varrho(y, G(t, x)) = \sup_{i\in\mathbb{N}_B} \varrho(y, G(t, x_i)) \qquad \text{for every } t \in A.$$

Let $t \in A$. For every $x \in B$ and every $\varepsilon > 0$, by using Proposition 1 of [15] and the density of D in B, there exists $i_0 \in \mathbb{N}_B$ such that

$$\varrho(y, G(t, x)) - \varepsilon < \varrho(y, G(t, x_{i_0})) \le \sup_{i\in\mathbb{N}_B} \varrho(y, G(t, x_i));$$

thus, $\varepsilon > 0$ being arbitrary,

$$\sup_{x\in B} \varrho(y, G(t, x)) \le \sup_{i\in\mathbb{N}_B} \varrho(y, G(t, x_i)).$$

The opposite inequality is obvious. ■

3. Main result. Let $\|\cdot\|_1, \|\cdot\|_2$ be two fixed norms on $\mathbb{R}^n$ (besides the already fixed norm $\|\cdot\|$, whose induced metric we have denoted by d). Define the norm $\|\cdot\|_{\mathbb{R}^n\times\mathbb{R}^n}$ on $\mathbb{R}^n \times \mathbb{R}^n$ by putting, for every $(x, z) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned}\|(x,z)\|_{\mathbb{R}^n\times\mathbb{R}^n} &:= \max\left\{\max\left\{1, \frac{4}{b-a}\right\}\|x\|_1, \max\left\{1, \frac{b-a}{4}\right\}\|z\|_2\right\} \\ &= \begin{cases} \max\left\{\dfrac{4}{b-a}\|x\|_1, \|z\|_2\right\} & \text{if } b-a \le 4, \\ \max\left\{\|x\|_1, \dfrac{b-a}{4}\|z\|_2\right\} & \text{if } b-a > 4. \end{cases}\end{aligned}$$

If c_1, c_2 are two positive constants such that

$$\|x\|_1 \le c_1\|x\| \quad \text{and} \quad \|z\|_2 \le c_2\|z\| \qquad \text{for all } (x, z) \in \mathbb{R}^n \times \mathbb{R}^n,$$

put $\gamma := \gamma(p) := \max\{c_1, c_2\}\gamma'$, where

$$\gamma' := \gamma'(p) := \begin{cases} \max\left\{1, \dfrac{b-a}{4}\right\}\left[\dfrac{(b-a)(p-1)}{2p-1}\right]^{1-1/p} & \text{if } 1 < p < \infty, \\ \lim\limits_{q\to1^+} \gamma'(q) = \max\left\{1, \dfrac{b-a}{4}\right\} & \text{if } p = 1, \\ \lim_{q\to\infty} \gamma'(q) = \max\left\{\dfrac{b-a}{2}, \dfrac{(b-a)^2}{8}\right\} & \text{if } p = \infty. \end{cases}$$

Recall that, if $M > 0$ is given and Γ^M denotes the cone $\{(t, x, z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n : \|(x, z)\|_{\mathbb{R}^n\times\mathbb{R}^n} \le Mt\}$, a function $h : E \to \mathbb{R}^n$, $E \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, is said to be *$\Gamma^M$-continuous* in E if for every $(t, x, z) \in E$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(h(t^*, v, w), h(t, x, z)) < \varepsilon$ for every $(t^*, v, w) \in E$ such that $t < t^* < t + \delta$ and $\|(v, w) - (x, z)\|_{\mathbb{R}^n\times\mathbb{R}^n} \le M(t^* - t)$.

The following is our main result.

THEOREM 1. *Let* $F : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$. *Suppose that there exist* $p, s \in [1, \infty]$ *with* $p \le s$, *a non-negative function* $m \in L^s([a,b], \mathbb{R})$, *and a positive number* $r \ge \|m\|_{L^p([a,b],\mathbb{R})}$ *such that*

(i) $F_{|[a,b]\times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n},\gamma r)}$ *has the lower Scorza Dragoni property*;

(ii) *for almost every* $t \in [a,b]$ *and every* $(x,z) \in B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$, *the set* $F(t,x,z)$ *is closed and* $F(t,x,z) \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t)) \neq \emptyset$.

Then there exists a multifunction $G : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ *with non-empty, closed and convex values such that*

(j) $G(\cdot, x, z)$ *is* $\mathcal{L}$*-measurable for every* $(x,z) \in \mathbb{R}^n \times \mathbb{R}^n$,

(jj) $G(t, \cdot, \cdot)$ *has closed graph for every* $t \in [a,b]$,

(jjj) $G(t,x,z) \subset B(\vartheta_{\mathbb{R}^n}, m(t))$ *for every* $(t,x,z) \in [a,b] \times \mathbb{R}^n \times \mathbb{R}^n$,

and every generalized solution u *of the problem* (P_G) *in* $W^{2,s}([a,b], \mathbb{R}^n)$ *is also a generalized solution of* (P_F) *and satisfies* $\|u''(t)\| \le m(t)$ *a.e. in* $[a,b]$.

Proof. Put $(X, \|\cdot\|_X) := (\mathbb{R}^n \times \mathbb{R}^n, \|\cdot\|_{\mathbb{R}^n\times\mathbb{R}^n})$, $(Y, \|\cdot\|_Y) := (\mathbb{R}^n, \|\cdot\|)$ and denote by ϑ_X and ϑ_Y the zero elements of X and Y respectively. Moreover, identify $(t,(x,z)) \in [a,b] \times X$ with $(t,x,z) \in [a,b] \times \mathbb{R}^n \times \mathbb{R}^n$ and $((x,z),y) \in X \times Y$ with $(x,z,y) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$.

Let N be the set of all $t \in [a,b]$ such that, for some $(x,z) \in B(\vartheta_X, \gamma r)$, the set $F(t,x,z)$ is not closed or $F(t,x,z) \cap B^\circ(\vartheta_Y, m(t)) = \emptyset$. By (ii), we have $\mu(N) = 0$.

Define $H : [a,b] \times X \to 2^Y$ by putting, for every $(t,x,z) \in [a,b] \times X$,

$$H(t,x,z) := \begin{cases} \overline{F(t,x,z) \cap B^\circ(\vartheta_Y, m(t))} & \\ & \text{if } (t,x,z) \in ([a,b] \setminus N) \times B(\vartheta_X, \gamma r), \\ B(\vartheta_Y, m(t)) & \text{if } (t,x,z) \notin ([a,b] \setminus N) \times B(\vartheta_X, \gamma r). \end{cases}$$

We claim that H, which obviously has non-empty and closed values, has the lower Scorza Dragoni property.

For $\varepsilon > 0$ fixed, let T_ε be a compact subset of $[a,b] \setminus N$, with $\mu([a,b] \setminus T_\varepsilon) < \varepsilon$, such that $F_{|T_\varepsilon \times B(\vartheta_X,\gamma r)}$ is lower semicontinuous and $m_{|T_\varepsilon}$ is continuous. Such a set exists since $F_{|[a,b]\times B(\vartheta_X,\gamma r)}$ has the lower Scorza Dragoni property, m satisfies the assumption of Lusin's theorem, and $\mu(N) = 0$.

Let Ω be an open subset of Y. Then

$$\begin{aligned} H_{|T_\varepsilon\times X}{}^-(\Omega) = \{(t,x,z) \in T_\varepsilon \times B(\vartheta_X, \gamma r) : \\ F(t,x,z) \cap B^\circ(\vartheta_Y, m(t)) \cap \Omega \neq \emptyset\} \\ \cup [\{t \in T_\varepsilon : B^\circ(\vartheta_Y, m(t)) \cap \Omega \neq \emptyset\} \times (X \setminus B(\vartheta_X, \gamma r))] \end{aligned}$$

and, as $(t,x,z) \mapsto F(t,x,z) \cap B^\circ(\vartheta_Y, m(t))$ is lower semicontinuous in $T_\varepsilon \times B(\vartheta_X, \gamma r)$ by Lemma 5, and $m_{|T_\varepsilon}$ is continuous, it is simple to show that the last set is open in $T_\varepsilon \times X$. Thus H has the lower Scorza Dragoni property.

Now, by using a standard argument, we can find a sequence $(E_i)_i$, $i = 0, 1, \dots$, of pairwise disjoint subsets of $[a, b]$ such that $[a, b] = \bigcup_{i=0}^{\infty} E_i$, $\mu(E_0) = 0$, and, for every $i = 1, 2, \dots$, E_i is compact, $H_{|E_i \times X}$ is lower semicontinuous and $m_{|E_i}$ is continuous.

For each $i = 1, 2, \dots$, put $m_i := \max\{m(t) : t \in E_i\}$ and choose $M_i > 0$ such that (if $p = 1$ or $p = \infty$, we assume the indeterminate expressions to be read as $\lim_{p\to 1^+}$ or $\lim_{p\to\infty}$ respectively)

$$
(8) \qquad M_i > \max\left\{ c_1 \left[\frac{(b-a)(p-1)}{2p-1}\right]^{1-1/p} r,\ \frac{4c_1}{(b-a)^{1/p}} \left(\frac{p-1}{2p-1}\right)^{1-1/p} r,\right.
$$
$$
\left. c_2(1+m_i),\ \frac{c_2(b-a)}{4}(1+m_i) \right\}.
$$

By Theorem 2.1 of [6], $H_{|E_i \times X}$ has a Γ^{M_i}-continuous selection h_i. Moreover, for $i = 0$, by the axiom of choice, $H_{|E_0 \times X}$ has a selection h_0. Define $h : [a, b] \times X \to Y$ by putting, for every $(t, x, z) \in [a, b] \times X$,

$$h(t, x, z) := h_i(t, x, z) \qquad \text{if } t \in E_i,\ i \in \mathbb{N}.$$

The definition is correct, since the sets E_i, $i = 0, 1, \dots$, are pairwise disjoint and $[a, b] = \bigcup_{i=0}^{\infty} E_i$.

Now, define $G : [a, b] \times X \to 2^Y$ by putting, for every $(t, x, z) \in [a, b] \times X$,

$$G(t, x, z) := \bigcap_{\varepsilon > 0} \overline{\text{co}}\{h(t, v, w) : \|(v, w) - (x, z)\|_X < \varepsilon\},$$

where, as usual, $\overline{\text{co}}$ denotes the closed convex closure operator.

G, obviously, has non-empty, closed and convex values and satisfies (jjj).

Moreover, arguing for example as in [9], pp. 69–70, it can be easily proved that G also satisfies (j) and (jj).

Now, let u be a generalized solution of the problem (P_G) in $W^{2,s}([a, b], Y)$. Obviously, $\|u''(t)\| \le m(t)$ a.e. in $[a, b]$. Thus, in particular,

$$(9) \qquad \|u''\|_{L^p([a,b],Y)} \le r.$$

Let us prove that $u''(t) = h(t, u(t), u'(t))$ a.e. in $[a, b]$ and $(u(t), u'(t)) \in B(\vartheta_X, \gamma r)$ a.e. in $[a, b]$, from which it follows that u is a generalized solution of (P_F).

As the second assertion is an easy consequence of (5) and (6), we only prove the first. Since $b - a = \mu(\bigcup_{i=1}^{\infty} E_i)$, it is sufficient to prove that $u''(t) = h_i(t, u(t), u'(t))$ a.e. in E_i for every $i = 1, 2, \dots$ Let T be the set of all $t \in E_i$ such that:

1) $u''(t) \in G(t, u(t), u'(t))$;

2) there exists a strictly decreasing sequence $(t_j)_j$ in E_i such that

$$t_j \xrightarrow{j} t, \qquad u''(t_j) \xrightarrow{j} u''(t), \qquad u''(t_j) \in G(t_j, u(t_j), u'(t_j)).$$

Then, by Lemma 4, $\mu(T) = \mu(E_i)$.

We prove that, for every $t \in T$, $u''(t) = h_i(t, u(t), u'(t))$.

Fix $\varepsilon > 0$. By the Γ^{M_i}-continuity of $h_{|E_i}$ in $(t, u(t), u'(t))$, there exists $\delta > 0$ such that, for every $(t^*, v, w) \in E_i \times X$ with $t < t^* < t + \delta$ and $\|(v, w) - (u(t), u'(t))\|_X \leq M_i(t^* - t)$, we have $d(h(t^*, v, w), h(t, u(t), u'(t))) < \varepsilon/2$.

Since $t_j \xrightarrow{j} t$, there exists $j_0 \in \mathbb{N}$ such that, for every $j \in \mathbb{N}$ with $j > j_0$, we have

$$t < t_j < t + \delta, \qquad d(u''(t_j), u''(t)) < \varepsilon/2, \qquad u''(t_j) \in G(t_j, u(t_j), u'(t_j))$$

and

$$\left\| \frac{u'(t_j) - u'(t)}{t_j - t} - u''(t) \right\| < 1.$$

Then, for every $j \in \mathbb{N}$ with $j > j_0$, we obtain

$$\begin{aligned} (10) \qquad \|u'(t_j) - u'(t)\| &= \left\| \frac{u'(t_j) - u'(t)}{t_j - t} \right\| (t_j - t) \\ &\leq \left(\left\| \frac{u'(t_j) - u'(t)}{t_j - t} - u''(t) \right\| + \|u''(t)\| \right) (t_j - t) \\ &\leq (1 + m_i)(t_j - t). \end{aligned}$$

Taking into account (7)–(10), it is simple to verify that

$$\|(u(t_j), u'(t_j)) - (u(t), u'(t))\|_X < M_i(t_j - t),$$

hence

$$G(t_j, u(t_j), u'(t_j)) \subset B(h_i(t, u(t), u'(t)), \varepsilon/2),$$

and thus

$$d(u''(t_j), h_i(t, u(t), u'(t))) \leq \varepsilon/2.$$

Therefore, we obtain

$$d(u''(t), h_i(t, u(t), u'(t))) \leq d(u''(t), u''(t_j)) + d(u''(t_j), h_i(t, u(t), u'(t))) < \varepsilon,$$

from which $u''(t) = h(t, u(t), u'(t))$ follows, since ε is arbitrary. ■

Remark 1. In Theorem 1 (and in Theorem 2 below), the hypothesis (ii) can be replaced by

(ii)′ for almost every $t \in [a, b]$ and every $(x, z) \in B(\vartheta_{\mathbb{R}^n \times \mathbb{R}^n}, \gamma r)$, the set $F(t, x, z)$ is closed and $\emptyset \neq F(t, x, z) \subset B(\vartheta_{\mathbb{R}^n}, m(t))$.

The proof differs from that of Theorem 1 only in the definition of H. Under (ii)′ one can use $H : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ defined by putting, for every $(t,x,z)\in[a,b]\times\mathbb{R}^n\times\mathbb{R}^n$,

$$H(t,x,z):=\begin{cases}F(t,x,z) & \text{if } (t,x,z)\in([a,b]\setminus N')\times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n},\gamma r),\\ B(\vartheta_{\mathbb{R}^n},m(t)) & \text{if } (t,x,z)\notin([a,b]\setminus N')\times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n},\gamma r),\end{cases}$$

where N' is the set of all $t\in[a,b]$ such that, for some $(x,z)\in B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n},\gamma r)$, $F(t,x,z)$ is empty or not closed or $F(t,x,z)\not\subset B(\vartheta_{\mathbb{R}^n},m(t))$.

It is not difficult to show that H has non-empty and closed values and has the lower Scorza Dragoni property.

REMARK 2. It is well known that $F_{|[a,b]\times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n},\gamma r)}$ has the lower Scorza Dragoni property if, for example, it is $\mathcal{L}\otimes\mathcal{B}(B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n},\gamma r))$-measurable and lower semicontinuous in (x,z), or if it is $\mathcal{L}$-measurable in t and continuous in (x,z).

There is extensive literature on this topic (see, for example, [2]–[4], [7], [12] and the recent survey [1]).

Also mixed properties of the multifunction guarantee the lower Scorza Dragoni property. We mention here Theorem 2 of [4].

When the multifunction F is weakly integrably bounded by a function $m^*\in L^s([a,b],\mathbb{R})$, the following result is a corollary of Theorem 1.

COROLLARY 1. *Let $F:[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\to 2^{\mathbb{R}^n}$ have the lower Scorza Dragoni property. Suppose that there exist $s\in[1,\infty]$ and a non-negative function $m^*\in L^s([a,b],\mathbb{R})$ such that*

(ii)$''$ *for almost every $t\in[a,b]$ and every $(x,z)\in\mathbb{R}^n\times\mathbb{R}^n$, the set $F(t,x,z)$ is closed and $F(t,x,z)\cap B(\vartheta_{\mathbb{R}^n},m^*(t))\neq\emptyset$.*

Then for each $\lambda>0$ there exists a multifunction $G_\lambda:[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\to 2^{\mathbb{R}^n}$ with non-empty, closed and convex values such that

(j) *$G_\lambda(\cdot,x,z)$ is $\mathcal{L}$-measurable for every $(x,z)\in\mathbb{R}^n\times\mathbb{R}^n$,*

(jj) *$G_\lambda(t,\cdot,\cdot)$ has closed graph for every $t\in[a,b]$,*

(jjj) *$G_\lambda(t,x,z)\subset B(\vartheta_{\mathbb{R}^n},m^*(t)+\lambda)$ for every $(t,x,z)\in[a,b]\times\mathbb{R}^n\times\mathbb{R}^n$,*

and every generalized solution u of the problem (P_{G_λ}) in $W^{2,s}([a,b],\mathbb{R}^n)$ is also a generalized solution of (P_F) and satisfies $\|u''(t)\|\le m^*(t)+\lambda$ a.e. in $[a,b]$.

P r o o f. Fix $\lambda>0$. Then F satisfies (ii) of Theorem 1 with $m:=m^*+\lambda$, $p=s$, and $r:=\|m^*+\lambda\|_{L^p([a,b],\mathbb{R})}$. ∎

4. Existence. In this section $\|\cdot\|$, d, $\|\cdot\|_{\mathbb{R}^n\times\mathbb{R}^n}$ and γ are as at the beginning of Section 3.

The following existence theorem follows at once from Theorem 1, Lemma 6 and Theorem 2.1 of [13].

Theorem 2. *Let F be a multifunction as in Theorem* 1 (*in which* (ii) *or* (ii)$'$ *can be used*; *see Remark* 1). *Then the problem* (P_F) *has at least one generalized solution $u \in W^{2,s}([a,b],\mathbb{R}^n)$ such that $\|u''(t)\| \le m(t)$ a.e. in $[a,b]$.*

Proof. Let G be the multifunction whose existence has been stated in Theorem 1, and $r \ge \|m\|_{L^p([a,b],\mathbb{R})}$ a positive number. By Lemma 6, $t \mapsto \sup\{d(\vartheta_{\mathbb{R}^n}, G(t,x,z)) : (x,z) \in B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)\}$ is $\mathcal{L}$-measurable. Thus, by (ii) and as $d(\vartheta_{\mathbb{R}^n}, G(t,x,z)) \le m(t)$ for every $(t,x,z) \in [a,b]\times\mathbb{R}^n\times\mathbb{R}^n$, it follows that $t \mapsto \sup\{d(\vartheta_{\mathbb{R}^n}, G(t,x,z)) : (x,z) \in B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)\}$ belongs to $L^s([a,b],\mathbb{R})$ and its norm in $L^p([a,b],\mathbb{R})$ is less than or equal to r.

Hence, we can use Theorem 2.1 of [13] to obtain a generalized solution u of (P_G) in $W^{2,s}([a,b],\mathbb{R}^n)$, which, by Theorem 1, is also a generalized solution of (P_F) such that $\|u''(t)\| \le m(t)$ a.e. in $[a,b]$. ∎

Corollary 2. *Let F be as in Corollary* 1. *Then for each $\lambda > 0$ the problem* (P_F) *has at least one generalized solution $u_\lambda \in W^{2,s}([a,b],\mathbb{R}^n)$ such that $\|u''_\lambda(t)\| \le m^*(t) + \lambda$ a.e. in $[a,b]$.*

Remark 3. Theorem 2 and Corollary 2 both improve Theorem 3 of [8], in which F has non-empty and compact values and is measurable in t, Hausdorff continuous in (x,z) and integrably bounded.

The following existence theorem is a consequence of our Theorem 2 and Theorem 11 of [18].

Theorem 3. *Let I be a non-empty subset of $\mathbb{R}$ and $F : I \to 2^{\mathbb{R}^n}$ a multifunction with non-empty and closed values such that*:

(α) $\mathrm{gr}(F)$ *is connected and locally connected*;

($\alpha\alpha$) *for every open set $\Omega \subset \mathbb{R}^n$, the set $F^-(\Omega) \cap \mathrm{int}(I)$ has no isolated points.*

Moreover, let $G : [a,b]\times\mathbb{R}^n\times\mathbb{R}^n \to 2^{\mathbb{R}}$ be a multifunction with non-empty values, $p, s \in [1,\infty]$, with $p \le s$, $m \in L^s([a,b],\mathbb{R})$ a non-negative function, and $r \ge \|m\|_{L^p([a,b],\mathbb{R})}$ a positive number such that:

(β) $G_{|[a,b]\times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n},\gamma r)}$ *has the lower Scorza Dragoni property*;

($\beta\beta$) *for almost every $t \in [a,b]$ and every $(x,z) \in B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$, the set $G(t,x,z)$ is a compact subset of I and $G(t,x,z) \cap F^+(B^\circ(\vartheta_{\mathbb{R}^n}, m(t))) \ne \emptyset$.*

Then there exists a generalized solution $u \in W^{2,s}([a,b],\mathbb{R}^n)$ of the problem

$$(\mathrm{P}_{F\circ G}) \qquad \begin{cases} u'' \in F(G(t,u,u')), \\ u(a) = u(b) = \vartheta_{\mathbb{R}^n}, \end{cases}$$

such that $\|u''(t)\| \le m(t)$ a.e. in $[a,b]$.

Proof. Thanks to our assumptions on F, we can apply Theorem 11 of [18]. Hence, there exist $\Phi_1, \Phi_2 : I \to 2^{\mathbb{R}^n}$ such that Φ_1 is lower semicontinuous, Φ_2 is upper semicontinuous with compact values, and $\emptyset \neq \Phi_1(v) \subset \Phi_2(v) \subset F(v)$ for every $v \in I$.

Let $N_0 \subset [a,b]$ with $\mu(N_0) = 0$ be such that, for every $(t,x,z) \in ([a,b] \setminus N_0) \times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$, the set $G(t,x,z)$ is a compact subset of I and $G(t,x,z) \cap F^+(B^\circ(\vartheta_{\mathbb{R}^n}, m(t))) \neq \emptyset$.

For every $(t,x,z) \in [a,b] \times \mathbb{R}^n \times \mathbb{R}^n$, put

$$\Gamma(t,x,z) := \begin{cases} \overline{\Phi_1(G(t,x,z))} & \text{if } (t,x,z) \in ([a,b] \setminus N_0) \times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r), \\ \mathbb{R}^n & \text{if } (t,x,z) \notin ([a,b] \setminus N_0) \times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r). \end{cases}$$

Obviously, the values of Γ are non-empty and closed and it is simple to see that Γ has the lower Scorza Dragoni property.

Moreover, for every $t \in ([a,b] \setminus N_0) \times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$, we have

$$\Gamma(t,x,z) \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t)) \neq \emptyset.$$

In fact, we have

$$G(t,x,z) \cap \Phi_1^+(B^\circ(\vartheta_{\mathbb{R}^n}, m(t))) \neq \emptyset,$$

hence

$$G(t,x,z) \cap \Phi_1^-(B^\circ(\vartheta_{\mathbb{R}^n}, m(t))) \neq \emptyset,$$

and then

$$\Phi_1(G(t,x,z)) \cap (B^\circ(\vartheta_{\mathbb{R}^n}, m(t))) \neq \emptyset,$$

which is equivalent to

$$\overline{\Phi_1(G(t,x,z))} \cap (B^\circ(\vartheta_{\mathbb{R}^n}, m(t))) \neq \emptyset.$$

By Theorem 2 (with the hypothesis (ii)) applied to Γ, there exists $u \in W^{2,s}([a,b], \mathbb{R}^n)$ such that

$$\begin{cases} u''(t) \in \Gamma(t, u(t), u'(t)) & \text{for a.e. } t \in [a,b], \\ u(a) = u(b) = \vartheta_{\mathbb{R}^n}, \end{cases}$$

and $\|u''(t)\| \leq m(t)$ a.e. in $[a,b]$. The function u is our solution.

In fact, by (5) and (6), we have $(u(t), u'(t)) \in B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$ for every $t \in [a,b]$. Then

$$\begin{cases} u''(t) \in \overline{\Phi_1(G(t, u(t), u'(t)))} & \text{for a.e. } t \in [a,b], \\ u(a) = u(b) = \vartheta_{\mathbb{R}^n}, \end{cases}$$

and, for almost every $t \in [a,b]$, $G(t, u(t), u'(t))$ is a compact subset of I, hence, by Theorem 2.1 of [11], $\Phi_2(G(t, u(t), u'(t)))$ is compact.

Thus, for almost every $t \in [a,b]$, we have

$$\Phi_1(G(t, u(t), u'(t))) \subset \Phi_2(G(t, u(t), u'(t))) = \overline{\Phi_2(G(t, u(t), u'(t)))},$$

therefore

$$\overline{\Phi_1(G(t,u(t),u'(t)))} \subset F(G(t,u(t),u'(t))),$$

from which the conclusion follows. ∎

REMARK 4. In Theorem 3 the hypothesis $(\beta\beta)$ can be replaced by

$(\beta\beta)'$ for almost every $t \in [a,b]$ and every $(x,z) \in B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$, the set $G(t,x,z)$ is a compact subset of I and $F(G(t,x,z)) \subset B(\vartheta_{\mathbb{R}^n}, m(t))$.

In fact the multifunction Γ defined in the proof of Theorem 3 satisfies the assumptions of Theorem 2 with (ii)$'$ in place of (ii).

REMARK 5. An existence result for the Cauchy problem for a first order differential inclusion with right-hand side of the type $F\circ G$ has recently been given in [5].

5. Applications. In this section, we give some applications of our results to the existence of solutions for a boundary value problem for second-order implicit equations. $\|\cdot\|$, $\|\cdot\|_{\mathbb{R}^n\times\mathbb{R}^n}$ and γ are as at the beginning of Section 3.

THEOREM 4. *Let Y be a non-empty, connected and locally connected subset of $\mathbb{R}^n$ and $f : [a,b]\times\mathbb{R}^n\times\mathbb{R}^n\times Y \to \mathbb{R}$. Assume that there exist $p, s \in [1,\infty]$, with $p \le s$, a non-negative function $m \in L^s([a,b],\mathbb{R})$, and a positive number $r \ge \|m\|_{L^p([a,b],\mathbb{R})}$, such that:*

(k) *for almost every $t \in [a,b]$ and every $(x,z) \in B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$, the function $f(t,x,z,\cdot)$ is continuous, $0 \in \operatorname{int}(f(t,x,z,Y\cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t))))$, and $\{y \in Y : f(t,x,z,y) = 0\}$ has empty interior in Y;*

(kk) *for every $\varepsilon > 0$ there exists a compact set $T_\varepsilon \subset [a,b]$ with $\mu([a,b]\setminus T_\varepsilon) < \varepsilon$ and a set $D_\varepsilon \subset Y\times Y$ with $\overline{D}_\varepsilon \supset Y\times Y$ such that, for every $(y',y'') \in D_\varepsilon$, the set $\{(t,x,z) \in T_\varepsilon \times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r) : f(t,x,z,y') < 0 < f(t,x,z,y'')\}$ is open in $T_\varepsilon \times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$;*

(kkk) *for almost every $t \in [a,b]$, the set $Y \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t))$ is connected.*

Then the problem

$$(\mathrm{P}^i_f) \qquad \begin{cases} f(t,u,u',u'') = 0, \\ u(a) = u(b) = \vartheta_{\mathbb{R}^n}, \end{cases}$$

has at least one generalized solution u in $W^{2,s}([a,b],\mathbb{R}^n)$ such that $\|u''(t)\| \le m(t)$ a.e. in $[a,b]$.

Proof. Define $Q : [a,b]\times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r) \to 2^Y$ by putting, for every $(t,x,z) \in [a,b]\times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$,

$$Q(t,x,z) := \{y \in Y : f(t,x,z,y) = 0,\ y \text{ is not a local extremum point for } f(t,x,z,\cdot)\}.$$

For every $\varepsilon > 0$, let T_ε be a compact subset of $[a,b]$ as in (kk) such that, for every $(t,x,z) \in T_\varepsilon \times B(\vartheta_{\mathbb{R}^n \times \mathbb{R}^n}, \gamma r)$, the function $f(t,x,z,\cdot)$ is continuous, $0 \in \operatorname{int}(f(t,x,z,Y \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t))))$ and the set $\{y \in Y : f(t,x,z,y) = 0\}$ has empty interior in Y.

By Théorème 1.1 of [17], $Q_{|T_\varepsilon \times B(\vartheta_{\mathbb{R}^n \times \mathbb{R}^n}, \gamma r)}$ (has non-empty and closed values (in Y) and) is lower semicontinuous. Thus Q has the lower Scorza Dragoni property.

We claim that, for almost every $t \in [a,b]$ and every $(x,z) \in B(\vartheta_{\mathbb{R}^n \times \mathbb{R}^n}, \gamma r)$, the set $Q(t,x,z)$ is closed and $Q(t,x,z) \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t)) \neq \emptyset$.

Let T be the set of all $t \in [a,b]$ such that $Y \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t))$ is connected and such that, for every $(x,z) \in B(\vartheta_{\mathbb{R}^n \times \mathbb{R}^n}, \gamma r)$, the function $f(t,x,z,\cdot)$ is continuous, $0 \in \operatorname{int}(f(t,x,z,Y \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t))))$, and $\{y \in Y : f(t,x,z,y) = 0\}$ has empty interior in Y and, thus, also in $Y \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t))$. Clearly, $\mu(T) = b - a$.

Let $(t,x,z) \in T \times B(\vartheta_{\mathbb{R}_n \times \mathbb{R}^n}, \gamma r)$.

If $\overline{y} \in \overline{Q(t,x,z)}$, then $f(t,x,z,\overline{y}) = 0$ since $f(t,x,z,\cdot)$ is continuous; moreover, for every open neighborhood Ω of $\overline{y}$, there is $y^* \in Q(t,x,z) \cap \Omega$. Thus, since y^* is not a local extremum point for $f(t,x,z,\cdot)$ and $f(t,x,z,\overline{y}) = f(t,x,z,y^*) = 0$, also $\overline{y}$ is not a local extremum point for $f(t,x,z,\cdot)$, that is, $\overline{y} \in Q(t,x,z)$. Hence $Q(t,x,z)$ is closed.

Let $y \in Y \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t))$ be such that $f(t,x,z,y) = 0$. If y is not a local extremum point for $f(t,x,z,\cdot)$, then $y \in Q(t,x,z) \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t))$. If y is a local extremum point for $f(t,x,z,\cdot)$, then, by Lemma 3.1 of [19], there exists another point $y^* \in Y \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t))$ such that $f(t,x,z,y^*) = 0$ and y^* is not a local extremum point for $f(t,x,z,\cdot)$, that is, $y^* \in Q(t,x,z) \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t))$. Hence, $Q(t,x,z) \cap B^\circ(\vartheta_{\mathbb{R}^n}, m(t))$ is non-empty and the claim is proved.

Finally, define $F : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ by putting, for every $(t,x,z) \in [a,b] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$F(t,x,z) := \begin{cases} Q(t,x,z) & \text{if } (x,z) \in B(\vartheta_{\mathbb{R}^n \times \mathbb{R}^n}, \gamma r), \\ \mathbb{R}^n & \text{if } (x,z) \notin B(\vartheta_{\mathbb{R}^n \times \mathbb{R}^n}, \gamma r). \end{cases}$$

F satisfies the hypotheses of Theorem 2. Thus, (P_F) has at least one generalized solution u in $W^{2,s}([a,b], \mathbb{R}^n)$ such that $\|u''(t)\| \leq m(t)$ a.e. in $[a,b]$. Taking into account (5), (6), it is simple to show that $(u(t), u'(t)) \in B(\vartheta_{\mathbb{R}^n \times \mathbb{R}^n}, \gamma r)$ for every $t \in [a,b]$, so that $u''(t) \in Q(t, u(t), u'(t))$ a.e. in $[a,b]$, that is, $f(t, u(t), u'(t), u''(t)) = 0$ a.e. in $[a,b]$. ∎

In the following Theorem 5, we put $p = s = \infty$ and $\gamma := \gamma(\infty)$.

Theorem 5. *Let Y be a non-empty, connected and locally connected subset of $\mathbb{R}^n$, and $f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times Y \to \mathbb{R}$. Assume that there exists $r > 0$ such that $Y \subset B(\vartheta_{\mathbb{R}^n}, r)$ and:*

(k)′ *for almost every* $t \in [a,b]$ *and every* $(x,z) \in B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$, *the function* $f(t,x,z,\cdot)$ *is continuous*, $0 \in \mathrm{int}(f(t,x,z,Y))$, *and* $\{y \in Y : f(t,x,z,y) = 0\}$ *has empty interior in* Y;

(kk) *for every* $\varepsilon > 0$ *there exists a compact set* $T_\varepsilon \subset [a,b]$ *with* $\mu([a,b]\setminus T_\varepsilon) < \varepsilon$ *and a set* $D_\varepsilon \subset Y \times Y$ *with* $\overline{D}_\varepsilon \supset Y \times Y$ *such that*, *for every* $(y', y'') \in D_\varepsilon$, *the set* $\{(t,x,z) \in T_\varepsilon \times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r) : f(t,x,z,y') < 0 < f(t,x,z,y'')\}$ *is open in* $T_\varepsilon \times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$.

Then the problem (P^i_f) *has at least one generalized solution* u *in the space* $W^{2,\infty}([a,b],\mathbb{R}^n)$ *such that* $\|u''(t)\| \le r$ *a.e. in* $[a,b]$.

Proof. Put $m(t) := r$ for every $t \in [a,b]$, define Q and F as in Theorem 4, and use Theorem 2 with (ii)′ instead of (ii). ■

Remark 6. In Theorems 4 and 5, the hypothesis (kk) is satisfied, in particular, when $f(\cdot,\cdot,\cdot,y)_{|[a,b]\times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n},\gamma r)}$ has the Scorza Dragoni property for every y in a dense subset of Y.

We observe that the hypothesis (kk) in Theorem 5 could be substituted with (ii) and (iii) of Theorem 2.1 of [14]; in fact, with these hypotheses the multifunction Q is $\mathcal{L} \otimes \mathcal{B}(B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r))$-measurable and lower semicontinuous in (x,z), thus (see Remark 2) it has the lower Scorza Dragoni property. In any case, in Theorem 2.1 of [14] the set Y is also compact.

Theorem 6. *Let* Y *be a non-empty subset of* $\mathbb{R}^n$ *and* $f : Y \to \mathbb{R}$ *such that*:

(α) $\mathrm{gr}(f)$ *is connected and locally connected*;

($\alpha\alpha$)′ *for every* $v \in \mathrm{int}(f(Y))$, *the set* $f^{-1}(v)$ *has empty interior in* Y;

($\alpha\alpha\alpha$) *for every* $v \in f(Y)$, *the set* $f^{-1}(v)$ *is closed in* $\mathbb{R}^n$.

Moreover, *let* $g : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $p, s \in [1,\infty]$, *with* $p \le s$, $m \in L^s([a,b],\mathbb{R})$ *a non-negative function*, *and* $r \ge \|m\|_{L^p([a,b],\mathbb{R})}$ *a positive number such that*:

(β) $g_{|[a,b]\times B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n},\gamma r)}$ *has the Scorza Dragoni property*;

($\beta\beta$)′ *for almost every* $t \in [a,b]$ *and every* $(x,z) \in B(\vartheta_{\mathbb{R}^n\times\mathbb{R}^n}, \gamma r)$, *we have* $\emptyset \ne f^{-1}(g(t,x,z)) \subset B(\vartheta_{\mathbb{R}^n}, m(t))$.

Then the problem

$$(\mathrm{P}^i_{f,g}) \qquad \begin{cases} f(u'') = g(t,u,u'), \\ u(a) = u(b) = \vartheta_{\mathbb{R}^n}, \end{cases}$$

has at least one generalized solution u *in* $W^{2,s}([a,b],\mathbb{R}^n)$ *such that* $\|u''(t)\| \le m(t)$ *a.e. in* $[a,b]$.

Proof. Put $I := f(Y)$, $F(v) := f^{-1}(v)$ for every $v \in I$, and $G(t,x,z) := \{g(t,x,z)\}$ for every $(t,x,z) \in [a,b] \times \mathbb{R}^n \times \mathbb{R}^n$.

The multifunctions F and G satisfy all the assumptions of Theorem 3, in which $(\beta\beta)'$ is used instead of $(\beta\beta)$.

This follows easily in the particular case when f is constant.

If f is not constant, then we can suppose that $\operatorname{int}(f(Y))$ is a non-empty open interval, and the only thing to prove is $(\alpha\alpha)$, which is equivalent to saying that, for every open set $\Omega \subset \mathbb{R}^n$, the set $f(\Omega \cap Y)$ has no isolated points. Suppose, on the contrary, that there exist $x_0 \in \Omega \cap Y$ and $\varepsilon > 0$ such that $]f(x_0) - \varepsilon, f(x_0) + \varepsilon[\cap f(\Omega \cap Y) = \{f(x_0)\}$. Let Ω' and Ω'' be open subsets of $\mathbb{R}^n$ such that $x_0 \in \Omega' \subset \overline{\Omega'} \subset \Omega'' \subset \overline{\Omega''} \subset \Omega$. Taking into account $(\alpha\alpha)'$, it is simple to verify that the sets $((\Omega' \cap Y) \times]f(x_0) - \varepsilon/3, f(x_0) + \varepsilon/3[) \cap \operatorname{gr}(f)$ and $[(Y \times \mathbb{R}) \setminus ((\overline{\Omega''} \cap Y) \times [f(x_0) - \varepsilon/2, f(x_0) + \varepsilon/2])] \cap \operatorname{gr}(f)$ are open in $\operatorname{gr}(f)$ and form a partition of $\operatorname{gr}(f)$, which is a contradiction.

Thus, the problem $(\mathrm{P}_{F\circ G})$ has a generalized solution u in $W^{2,s}([a, b], \mathbb{R}^n)$ such that $\|u''(t)\| \le m(t)$ a.e. in $[a, b]$. The function u is solution. ∎

Remark 7. We point out that, as the example on p. 227 of [18] shows, there are discontinuous functions f satisfying the hypotheses (α), $(\alpha\alpha)'$ and $(\alpha\alpha\alpha)$ of Theorem 6.

Remark 8. Theorem 6 improves Theorem 2.2 of [14], in which the continuity of f and g is required, Y is a non-empty, compact, connected and locally connected subset of $\mathbb{R}^n$ and generalized solutions in $W^{2,\infty}([a, b], \mathbb{R}^n)$ can only be obtained.

Finally, we stress that Theorems 4 and 6 can give existence of solutions also for boundary value problems with no solutions in $W^{2,\infty}([a, b], \mathbb{R}^n)$ as the following example shows.

Example 1. Consider the following boundary value problem:

$$(\mathrm{P}) \qquad \begin{cases} u''(2 + \sin u'') = \dfrac{1}{4\sqrt{t}}\left(|u| + \dfrac{3}{4}\right)\left(\dfrac{|u'| + 1}{2}\right), \\ u(0) = u(1) = \vartheta_{\mathbb{R}}. \end{cases}$$

Put $\|\cdot\| = \|\cdot\|_1 = \|\cdot\|_2 = |\cdot|$ and $c_1 = c_2 = 1$. Theorem 4 or Theorem 6 can be used to prove existence of generalized solutions in $W^{2,1}([0, 1], \mathbb{R})$.

In fact, put $Y := \mathbb{R}$, $p := s := 1$,

$$m(t) := \begin{cases} 1/(2\sqrt{t}) & \text{if } t \in]0, 1], \\ 0 & \text{if } t = 0, \end{cases}$$

and $r := \|m\|_{L^1([0,1],\mathbb{R})}$.

It is not difficult to verify that Theorem 4 can be used if we define, for every $(t, x, z, y) \in [0, 1] \times \mathbb{R} \times \mathbb{R} \times Y$,

$$f(t, x, z, y) := \begin{cases} y(2 + \sin y) - \dfrac{1}{2\sqrt{t}}\left(|u| + \dfrac{3}{4}\right)\left(\dfrac{|u'| + 1}{2}\right) & \text{if } t \in]0, 1], \\ y(2 + \sin y) & \text{if } t = 0. \end{cases}$$

In a similar way, Theorem 6 can be used if we define $f(y) := y(2+\sin y)$ for every $y \in Y$ and we put, for every $(t,x,z) \in [0,1] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$g(t,x,z) := \begin{cases} \dfrac{1}{2\sqrt{t}}\left(|u| + \dfrac{3}{4}\right)\left(\dfrac{|u'|+1}{2}\right) & \text{if } t \in \,]\,0,1\,], \\ 0 & \text{if } t = 0. \end{cases}$$

Nevertheless, $\lim_{t\to 0^+} u''(t) = \infty$ for every generalized solution of problem (P), thus problem (P) has no generalized solutions in $W^{2,\infty}([0,1],\mathbb{R})$.

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