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On the delay differential equation $y'(x) = ay(\tau(x)) + by(x)$

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Abstract. The paper discusses the asymptotic properties of solutions of the scalar functional differential equation

$$y'(x) = ay(\tau(x)) + by(x), \quad x \in [x_0, \infty].$$

Asymptotic formulas are given in terms of solutions of the appropriate scalar functional nondifferential equation.

1. Introduction. The linear functional differential equation

$$y'(x) = ay(\tau(x)) + by(x), \quad x \in I = [x_0, \infty), \tag{1.1}$$

has been studied, under special hypotheses, in many papers, for theoretical reasons as well as with a view to applications. In these problems a and b are usually real constants and y is a real-valued function. However, in our paper we also allow complex values for a and y.

Equations (1.1) with bounded $r(x) = x - \tau(x)$ are fairly well understood, whereas for $r(x)$ unbounded the theory is less developed. Among papers dealing with such equations we can mention, e.g., [6] and [5]. The paper [6] discusses the asymptotic behaviour of solutions of (1.1) with $\tau(x) = \lambda x$, $\lambda > 0, \lambda \neq 1$. The paper [5] is devoted to relating the asymptotic properties of solutions of the delay equation (1.1) to the behaviour of solutions of the linear functional nondifferential equation

$$a\psi(\tau(x)) + b\psi(x) = 0, \quad x \in I. \tag{1.2}$$

The resemblance between the asymptotic behaviour of solutions of (1.1) and (1.2) has been shown for $b < 0$ and holds for certain delay equations (1.1) with $r(x)$ unbounded. Our aim is to show this asymptotic resemblance also in the case $b > 0$. Similarly to [5] we study delay equations (1.1) with $r(x)$ unbounded.

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Let us remark that the idea of establishing estimates of solutions of linear functional differential equations by means of solutions of auxiliary functional nondifferential equations has also been used in some other cases (see, e.g., [2]). Nevertheless, as remarked above, it is usually required that the function $r(x) = x - \tau(x)$ is constant (or at least bounded).

Throughout this paper we assume that τ is an increasing differentiable function on I such that $\tau(x) < x$ for every $x \in I$ and $\lim_{x\to\infty} \tau(x) = \infty$. Nevertheless, our results are also valid for equations (1.1) with $\tau(x_0) = x_0$ (the proofs require only small modifications).

We say that a real- or complex-valued function y is a *solution* of (1.1) if $y \in C^0([\tau(x_0), \infty)) \cap C^1([x_0, \infty))$ and satisfies (1.1) on $[x_0, \infty)$. The symbol τ^n, where $n \in \mathbb{Z}$, stands for the nth iterate of τ (for $n > 0$) or the $(-n)$th iterate of the inverse function τ^{-1} (for $n < 0$); we put $\tau^0 = \mathrm{id}$.

2. Main results. We introduce a parameter λ defined as

$$\lambda = \sup\{\tau'(x) : x \in I\}.$$

Further, we consider the Schröder functional equation

$$\varphi(\tau(x)) = \lambda\varphi(x), \qquad x \in I, \tag{2.1}$$

where τ is known, λ is defined above, and φ is unknown. A survey of results concerning this equation can be found in [7]. Here we only state the result that we need.

Proposition 1. *Let $\tau \in C^r(I)$, $r \geq 1$, be such that $\tau'(x) > 0$ for every $x \in I$ and $\lambda < 1$. Further, let $\varphi_0 \in C^r(I_0)$, where $I_0 = [\tau(x_0), x_0]$, be a positive function with a positive derivative on I_0 satisfying*

$$(\varphi_0 \circ \tau)^{(k)}(x_0) = \lambda\varphi_0^{(k)}(x_0), \qquad k = 0, 1, \ldots, r.$$

Then there exists a unique positive solution $\varphi \in C^r(I)$ of (2.1) such that φ' is positive and bounded on I and $\varphi(x) = \varphi_0(x)$ for every $x \in I_0$. This solution is given by the formula

$$\varphi(x) = \lambda^{-n}\varphi_0(\tau^n(x)), \qquad \tau^{-n+1}(x_0) \leq x \leq \tau^{-n}(x_0),\ n = 0, 1, 2, \ldots \tag{2.2}$$

Remark 1. Looking for a solution ψ of (1.2) by means of a solution φ of (2.1) we suppose ψ has the form $\psi(x) = (\varphi(x))^\alpha$, $x \in I$. Substituting this into (1.2) we obtain

$$a\lambda^\alpha(\varphi(x))^\alpha + b(\varphi(x))^\alpha = 0,$$

i.e., $\alpha = \log(a/(-b))/\log\lambda^{-1}$. Note that log will always mean the principal branch of the corresponding logarithm.

PROPOSITION 2. *Let τ satisfy the hypotheses of Proposition* 1 *and let φ be given by* (2.2). *Then the function*

$$\psi(x) = (\varphi(x))^{\alpha}, \qquad \alpha = \frac{\log \frac{a}{-b}}{\log \lambda^{-1}},$$

is a solution of (1.2) *such that $|\psi(x)| > 0$ for every $x \in I$.*

LEMMA. *Let $a \neq 0$, $b > 0$ be scalars and let $\tau \in C^1(I)$ be such that $\lambda < 1$. If y is any solution of* (1.1), *then $e^{-bx}y(x)$ tends to a finite* (*possibly zero*) *constant as $x \to \infty$. Moreover, there exists a $\sigma \geq x_0$ and a solution y^* of* (1.1) *defined on $[\tau(\sigma), \infty)$ such that $e^{-bx}y^*(x)$ tends to* 1 *as $x \to \infty$.*

Proof. We introduce a change of variables $z(x) = e^{-bx}y(x)$ in (1.1) to obtain

$$z'(x) = p(x)z(\tau(x)), \qquad x \in I, \tag{2.3}$$

where $p(x) = ae^{b(\tau(x)-x)}$, $x \in I$. Every solution z of (2.3) tends to a finite constant provided

$$\int_{x_0}^{\infty} |p(s)|\, ds < \infty \tag{2.4}$$

(see, e.g., [9]). By our assumptions on b and τ condition (2.4) is satisfied, which implies the first assertion.

Now set

$$p^-(x) = \max(0, -p(x)), \qquad x \in I.$$

In addition to (2.4) assume that

$$\int_{x_0}^{\infty} p^-(s)\, ds < 1. \tag{2.5}$$

Then the converse statement holds as well, i.e., for every $\xi \in \mathbb{R}$ there exists a solution z^* of (2.3) such that $\lim_{x\to\infty} z^*(x) = \xi$ (see [4]). Therefore, once $\sigma \geq x_0$ has been chosen large enough we can obtain

$$\int_{\sigma}^{\infty} p^-(s)\, ds < 1$$

and there exists a solution z^* of (2.3) defined on $[\tau(\sigma), \infty)$ and tending to 1 as $x \to \infty$. We put $y^*(x) = e^{bx}z^*(x)$, $x \in [\tau(\sigma), \infty)$, and the lemma is proved.

REMARK 2. If, moreover, $a/b > \lambda - 1$, then there exists a solution $y^*(x)$ of (1.1) asymptotic to e^{bx} as $x \to \infty$ and defined on $[\tau(x_0), \infty)$, i.e., we can put $\sigma = x_0$.

Indeed, let $a > 0$. Then p^- is identically zero on I and (2.5) is satisfied.

Let $a < 0$. Then

$$\int_{x_0}^{\infty} p^-(s)\,ds = -a \int_{x_0}^{\infty} e^{b(\tau(s)-s)}\,ds \le \frac{-a}{b(1-\lambda)}$$

and (2.5) holds if $a/b > \lambda - 1$.

Notice that the equation

$$y'(x) = b(y(x) - y(\lambda x)), \qquad x \in [0, \infty),$$

where $b > 0$, $0 < \lambda < 1$, has a one-parameter family of solutions $y = c$, hence no solution $y(x)$ defined on $[0, \infty)$ is asymptotic to e^{bx} as $x \to \infty$.

Theorem 1. *Let $a \neq 0$, $b > 0$ be scalars, $\tau \in C^1(I)$ be such that τ' is positive and nonincreasing on I, $\lambda = \tau'(x_0) < 1$, and let y^* be given by the Lemma. Then for any solution y of* (1.1) *there exists a constant $c \in \mathbb{R}$ and a continuous periodic function g of period* $\log \lambda^{-1}$ *such that*

$$y(x) = cy^*(x) + (\varphi(x))^{\alpha} g(\log \varphi(x)) + O\{(\varphi(x))^{\alpha_r - 1}\} \quad \text{as } x \to \infty, \tag{2.6}$$

where φ is a solution of (2.1) *given by* (2.2), $\alpha = \frac{\log \frac{a}{-b}}{\log \lambda^{-1}}$, $\alpha_r = \operatorname{Re} \alpha$.

Remark 3. It can be easily verified that the function $(\varphi(x))^{\alpha} g(\log \varphi(x))$ is a solution of (1.2). Consequently, the asymptotic formula (2.6) essentially says that the difference of any two solutions y_1, y_2 of (1.1) satisfying

$$\lim_{x\to\infty} e^{-bx} y_i(x) = c \in \mathbb{R} \quad \text{as } x \to \infty, \qquad i = 1, 2, \tag{2.7}$$

with the same constant c approaches a solution of (1.2).

Proof (of Theorem 1). Suppose that y_1, y_2 are solutions of (1.1) satisfying (2.7) and put $y = y_1 - y_2$. We show that if $\alpha \neq 0$ then

$$y^{(n)}(x) = O\{|\psi^{(n)}(x)|\} \quad \text{as } x \to \infty, \qquad n = 0, 1, \tag{2.8}$$

where $\psi(x) = (\varphi(x))^{\alpha}$, $x \in I$, is a solution of (1.2). If $\alpha = 0$ then we replace (2.8) by

$$y(x) = O\{1\} \quad \text{as } x \to \infty, \qquad y'(x) = O\left\{\frac{\varphi'(x)}{\varphi(x)}\right\} \quad \text{as } x \to \infty. \tag{2.9}$$

Once (2.8) and (2.9) have been proved we can set

$$t = \log \varphi(x), \quad w(t) = (\varphi(x))^{-\alpha} y(x).$$

Then $w(t)$ satisfies the equation

$$\frac{1}{b} s(t)\dot{w}(t) = \left(1 - \frac{\alpha}{b} s(t)\right) w(t) - w(t + \log \lambda),$$

where $s(t) = \varphi'(\varphi^{-1}(e^t))/e^t$. By Proposition 1, $s(t) = O\{e^{-t}\}$ as $t \to \infty$, hence in view of (2.8) and (2.9),

$$w(t) - w(t + \log \lambda) = O\{e^{-t}\} \quad \text{as } t \to \infty.$$

Repeating this process we can obtain the uniform convergence of the sequence $\{w(t - n \log \lambda)\}_{n=1}^{\infty}$ to a continuous periodic function g of period $\log \lambda^{-1}$ such that

$$w(t) = g(t) + O\{e^{-t}\} \quad \text{as } x \to \infty.$$

Substituting back $y(x)$ into the last relation we have

$$y(x) = y_1(x) - y_2(x) = (\varphi(x))^{\alpha} g(\log \varphi(x)) + O\{(\varphi(x))^{\alpha_r - 1}\} \quad \text{as } x \to \infty.$$

It remains to prove the asymptotic relations (2.8) and (2.9). Since both can be proved in the same way we consider only (2.8) for $n = 0, 1$.

First let $n = 0$. Obviously $\lim_{x\to\infty} e^{-bx} y(x) = 0$. Multiplying both sides of (1.1) by e^{-bx} we obtain

$$\frac{d}{dx}[e^{-bx} y(x)] = a e^{-bx} y(\tau(x)).$$

Integrating this we get

$$y(x) = -a e^{bx} \int_x^{\infty} (e^{-bs} y(\tau(s)))\, ds.$$

Since $|y(x)| \le M e^{bx}$ for $x \ge x_0$, where $M > 0$ is a suitable constant, we have, in view of $\tau' \le \lambda$ on I,

$$|y(x)| \le M|a| e^{bx} \int_x^{\infty} e^{b(\tau(s) - s)}\, ds \le M \frac{|a|}{b(1-\lambda)} e^{b\tau(x)} \quad \text{for } x \ge \tau^{-1}(x_0).$$

Repeating this idea we obtain

$$|y(x)| \le M \frac{|a|^n e^{b\tau^n(x)}}{b^n (1-\lambda) \dots (1-\lambda^n)} \quad \text{for } x \ge \tau^{-n}(x_0),\ n = 1, 2, \dots$$

Further, $e^{b\tau^n(x)} \le e^{b\tau^{-1}}(x_0)$ for $x \le \tau^{-n-1}(x_0)$, i.e.,

$$|y(x)| \le M_1 \frac{|a|^n}{b^n} \quad \text{for } \tau^{-n}(x_0) \le x \le \tau^{-n-1}(x_0),\ n = 1, 2, \dots, \tag{2.10}$$

where $M_1 = M e^{b\tau^{-1}}(x_0) (\prod_{j=1}^{\infty} (1 - \lambda^j))^{-1}$.

Now let $N_1 = \inf\{|\psi(x)| : x \in [x_0, \tau^{-1}(x_0)]\} > 0$. Then

$$|\psi(x)| = |\psi(\tau^{-n}(\tau^n(x)))| \ge N_1 \frac{|a|^n}{b^n} > 0$$

for $\tau^{-n}(x_0) \le x \le \tau^{-n-1}(x_0)$ and it is easy to deduce that this estimate together with relation (2.10) implies $y(x) = O\{|\psi(x)|\}$ as $x \to \infty$.

Now we show that (2.8) holds for $n = 1$ as well. Since y' is a solution of

$$y''(x) = a\tau'(x) y'(\tau(x)) + b y'(x),$$

it is easy to check that $\lim_{x\to\infty} e^{-bx}y'(x) = 0$. Similarly to the previous part we can estimate y' as

$$|y'(x)| \le M_2 \frac{|a|^n(\tau^n)'(x)}{b^n} \qquad \text{for } \tau^{-n}(x_0) \le x \le \tau^{-n-1}(x_0),\ n = 1, 2, \dots,$$

by use of the fact that τ' is nonincreasing on I.

Further, let $N_2 = \inf\{|\psi'(x)| : x \in [x_0, \tau^{-1}(x_0)]\} > 0$. Then

$$|\psi'(x)| = |\psi'(\tau^{-n}(\tau^n(x)))| \ge N_2 \frac{|a|^n(\tau^n)'(x)}{b^n} > 0$$

for $\tau^{-n}(x_0) \le x \le \tau^{-n-1}(x_0)$, i.e., $y'(x) = O\{|\psi'(x)|\}$ as $x \to \infty$, which completes the proof.

As remarked above, the case $a \ne 0$, $b < 0$ has been studied in [5]. We state here the relevant result.

Theorem 2 [5, Theorem 3.1]. *Let $a \ne 0$, $b < 0$ be scalars, $\tau \in C^2(I)$ be such that τ' is positive and decreasing on I and $\lambda = \tau'(x_0) < 1$. Then for any solution y of* (1.1) *there exists a continuous periodic function g of period $\log \lambda^{-1}$ such that*

$$y(x) = (\varphi(x))^\alpha g(\log \varphi(x)) + O\{(\varphi(x))^{\alpha_r - 1}\} \qquad \text{as } x \to \infty, \tag{2.11}$$

where φ is the solution of (2.1) *given by* (2.2), $\alpha = \log(a/(-b))/\log \lambda^{-1}$ *and* $\alpha_r = \operatorname{Re} \alpha$.

Remark 4. Considering the case $a \ne 0$, $b = 0$ we cannot expect any connection between the asymptotic behaviour of solutions of (1.1) and (1.2) because equation (1.2) then admits only the zero solution. It is perhaps curious that the case $b = 0$ is in many ways more difficult to discuss than the case $b \ne 0$.

We make a few remarks about the transformation approach which can help us in the study of asymptotic properties of solutions of (1.1) with $b = 0$, i.e.

$$y'(x) = ay(\tau(x)). \tag{2.12}$$

Using this approach we can convert equation (2.12) via the transformation $t = \frac{\log \varphi(x)}{\log \lambda^{-1}}$, $w(t) = y(x)$ into the equation

$$s(t)\dot{w}(t) = aw(t - 1), \tag{2.13}$$

where φ is the solution of (2.1) given by (2.2) and

$$s(t) = \varphi'(\varphi^{-1}(\lambda^{-t}))/(\lambda^{-t} \log \lambda^{-1})$$

(see also [8] and [5]). The asymptotic behaviour of equation (2.13) has been studied, under special hypotheses, in many papers. Therefore we can use some of these results to obtain asymptotic results for certain equations of type (2.12). As an example we consider equation (2.12) with the retarded argument $\tau(x) = \lambda x$ $(0 < \lambda < 1)$, i.e.,

$$y'(x) = ay(\lambda x). \tag{2.14}$$

Then $\varphi(x) = x$ is obviously a solution of (2.1) and the substitution $t = \log x / \log \lambda^{-1}$, $w(t) = y(x)$ converts (2.14) into

$$\frac{\lambda^t}{\log \lambda^{-1}} \dot{w}(t) = aw(t-1). \tag{2.15}$$

A very complete account of the asymptotic results concerning equation (2.15) has been given in [1]. Then it is not difficult to restate these results in the form corresponding to (2.14).

3. Applications

Example 1. First we consider the equation

$$y'(x) = ay(\lambda x) + by(x), \qquad x \in [0, \infty), \tag{3.1}$$

where a, b, λ are constants, $a \neq 0$, $0 < \lambda < 1$. The asymptotic behaviour of solutions of (3.1) has been deeply investigated in [6]. Applying our previous results to this equation (with $b \neq 0$) we note that the deviation $\tau(x) = \lambda x$ satisfies all the required assumptions except $\tau(x) \neq x$ for each $x \in [0, \infty)$. Nevertheless, using a small modification in Proposition 1 we find that the results of the previous sections are also valid for delays τ intersecting the identity function at the initial point. Schröder equation (2.1) then becomes

$$\varphi(\lambda x) = \lambda \varphi(x), \qquad x \in [0, \infty),$$

and admits the identity $\varphi(x) = x$ as the required solution. Substituting this φ into (2.6) and (2.11) we obtain the coincidence between our asymptotic formulas and the corresponding results of [6].

We note that the case $b = 0$ has been dealt with in Remark 4.

Example 2. Now we discuss the asymptotic behaviour of solutions of the equation

$$y'(x) = ay(x^\gamma) + by(x), \qquad x \in [1, \infty), \tag{3.2}$$

where a, b, γ are constants, $a \neq 0$, $0 < \gamma < 1$. Schröder equation (2.1) has the form

$$\varphi(x^\gamma) = \gamma \varphi(x), \qquad x \in [1, \infty),$$

with $\varphi(x) = \log x$ being the required solution.

In the sequel we put $\alpha = \log(a/(-b))/\log \lambda^{-1}$, $\alpha_r = \operatorname{Re} \alpha$ and distinguish three cases:

Let $b > 0$. Then by the Lemma and Theorem 1 there exists a solution y^* of (3.2) defined on $[\tau(\sigma), \infty)$, $\sigma \geq x_0$, such that $\lim_{x \to \infty} e^{-bx} y^*(x) = 1$.

Moreover, for any solution y of (3.2) there exists a constant $c \in \mathbb{R}$ and a continuous periodic function g of period $\log \gamma^{-1}$ such that

$$y(x) = cy^*(x) + (\log x)^\alpha g(\log\log x) + O\{(\log x)^{\alpha_r - 1}\} \quad \text{as } x \to \infty.$$

Let $b < 0$. Then by Theorem 2 for any solution y of (3.2) there exists a continuous periodic function g of period $\log \gamma^{-1}$ such that

$$y(x) = (\log x)^\alpha g(\log\log x) + O\{(\log x)^{\alpha_r - 1}\} \quad \text{as } x \to \infty.$$

Finally, let $b = 0$. Then equation (3.2) becomes

$$y'(x) = ay(x^\gamma), \quad x \in [1, \infty). \tag{3.3}$$

In accordance with Remark 4 we can convert (3.3) via the transformation $t = \log\log x / \log \gamma^{-1}$, $w(t) = y(x)$ into

$$\frac{\dot{w}(t)}{e^{\lambda^{-t}} \lambda^{-t} \log \lambda^{-1}} = aw(t-1)$$

(see also [8, Example]). However, we have no precise asymptotic results concerning this equation as we had in the case of equation (2.15). Therefore we need to proceed differently and introduce a transformation converting equation (3.3) into an equation of the type (3.1) considered in [6] and in Example 1. Indeed, let $\mu = 1/(\gamma - 1) < 0$ and y be any solution of (3.3). Then the function $w(t) = e^{\mu t} y(e^t)$ satisfies

$$\dot{w}(t) = aw(\gamma t) + \mu w(t), \quad t \in [0, \infty). \tag{3.4}$$

Applying Theorem 2 to (3.4) we see, in view of $y(x) = w(\log x) x^{1/(1-\gamma)}$, that for any solution y of (3.3) there exists a continuous periodic function g of period $\log \gamma^{-1}$ such that

$$y(x) = x^{1/(1-\gamma)} (\log x)^\varrho g(\log\log x) + O\{(\log x)^{\varrho_r - 1}\} \quad \text{as } x \to \infty,$$

where $\varrho = \log a(1-\gamma)/\log \gamma^{-1}$ and $\varrho_r = \operatorname{Re} \varrho$.

Example 3. Consider the equation

$$y'(x) = b[y(x) - y(\tau(x))], \quad x \in I, \tag{3.5}$$

where b is a nonzero constant and τ satisfies the assumptions introduced in the previous sections. Since obviously $\alpha = 0$ it is easy to restate (2.6) and (2.11) in the corresponding simplified form.

In particular, let $b > 0$ and $\tau(x) < x$ for $x \geq x_0$. Then the Lemma and Theorem 1 imply that for any solution y of (3.5) there exists a constant $c \in \mathbb{R}$ and a continuous periodic function g of period $\log \lambda^{-1}$ such that

$$y(x) = cy^*(x) + g(\log \varphi(x)) + O\{(\varphi(x))^{-1}\} \quad \text{as } x \to \infty, \tag{3.6}$$

where y^* is given by the Lemma. In addition to the Lemma and Theorem 1 we show that there exists a solution $y^*(x)$ of (3.5) asymptotic to e^{bx} and defined on $[\tau(x_0), \infty)$ (i.e., we can put $\sigma = x_0$). Suppose not and consider a solution y of (3.5) defined on $[\tau(x_0), \infty)$ such that in the initial interval $[\tau(x_0), x_0]$ both $y' > 0$ and $y'' > 0$. Notice that then $c = 0$ in (3.6), i.e., y is bounded. However, since

$$y''(x) = b(y'(x) - \tau'(x)y'(\tau(x))), \qquad x \geq x_0,$$

we have $y'' > 0$ on I, hence y' is positive and increasing on I. Then y is unbounded, which is impossible.

Now together with (3.5) we consider the equation

$$\dot{w}(t) = \beta(t)(w(t) - w(t-1)), \qquad t \in J = [t_0, \infty). \tag{3.7}$$

Setting $b = 1$, $t = k(x) = \log\varphi(x)/\log\lambda^{-1}$ and $w(t) = y(x)$ in (3.5) we obtain (3.7) with $\beta(t) = \dot{h}(t) > 0$ for all $t \in J$, where $h = k^{-1}$ on J. The asymptotic behaviour of solutions of (3.7) with β continuous and positive has been discussed, e.g., in [3]. Using our transformation approach we can obtain further asymptotic properties of solutions of (3.5) as well as of (3.7).

References

[1] N. G. de Bruijn, *The difference-differential equation* $F'(x) = e^{\alpha x+\beta}F(x-1)$, *I*, *II*, Nederl. Akad. Wettensch. Proc. Ser. A 56 = Indag. Math. 15 (1953), 449–464.

[2] J. Diblík, *Asymptotic behaviour of solutions of linear differential equations with delay*, Ann. Polon. Math. 58 (1993), 131–137.

[3] —, *Asymptotic representation of solutions of equation* $\dot{y}(t) = \beta(t)[y(t) - y(t-\tau(t))]$, J. Math. Anal. Appl. 217 (1998), 200–215.

[4] I. Győri and M. Pituk, *Comparison theorems and asymptotic equilibrium for delay differential and difference equations*, Dynam. Systems Appl. 5 (1996), 277–302.

[5] M. L. Heard, *A change of variables for functional differential equations*, J. Differential Equations 18 (1975), 1–10.

[6] T. Kato and J. B. McLeod, *The functional differential equation* $y'(x) = ay(\lambda x) + by(x)$, Bull. Amer. Math. Soc. 77 (1971), 891–937.

[7] M. Kuczma, B. Choczewski and R. Ger, *Iterative Functional Equations*, Encyclopedia Math. Appl., Cambridge Univ. Press, 1990.

[8] F. Neuman, *On transformations of differential equations and systems with deviating argument*, Czechoslovak Math. J. 31 (1981), 87–90.

[9] M. Pituk, *On the limits of solutions of functional differential equations*, Math. Bohemica 118 (1993), 53–66.

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