Completeness of the Bergman metric on non-smooth pseudoconvex domains

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Abstract. We prove that the Bergman metric on domains satisfying condition **S** is complete. This implies that any bounded pseudoconvex domain with Lipschitz boundary is complete with respect to the Bergman metric. We also show that bounded hyperconvex domains in the plane and convex domains in $\mathbb{C}^n$ are Bergman comlete.

1. Introduction. Let $D \subset \mathbb{C}^n$ be a bounded domain and let $K_D(z, w)$ be the Bergman kernel. The Bergman metric on D is defined as follows:

$$ds_D^2 = \sum_{j,k=1}^{n} \frac{\partial^2 \log K_D(z, z)}{\partial z_j \partial \overline{z}_k} dz_j d\overline{z}_k.$$

In [11] Kobayashi posed an interesting question:

Which bounded pseudoconvex domains in $\mathbb{C}^n$ are complete with respect to the Bergman metric?

The assumption of pseudoconvexity is necessary (cf. [4]). It is quite clear that any bounded pseudoconvex domain with C^∞-boundary is complete w.r.t. ds_D^2 (cf. [14]). In [13] Ohsawa has proved that the Bergman metric of any pseudoconvex domain with C^1-boundary is complete. In this article, we first study a class of pseudoconvex domains defined as follows:

DEFINITION. We say that a domain D in $\mathbb{C}^n$ *satisfies condition* **S** if there exists a sequence $\{D_j\}$ of pseudoconvex domains with $D \subset\subset D_j$ such that

(1) $\Lambda_j = \sup_{z \in \partial D} d_{D_j}(z) \to 0$ as $j \to \infty$, where $d_{D_j}(z) = d(z, \partial D_j)$ is the Euclidian distance from z to ∂D_j;

(2) there exist reals $r \geq 1$ and $0 < \alpha \leq 1$ such that $\Lambda_j \leq r\lambda_j^\alpha$, where $\lambda_j = \inf_{z \in \partial D} d_{D_j}(z)$.

We say that D *satisfies condition* **S** *locally* if for each $z^0 \in \partial D$, there exists a ball $B(z^0, r_0)$ such that $D \cap B(z^0, r_0)$ satisfies condition **S**.

One can easily conclude that D is pseudoconvex if D satisfies condition **S** locally.

We have the following main result:

THEOREM 1.1. *Let D be a domain which satisfies condition **S** locally. Then D is complete w.r.t. the Bergman metric.*

From Theorem 1.1 we can deduce

THEOREM 1.2. *The Bergman metric of any bounded pseudoconvex domain with Lipschitz boundary is complete.*

It is known from [5] that any bounded pseudoconvex domain with Lipschitz boundary is hyperconvex, that is, it admits a bounded continuous plurisubharmonic (psh) exhaustion function. Naturally, one would ask:

(a) *Is any bounded hyperconvex domain in $\mathbb{C}^n$ complete w.r.t. the Bergman metric?*

(b) *Are the hyperconvexity and the Bergman completeness equivalent?*

In Section 5 we will construct a domain which is not hyperconvex but complete w.r.t. the Bergman metric. Question (a) seems to be quite difficult. However, we can prove that the answer is affirmative for some special cases.

Given a domain $D \subset \mathbb{C}^n$ and $\zeta \in D$, we consider the function

$$u_D(\zeta, z) = u_\zeta(z) = \sup\{v(z) \mid v \text{ is psh on } D,$$
$$v < 0 \text{ and } v(w) \leq \log|w - \zeta| + O(1), w \in D\},$$

which is the pluricomplex Green function on D with logarithmic pole at ζ (cf. [1], [5]). We obtain another main result:

THEOREM 1.3. *Let D be a bounded hyperconvex domain in $\mathbb{C}^n$ and suppose that $u_D(\zeta, z)$ is symmetric. Then the Bergman metric on D is complete.*

Since the complex Green function of hyperconvex domains in $\mathbb{C}$ is symmetric (cf. [12]), Theorem 1.3 implies

COROLLARY 1.4. *Any bounded hyperconvex domain in $\mathbb{C}$ is complete w.r.t. the Bergman metric.*

It is known from [3] that any bounded convex domain in $\mathbb{C}^n$ is hyperconvex, and the pluricomplex Green function is symmetric (cf. [15]). Immediately we obtain

COROLLARY 1.5. *The Bergman metric of any bounded convex domain in $\mathbb{C}^n$ is complete.*

The proofs of the theorems are based on the techniques of L^2-estimates for the $\overline{\partial}$-equation on complete Kähler manifolds due to Diederich and Ohsawa.

2. Preliminaries. Let X be a complete manifold of dimension n and ds^2 be a Hermitian metric on X. Let $C_0^{p,q}(X)$ be the set of C^∞-differentiable (p, q)-forms on X with compact support, and let $\varphi : X \to \mathbb{R}$ be a continuous function. We define an inner product by

$$(u, v)_{ds^2, \varphi} := \int_X e^{-\varphi} u \wedge *_{ds^2} \overline{v}$$

for $u, v \in C_0^{p,q}(X)$. Here $*_{ds^2}$ denotes the Hodge star operator associated with ds^2. Put

$$\|u\|_{ds^2, \varphi} := (u, u)_{ds^2, \varphi}^{1/2}$$

and denote by $L^{p,q}(X, ds^2, \varphi)$ the space of all square integrable (p, q)-forms on X, i.e., the completion of $C_0^{p,q}(X)$ with respect to the norm $\|\cdot\|_{ds^2, \varphi}$. Now we recall the following useful result:

PROPOSITION 2.1 (cf. Theorem 3 in [6]). *Let X be a complex manifold that admits a complete Kähler metric with a positive C^∞ global potential function η, and let $\psi : X \to \mathbb{R}$ be another C^∞ strictly psh function on X satisfying the estimate $\partial \overline{\partial} \psi \geq \partial \psi \overline{\partial} \psi$. Furthermore, let φ be any C^∞ psh function on X. Then, for any $\overline{\partial}$-closed $(n, 1)$-form g on X satisfying $\|g\|_{\partial \overline{\partial} \psi, \varphi} < \infty$, there is a measurable $(n, 0)$-form h satisfying $\overline{\partial} h = g$ and $\|h\|_\varphi \leq C\|g\|_{\partial \overline{\partial} \psi, \varphi}$, where C is a numerical constant (independent of X, ψ, φ, g) and $\|h\|_\varphi^2 = |\int_X e^{-\varphi} h \wedge \overline{h}|$.*

3. Proofs of Theorems 1.1–1.2. Let D be a bounded domain in $\mathbb{C}^n$. We denote by $H^2(D)$ the space of all square integrable holomorphic functions on D and by $H(\overline{D})$ the space of all functions holomorphic in a neighbourhood of $\overline{D}$. The L^2-norm is denoted by $\|\cdot\|_D$, and $\|f\|_{D, \varphi}$ means the L^2-norm of f with weight $e^{-\varphi}$, where φ is a continuous real function on D. We claim

LEMMA 3.1. *Let D be a domain satisfying condition $\mathbf{S}$. Then $H(\overline{D})$ is dense in $H^2(D)$.*

Proof. Let $f \in H^2(D)$. Without loss of generality we can assume $\|f\|_D \leq 1$. We define $D_{j,t} = \{z \in D_j \mid d_{D_j}(z) > t\}$, $t > 0$. Then $D_{j,t}$ is pseudoconvex since $-\log d_{D_j}(z)/t$ is a psh exhaustion function on $D_{j,t}$, and we have

$$D_{j, \Lambda_j} \subset D \subset\subset D_{j,t} \quad \text{for all } t < \lambda_j.$$

Put

$$\beta = \begin{cases} 1/2, & \alpha = 1, \\ \dfrac{1-\alpha}{1-\alpha/2}, & 0 < \alpha < 1. \end{cases}$$

Let $\chi : \mathbb{R} \to [0,1]$ be a C^∞ function satisfying $\chi|_{(-\infty, 1+1/(2\log\beta))} = 1$ and $\chi|_{[1,\infty)} = 0$. Since each continuous psh function can be approximated by a decreasing sequence of C^∞ strictly psh functions, there exist for each j a number ν_j with $\lambda_j/2 < \nu_j < \lambda_j$ and a C^∞ strictly psh function u_j defined on D_{j,ν_j} such that

(1) $-\log \dfrac{d_{D_j}(z)}{\lambda_j/2} < u_j(z) < 0$ on D_{j,ν_j},

(2) $u_j(z) + \log \dfrac{d_{D_j}(z)}{\lambda_j/2} < (\beta^{-1} - \beta^{-1/2}) \log \dfrac{\Lambda_j}{\lambda_j/2}, \ \forall z \in D.$

We write $\widetilde{D}_j = D_{j,\nu_j}$ and put

$$\psi_j = -\log(-u_j), \qquad \varrho_j = \chi\left(\psi_j + \log\log \frac{\Lambda_j}{\lambda_j/2} + 1\right).$$

Then ψ_j is a C^∞ strictly psh function on $\widetilde{D}_j$ satisfying

$$\partial\overline{\partial}\psi_j \geq \partial\psi_j\overline{\partial}\psi_j.$$

This gives

$$|\overline{\partial}\varrho_j|_{\partial\overline{\partial}\psi_j} \leq \sup|\chi'|,$$

where $|\overline{\partial}\varrho_j|_{\partial\overline{\partial}\psi_j}$ is the point-norm of $\overline{\partial}\varrho_j$ w.r.t. the metric $\partial\overline{\partial}\psi_j$.

From (1) we obtain

$$\operatorname{supp}\varrho_j \subset D_{j,\Lambda_j} \subset D.$$

So we can define a C^∞ $\overline{\partial}$-closed $(n,1)$-form on $\widetilde{D}_j$ as follows:

$$g_j = \overline{\partial}(\varrho_j f) \wedge dz_1 \wedge \ldots \wedge dz_n.$$

Since

$$\operatorname{supp}\overline{\partial}\varrho_j \subset \left\{ z \in D \,\middle|\, u_j(z) > -\beta^{-1/2}\log\frac{\Lambda_j}{\lambda_j/2} \right\}$$

$$\subset \left\{ z \in D \,\middle|\, -\log\frac{d_{D_j}(z)}{\lambda_j/2} > -\beta^{-1}\log\frac{\Lambda_j}{\lambda_j/2} \right\}$$

$$= D \setminus D_{j,\mu_j},$$

where

$$\mu_j = \begin{cases} 2r\Lambda_j, & \alpha = 1, \\ (2^\alpha r)^{1/(2(1-\alpha))}\Lambda_j^{1/2}, & 0 < \alpha < 1, \end{cases}$$

we obtain

$$\|g_j\|^2_{\partial\overline{\partial}\psi_j,0} = 2^n \int\limits_{\widetilde{D}_j} |\overline{\partial}\varrho_j|^2_{\partial\overline{\partial}\psi_j} |f|^2 \, dV_n \le 2^n \sup|\chi'|^2 \|f\|^2_{D\setminus D_{j,\mu_j}},$$

where dV_k denotes the $2k$-dimensional Lebesgue measure.

Since any pseudoconvex domain admits a complete Kähler metric with a positive global potential (cf. [7], p. 49), by Proposition 2.1 there exists an $(n,0)$-form $h_j = \widetilde{h}_j dz_1 \wedge \ldots \wedge dz_n$ on $\widetilde{D}_j$ satisfying $\overline{\partial}h_j = g_j$ and

$$\|\widetilde{h}_j\|_{\widetilde{D}_j} \le C \sup|\chi'| \cdot \|f\|_{D\setminus D_{j,\mu_j}},$$

where C is a numerical constant (independent of f, j, α, D).

Hence $f_j = \varrho_j f - \widetilde{h}_j$ is holomorphic on $\widetilde{D}_j$ and satisfies

$$\|f_j - f\|_D \le \|(1 - \varrho_j)f\|_D + \|\widetilde{h}_j\|_D \le (1 + C)\sup|\chi'| \cdot \|f\|_{D\setminus D_{j,\mu_j}}.$$

Given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\|f\|_{D\setminus D_\delta} < \varepsilon,$$

where $D_t = \{z \in D \mid d_D(z) > t\}$, $t > 0$. Since $\Lambda_j \to 0$ as $j \to \infty$, one has $\mu_j < \delta/2$ for all sufficiently large j. This gives $D_{j,\mu_j} \supset D_{j,\delta/2}$ for all sufficiently large j. Since $d_{D_j}(z) \to d_D(z)$ on D, there exists a $j(\varepsilon)$ such that

$$D_{j(\varepsilon),\mu_{j(\varepsilon)}} \supset D_\delta.$$

Thus

$$\|f_{j(\varepsilon)} - f\|_D \le (1 + C)\varepsilon \sup|\chi'|.$$

The proof is complete.

LEMMA 3.2. $\lim_{z\to\partial D} K_D(z,z) = \infty$ *if D is a domain which satisfies condition* **S** *locally.*

Proof. We can easily conclude that D satisfies the cone condition in the sense of Pflug (cf. [14], p. 399), hence we have $\lim_{z\to\partial D} K_D(z,z) = \infty$.

The following localization lemma for the Bergman metric is perhaps known:

LEMMA 3.3. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and z^0 be any point in ∂D. Suppose that U, V are two open neighbourhoods of z^0 with $V \subset\subset U$. Then*

$$ds^2_D(\zeta;X) \ge C ds^2_{D\cap U}(\zeta;X), \quad \forall X \in T^{1,0}(\mathbb{C}^n), \ \zeta \in V \cap D,$$

where C is a constant (independent of ζ, X).

Proof. From the original work of Bergman (cf. [2]) we know that for any $X \in T^{1,0}(\mathbb{C}^n)$ and $\zeta \in D$ we have

$$ds^2_D(\zeta;X) = K_D^{-1}(\zeta,\zeta) \sup\{|Xf|^2 \mid f \in H^2(D), \ \|f\|_D \le 1, \ f(\zeta) = 0\}.$$

We choose a C^∞ function $\eta : \mathbb{C}^n \to [0,1]$ such that $\eta = 1$ on a neighbourhood of $\overline{V}$, $\eta = 0$ on $\mathbb{C}^n \setminus U$. Fix any point $\zeta \in D \cap V$ and for any $f \in H^2(D \cap U)$ with $\|f\|_{D \cap U} \le 1$ and $f(\zeta) = 0$ put $v = \overline{\partial}(\eta f)$. Thus we have defined a C^∞ $\overline{\partial}$-closed $(0,1)$-form on D. Let $\psi(z) = 2(n+1)\log|z - \zeta|$. We have

$$\int_D |v|^2 e^{-\psi}\, dV_n \le C_1,$$

where C_1 is a constant (independent of ζ, f). By a well known Hörmander theorem, there exists a function $u(z)$ which satisfies $\overline{\partial}u = v$ and

$$\int_D |u|^2 |z - \zeta|^{-2(n+1)}\, dV_n \le C_2,$$

where C_2 is a constant (independent of ζ, f). Then $F(z) = \eta(z)f(z) - u(z)$ is a holomorphic function on D and $\|F\|_D \le 1 + C_3$, where C_3 is also independent of ζ, f. Since $\eta = 1$ on V, u is holomorphic on V and we have

$$u(\zeta) = 0, \qquad \frac{\partial u}{\partial z_j}(\zeta) = 0, \qquad \forall 1 \le j \le n.$$

So we have

$$F(\zeta) = -u(\zeta) = 0, \qquad \frac{\partial F}{\partial z_j}(\zeta) = \frac{\partial f}{\partial z_j}(\zeta), \qquad \forall 1 \le j \le n.$$

Hence

$$ds_D^2(\zeta; X) \ge C ds_{D \cap U}^2(\zeta; X), \qquad \forall X \text{ and } \zeta \in D \cap V,$$

where $C = (1 + C_3)^{-2}$.

Theorem 1.1 is immediately derived from Lemmas 3.1–3.3 and the following proposition:

PROPOSITION 3.4 (cf. [9]). *Let D be a bounded domain in $\mathbb{C}^n$ and assume*

(1) *the bounded holomorphic functions in D are dense in $H^2(D)$;*
(2) $\lim_{z \to \partial D} K_D(z, z) = \infty$.

Then the Bergman metric of D is complete.

REMARK. In fact, we can replace assumption (1) by a weaker condition:

For each $z^0 \in \partial D$ and $f \in H^2(D)$, f can be approximated in L^2-norm by a family of holomorphic functions on D which are bounded in some neighbourhood of z^0.

This can be proved easily with the method of [14] (cf. p. 409).

Proof of Theorem 1.2. Let D be a bounded pseudoconvex domain with Lipschitz boundary. For each $z^0 \in \partial D$, there exist reals $\varepsilon_0, r_0 > 0$ and a vector T_0 that points outside of D such that $z + \varepsilon T_0 \in \mathbb{C}^n \setminus \overline{D}$ for all $z \in (\mathbb{C}^n \setminus \overline{D}) \cap B(z^0, r_0)$ and $0 < \varepsilon < \varepsilon_0$.

It is known from [5] that D admits a C^∞ strictly psh function u on D such that $u < 0$ on D and $\lim_{z \to \partial D} u(z) = 0$. For $\varepsilon, r > 0$, we set

$$D_r^\varepsilon = \{z \in B(z^0, r) \mid u(z - \varepsilon T_0) < 0\}.$$

Put $D_\varepsilon = D_{r_0/2+\varepsilon}^\varepsilon$ for $\varepsilon < \min\{\varepsilon_0, r_0/4\}$. Then:

(1) D_ε is pseudoconvex;
(2) $D \cap B(z^0, r_0/2) \subset\subset D_\varepsilon$;
(3) there exists a constant $0 < C < 1$ such that

$$C\varepsilon < d_{D_\varepsilon}(z) < \varepsilon$$

for all $z \in \partial(D \cap B(z^0, r_0/2))$. Hence D locally satisfies condition **S**, and therefore Theorem 1.2 is a direct consequence of Theorem 1.1.

4. Bergman metric on hyperconvex domains. Let D be a bounded hyperconvex domain in $\mathbb{C}^n$, and $u_D(\zeta, z)$ be the pluricomplex Green function on D. From [10] we know that there exists a C^∞ strictly psh function u on D such that $u < 0$ on D and $\lim_{z \to \partial D} u(z) = 0$. We set $D_t = \{z \in D \mid u(z) < t\}$. The following fact is due to Demailly:

PROPOSITION 4.1 (cf. [5], p. 531). *For each $\zeta \in D$, $u_D(\zeta, \cdot)$ is a continuous psh exhaustion function on D with values in $[-\infty, 0)$ and $u_D(\zeta, z) \sim \log|z - \zeta|$ as $z \to \zeta$.*

Now we prove the following two lemmas:

LEMMA 4.2. *Suppose that D is a bounded hyperconvex domain in $\mathbb{C}^n$. Then for each $\zeta \in D$ and $f \in H^2(D)$, there exists a function $F \in H^2(D)$ satisfying $F(\zeta) = 0$ and*

$$\|F - f\|_D \leq C\|f\|_{D_{\zeta,-1}},$$

where $D_{\zeta,t} = \{z \in D \mid u_D(\zeta, z) < t\}$ and C is a constant (independent of ζ, f).

P r o o f. Fix ζ. There are negative C^∞ strictly psh functions u_ε defined on $D_{-a(\varepsilon)}$, where $a(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that $u_\varepsilon \downarrow u_D(\zeta, \cdot)$ as $\varepsilon \downarrow 0$.

Let $0 \leq \kappa \leq 1$ be a C^∞ function on $\mathbb{R}$ such that $\kappa(t) = 1$ on $(-\infty, 1 - \log 2)$ and $\kappa(t) = 0$ on $[1, \infty)$. Put

$$\psi_\varepsilon = -\log(-u_\varepsilon), \qquad \varrho_\varepsilon(z) = 1 - \kappa(\psi_\varepsilon(z) + 1).$$

Again we have

$$|\overline{\partial}\varrho_\varepsilon|_{\partial\overline{\partial}\psi_\varepsilon} \leq \sup|\kappa'|.$$

Put $g_\varepsilon = \overline{\partial}(\varrho_\varepsilon f) \wedge dz_1 \wedge \ldots \wedge dz_n$. One has

$$\|g_\varepsilon\|^2_{\partial\overline{\partial}\psi_\varepsilon, 2nu_\varepsilon} \le 2^n \sup |\kappa'|^2 \int\limits_{-2 < u_\varepsilon < -1} |f|^2 e^{-2nu_\varepsilon} \, dV_n$$

$$\le 2^n e^{4n} \sup |\kappa'|^2 \|f\|^2_{D_{\zeta,-1}}$$

since $u_\varepsilon \ge u_D(\zeta, \cdot)$. By Proposition 2.1, there exists an $(n,0)$-form $h_\varepsilon = \widetilde{h}_\varepsilon dz_1 \wedge \ldots \wedge dz_n$ on $D_{-a(\varepsilon)}$ such that $\overline{\partial}h_\varepsilon = g_\varepsilon$ and

$$\|\widetilde{h}_\varepsilon\|_{D_{-a(\varepsilon)}, 2nu_\varepsilon} \le C_0 \|f\|_{D_{\zeta,-1}},$$

where C_0 is a constant (independent of ζ, ε, f). Then $F_\varepsilon = \varrho_\varepsilon f - \widetilde{h}_\varepsilon$ is holomorphic on $D_{-a(\varepsilon)}$ and

$$\|F_\varepsilon - f\|_{D_{-a(\varepsilon)}} \le \|(1 - \varrho_\varepsilon)f\|_{D_{-a(\varepsilon)}} + \|\widetilde{h}_\varepsilon\|_{D_{-a(\varepsilon)}, 2nu_\varepsilon} \le (1 + C_0)\|f\|_{D_{\zeta,-1}}.$$

We also have

$$\|F_\varepsilon\|_{D_{-a(\varepsilon)}, 2nu_\varepsilon} \le \|\varrho_\varepsilon f\|_{D_{-a(\varepsilon)}, 2nu_\varepsilon} + \|\widetilde{h}_\varepsilon\|_{D_{-a(\varepsilon)}, 2nu_\varepsilon} \le (e^{2n} + C_0)\|f\|_D.$$

Since $u_\varepsilon < 0$, we can choose a sequence $\varepsilon_j \to 0$ such that F_{ε_j} converges weakly to a holomorphic function F on D. Hence

$$\|F - f\|_D \le (1 + C_0)\|f\|_{D_{\zeta,-1}}.$$

Since u_ε decreases with ε, for each $b > 0$ and ε with $0 < a(\varepsilon) < b$ we have

$$\|F\|_{D_{-b}, 2nu_\varepsilon} \le (C_0 + e^{2n})\|f\|_D.$$

Letting $\varepsilon \to 0$ and then $b \to 0$, one gets

$$\|F\|_{D, 2nu_D(\zeta, \cdot)} \le (C_0 + e^{2n})\|f\|_D.$$

Since $u_D(\zeta, z) \sim \log|z - \zeta|$ as $z \to \zeta$, one has $F(\zeta) = 0$.

LEMMA 4.3. *Suppose that D is a bounded hyperconvex domain in $\mathbb{C}^n$, and $u_D(\zeta, z)$ is symmetric. Then the volume of $D_{\zeta,-1}$ tends to zero as $\zeta \to \partial D$.*

P r o o f. The proof is due to Ohsawa (cf. [12]). For any $\varepsilon > 0$, there exists a C^∞ function $\kappa_\varepsilon(z)$ such that $\operatorname{supp} \kappa_\varepsilon \subset D_{-\varepsilon/2}$ and $\kappa_\varepsilon|_{D_{-\varepsilon}} = 1$. Then we can find a constant $C = C(\varepsilon)$ such that $Cu(z) + \kappa_\varepsilon(z) \log|z - \zeta|$ is a negative psh function on D for all $\zeta \in D_{-\varepsilon}$. Hence there exists a $\delta > 0$ such that

$$u_D(\zeta, z) > -1, \quad \forall \zeta \in D_{-\varepsilon}, \ z \in D \setminus D_{-\delta}.$$

Since $u_D(\zeta, z) = u_D(z, \zeta)$, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$u_D(\zeta, z) > -1, \quad \forall \zeta \in D \setminus D_{-\delta}, \ z \in D_{-\varepsilon}.$$

This proves the lemma.

To prove Theorem 1.3, we need the following proposition:

PROPOSITION 4.4 (cf. [14], p. 408). *Let D be a bounded domain in $\mathbb{C}^n$, and $\{\zeta^j\}_{j=1}^\infty \subset D$ a Cauchy sequence w.r.t. the Bergman metric. Then there*

exist a subsequence $\{\zeta^{j_k}\}_{k=1}^{\infty}$ *and real numbers* θ_k *such that*

$$\left\{ \frac{K_D(\,\cdot\,,\zeta^{j_k})}{K_D^{1/2}(\zeta^{j_k},\zeta^{j_k})} e^{i\theta_k} \right\}_{k=1}^{\infty}$$

is a Cauchy sequence in $H^2(D)$.

Proof of Theorem 1.3. Suppose that D is not complete w.r.t. the Bergman metric. Then there exists a Cauchy sequence $\{\zeta^j\}_{j=1}^{\infty}$ w.r.t. the Bergman metric which converges to a boundary point ζ^0 of D as $j \to \infty$. Proposition 4.4 gives us a subsequence $\{\zeta^{j_k}\}_{k=1}^{\infty}$ and real numbers θ_k satisfying

$$\frac{K_D(\,\cdot\,,\zeta^{j_k})}{K_D^{1/2}(\zeta^{j_k},\zeta^{j_k})} e^{i\theta_k} \to f$$

in $H^2(D)$ with $\|f\|_D = 1$. This yields

$$\frac{|f(\zeta^{j_k})|}{K_D^{1/2}(\zeta^{j_k},\zeta^{j_k})} = \left| e^{-i\theta_k} \left(f, \frac{K_D(\,\cdot\,,\zeta^{j_k})}{K_D^{1/2}(\zeta^{j_k},\zeta^{j_k})} \right) \right| \to \|f\|^2 = 1.$$

By Lemma 4.2 there exists a sequence $\{F_k\}_{k=1}^{\infty}$ of holomorphic functions on D satisfying $F_k(\zeta^{j_k}) = 0$ and

$$\|F_k - f\|_D \leq C\|f\|_{D_{\zeta^{j_k},-1}} \overset{k\to\infty}{\longrightarrow} 0$$

because $\mathrm{Vol}(D_{\zeta^{j_k},-1}) \to 0$ by Lemma 4.3. On the other hand

$$\|F_k - f\|_D \geq \frac{|f(\zeta^{j_k})|}{K_D^{1/2}(\zeta^{j_k},\zeta^{j_k})} \to 1,$$

which is a contradiction.

5. A counterexample. Let Δ be the unit disc in $\mathbb{C}$ and define

$$D_N = \Delta \setminus \overline{\bigcup_{k=1}^{\infty} \Delta(2^{-k}, 2^{-kN(k)})},$$

where $\Delta(x, r)$ is the disc centred at x with radius $r > 0$. Assume $N(k) > 2$ for all k. Then $\Delta(2^{-k}, 2^{-kN(k)})$ are disjoint from each other for all k. We have the following criterion for the hyperconvexity of D_N (cf. [12], p. 50):

PROPOSITION 5.1. D_N *is hyperconvex if and only if* $\sum_{k=1}^{\infty} N(k)^{-1} = \infty$.

We first prove the following:

LEMMA 5.2. *Let D be a bounded domain in $\mathbb{C}$ satisfying*:

(∗) *For each $z_0 \in \partial D$, there exists a sequence $\{z_k\}_{k=1}^{\infty} \subset \mathbb{C} \setminus \overline{D}$ with $z_k \to z_0$ as $k \to \infty$ such that*

$$|z_k - z_0|^2 \log d(z_k, \partial D) \to 0 \quad \text{as } k \to \infty.$$

Then the Bergman metric on D is complete.

Proof. First, we prove $K_D(z,z) \to \infty$ as $z \to \partial D$. For each $z_0 \in \partial D$, put $f_k(z) = (z - z_k)^{-1}$. Then f_k is holomorphic on D and satisfies

$$\|f_k\|_D^2 \leq \int\limits_{D \cap \{z \,|\, |z-z_k|>d(z_k,\partial D)\}} |z - z_k|^{-2}\, dV_1 = 2\pi(C_1 - \log d(z_k, \partial D)),$$

where C_1 is a constant depending only on the diameter of D.

For each $z \in D \cap \Delta(z_0, |z_k - z_0|)$, one has

$$K_D(z,z) \geq \frac{|f_k(z)|^2}{\|f_k\|_D^2} \geq \frac{1}{8\pi|z_k - z_0|^2(C_1 - \log d(z_k, \partial D))}.$$

Thus we obtain $K_D(z,z) \to \infty$ as $z \to z_0$.

Next we want to show that every $f \in H^2(D)$ with $\|f\|_D \leq 1$ can be approximated in L^2-norm by a family $\{f_\varepsilon\}_{\varepsilon>0}$ of holomorphic functions on D which are bounded in some open neighbourhood of z_0.

For simplicity we can assume $z_0 = 0$ and $D \subset\subset \Delta$. Put

$$\psi(z) = -\log(-\log|z|).$$

Then ψ is a C^∞ strictly psh function on $\Delta \setminus \{0\}$ satisfying $\partial\overline{\partial}\psi \geq \partial\psi\overline{\partial}\psi$. For each $0 < \varepsilon < 1$ we set

$$\eta_\varepsilon = \kappa(-\psi - \log\log(1/\varepsilon) + 1),$$

where κ is the function in the proof of Lemma 4.2. Put

$$D_\varepsilon = \{z \in \mathbb{C} \mid 0 < |z| < \varepsilon/2\} \cup D.$$

Then $\eta_\varepsilon f \in C^\infty(D_\varepsilon)$. Since every domain in $\mathbb{C}$ is pseudoconvex, we can find a function y_ε on D_ε such that $f_\varepsilon = \eta_\varepsilon f - y_\varepsilon$ is a holomorphic function on D_ε satisfying $\|y_\varepsilon\|_{D_\varepsilon} \leq C_2\|f\|_{D \cap \Delta(0,\varepsilon^{1/2})}$, where C_2 is a constant (independent of f, ε). This gives

$$\|f_\varepsilon - f\|_D \leq (1 + C_2)\|f\|_{D \cap \Delta(0,\varepsilon^{1/2})}$$

and $\|f\|_{D_\varepsilon} \leq 1 + C_2$. Then f_ε can be extended to a holomorphic function on $D \cup \Delta(0, \varepsilon^{1/2})$ and $\|f_\varepsilon - f\|_D \to 0$ as $\varepsilon \to 0$. The proof of Lemma 5.2 is complete by the remark after Proposition 3.4.

Lemma 5.2 implies

LEMMA 5.3. *Let D_N be defined as above. Suppose that*

$$kN(k)2^{-2k} \to 0 \qquad as\ k \to \infty.$$

Then the Bergman metric on D_N is complete.

Now we take $N(k) = k^2 + 1$. Then D_N is not hyperconvex by Proposition 5.1 and the Bergman metric on D_N is complete by Lemma 5.3.

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