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On the Hartogs-type series for harmonic functions on Hartogs domains in $\mathbb{R}^n \times \mathbb{R}^m$, $m \geq 2$

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Abstract. We study series expansions for harmonic functions analogous to Hartogs series for holomorphic functions. We apply them to study conjugate harmonic functions and the space of square integrable harmonic functions.

1. Introduction and the statements of results. If a domain D in $\mathbb{C}^n \times \mathbb{C}^m$ has the form

$$D = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}^m : |w| < \varphi(z)\}$$

then each function f holomorphic on D can be expressed as

$$f(z,w) = \sum_{|\alpha|=0}^{\infty} f_\alpha(z) w^\alpha.$$

The series on the right converges almost uniformly on D (i.e. uniformly on each compact subset of D). Such an expansion is called the *Hartogs series* of f.

In the present paper we consider analogous expansions for harmonic functions on Hartogs domains in $\mathbb{R}^n \times \mathbb{R}^m$, $n \geq 1$, $m \geq 2$.

Definition 1. A domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, $n \geq 1$, $m \geq 2$, is a *Hartogs domain* if $(x,y) \in \Omega$ implies that $(x,s) \in \Omega$ for every $s \in \mathbb{R}^m$ such that $|y| = |s|$. The symbol $|\cdot|$ denotes here (and in the sequel) the euclidean norm.

Definition 2. Let Ω be a Hartogs domain in $\mathbb{R}^n \times \mathbb{R}^m$, $m \geq 2$, and let Ω' denote the orthogonal projection of Ω onto $\mathbb{R}^n$. We define $\widehat{\Omega} \subset \Omega' \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$ in the following way:

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(a) If Ω does not intersect $\mathbb{R}^n \times \{0\}$ then $\widehat{\Omega} = p(\Omega)$, where $p(x,y) = (x,|y|)$.

(b) If $\Omega \cap (\mathbb{R}^n \times \{0\}) \neq \emptyset$ then

$$\widehat{\Omega} = p(\Omega) \cup \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : (x,-y) \in p(\Omega)\}.$$

For every $j \in \mathbb{N} \cup \{0\}$ let $\{P_{jr}\}_{r=1,\ldots,r(j)}$ be an orthonormal basis in the space of j-homogeneous polynomials equipped with the $L^2(S^{m-1})$ norm (i.e. the space of spherical harmonics of degree j on S^{m-1}). Note that

$$r(0) = 1, \qquad r(1) = m$$

and

$$r(j) = \binom{m+j-1}{m-1} - \binom{m+j-3}{m-1}$$

for $j \geq 2$ (see [2], p. 82).

We shall prove the following

Theorem 1. *Let h be a harmonic function on a Hartogs domain Ω in $\mathbb{R}^n \times \mathbb{R}^m$, $m \geq 2$. Then*

$$h(x,y) = \sum_{j=0}^{\infty} \sum_{r=1}^{r(j)} P_{jr}(y) u_{jr}(x,|y|)$$

where each u_{jr} is a real-analytic function on $\widehat{\Omega}$ and the series converges almost uniformly on Ω. For each j and r the function $u_{jr}(x,t)$ satisfies the equation

$$(*) \qquad \Delta u_{jr} + \frac{2j+m-1}{t} \frac{\partial u_{jr}}{\partial t} = 0.$$

If $\Omega \cap (\mathbb{R}^n \times \{0\}) \neq \emptyset$ then $u_{jr}(x,t) = u_{jr}(x,-t)$. (Note that $\Omega \cap (\mathbb{R}^n \times \{0\}) = \widehat{\Omega} \cap (\mathbb{R}^n \times \{0\})$.) This will be a direct consequence of

Theorem 2. *Let $\widehat{\Omega} \cap (\mathbb{R}^n \times \{0\}) \neq \emptyset$. Denote by V the sum of all open balls $B((x,0),\varrho)$ such that $B((x,0),\varrho\sqrt{2}) \subset \widehat{\Omega}$. Let $V' = V \cap (\mathbb{R}^n \times \{0\}) = \widehat{\Omega} \times (\mathbb{R}^n \times \{0\}) = \Omega \times (\mathbb{R}^n \times \{0\})$. Then for each j and r there exists a function f_{jr}, real-analytic on V', such that*

$$u_{jr}(x,t) = \sum_{\beta=1}^{\infty} c_{jr\beta} \Delta^\beta f_{jr}(x) t^{2\beta} \qquad \text{on } V$$

where the series converges almost uniformly on V.

Theorem 2 generalizes the fact proved for harmonic functions in [1].

If Ω does not intersect $\mathbb{R}^n \times \{0\}$ we can use Weinstein's formula from [10] to get

THEOREM 3. *Assume that $\widehat{\Omega}$ has connected vertical sections $\widehat{\Omega}_x = \{t \in \mathbb{R} : (x,t) \in \widehat{\Omega}\}$ for each $x \in \Omega'$. Then*

(a) *If m is odd, i.e. $m = 2l+1$, $l \geq 1$, then for every j, r there exists a function h_{jr} harmonic on $\widehat{\Omega}$ such that*

$$u_{jr}(x,t) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{j+l} h_{jr}(x,t)$$

and the expansion from Theorem 1 takes the form

$$h(x,y) = \sum_{j=0}^{\infty}\sum_{r=1}^{r(j)}\sum_{i=1}^{j+1} c_{jril}\frac{1}{|y|^{2(j+l)-i}}\left(\frac{\partial^i}{\partial t^i}h_{jr}\right)(x,|y|).$$

(b) *If m is even, i.e. $m = 2l$, $l \geq 1$, then there exists a function h_{jr} on $\widehat{\Omega}$ which satisfies the equation*

$$(**) \qquad \Delta h_{jr} - \frac{1}{t}\frac{\partial h_{jr}}{\partial t} = 0$$

and such that

$$u_{jr} = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{j+l} h_{jr}.$$

Hence in case (b) we can also write down a formula analogous to that of (a).

The situation for $m = 2l$ is however worse, because if h_{jr} satisfies $(**)$ then $\frac{\partial^i}{\partial t^i}h_{jr}$ may not satisfy it.

In the last part of our paper we shall use the expansions given above to study conjugate harmonic functions and the space of square integrable harmonic functions on Ω.

It should be mentioned here that the equations $(*)$ are the special case of the more general class of singular differential equations $\Delta u + \frac{2\alpha+1}{t}\frac{\partial u}{\partial t} = 0$. Those and similar equations were studied by many authors (see [7] for $n = 1$, [3] for $n > 1$ and [10]). Similar problems were studied in [4], [6], [9] and in the so-called axially symmetric potential theory.

2. Proofs

Proof of Theorem 1. Let $\Omega_1 = \Omega \setminus (\mathbb{R}^n \times \{0\})$. For $(x,y) \in \Omega_1$ consider the sphere $S = \{(x,s) \in \Omega_1 : |s| = |y|\}$. We can consider the function $\varphi(s/|s|) = h(x,s)$ on the unit sphere S_1 in $\mathbb{R}^m$. The function φ can be expressed as the sum of a series of spherical harmonics $P_{jr}|_{S_1}$ (see [8], Chapter III). Thus $\varphi = \sum_{j=0}^{\infty}\sum_{r=1}^{r(j)} a_{jr}P_{jr}$. The coefficients a_{jr} depend only on x and $|y|$. Hence we have

$$h(x,y)=\varphi\left(\frac{y}{|y|}\right)=\sum_{j=0}^{\infty}\sum_{r=1}^{r(j)}a_{jr}(x,|y|)P_{jr}\left(\frac{y}{|y|}\right)=\sum_{j=0}^{\infty}\sum_{r=1}^{r(j)}\frac{a_{jr}(x,|y|)}{|y|^j}P_{jr}(y).$$

We define

$$u_{jr}(x,|y|)=\frac{a_{jr}(x,|y|)}{|y|^j}=\frac{a_{jr}(x,t)}{t^j}\qquad(|y|=t).$$

The estimates from [8], Chapter III, p. 315, show that for each j and r, $u_{jr}(x,|y|)$ is locally bounded on Ω_1 and the series $\sum_{j=0}^{\infty}\sum_{r=1}^{r(j)}u_{jr}(x,|y|)\times P_{jr}(y)$ is convergent in $L^2(\Omega_1,\mathrm{loc})$ to $h(x,y)$. Let $\psi(x,t)\in C_0^\infty(\widehat{\Omega}_1)$. Then $P_{jr}(y)\psi(x,|y|)\in C_0^\infty(\Omega_1)$. Since $h(x,y)$ is harmonic on Ω_1 we have

$$\begin{aligned}
0&=\langle h(x,y),\Delta[P_{jr}(y)\psi(x,|y|)]\rangle_{\Omega_1}\\
&=\int_{\Omega_1}h(x,y)\overline{\Delta[P_{jr}(y)\psi(x,|y|)]}\,dV\\
&=\int_{\Omega_1}h(x,y)\overline{P_{jr}(y)}\,\overline{\left[\Delta\psi(x,t)+\frac{2j+m-1}{t}\frac{\partial\psi}{\partial t}(x,t)\right]_{t=|y|}}\,dV\\
&=\int_{\Omega_1}|P_{jr}(y)|^2\,\overline{\left[\Delta\psi+\frac{2j+m-1}{t}\frac{\partial\psi}{\partial t}\right](x,|y|)}\,u_{jr}(x,|y|)\,dV\\
&=\int_{\widehat{\Omega}_1}t^{2j+m-1}\overline{\left[\Delta\psi+\frac{2j+m-1}{t}\frac{\partial\psi}{\partial t}\right](x,t)}\,u_{jr}(x,t)\,dV\\
&=\left\langle u_{jr},t^{2j+m-1}\left(\Delta\psi+\frac{2j+m-1}{t}\frac{\partial\psi}{\partial t}\right)\right\rangle_{\widehat{\Omega}_1}.
\end{aligned}$$

This means that u_{jr} is a weak solution of the differential equation

$$t^{2j+m-1}\Delta u+t^{2j+m-2}(2j+m-1)\frac{\partial u}{\partial t}=0.$$

Since $t>0$, the operator on the left is strongly elliptic on $\widehat{\Omega}_1$ and has real-analytic coefficients. Hence, by the Friedrichs theorem and Weyl lemma each $u_{jr}(x,t)$ is real-analytic on $\widehat{\Omega}$. This implies that

$$\Delta u_{jr}(x,t)+\frac{2j+m-1}{t}\frac{\partial u_{jr}}{\partial t}=0\qquad\text{on }\widehat{\Omega}_1$$

and each function $P_{jr}(y)u_{jr}(x,|y|)$ is harmonic on Ω_1. The Harnack theorem implies that the series $\sum_{j=0}^{\infty}\sum_{r=1}^{r(j)}P_{jr}(y)u_{jr}(x,|y|)$ is convergent to $h(x,y)$ almost uniformly on Ω_1.

We must now check what is going on near the set $\Omega\cap(\mathbb{R}^n\times\{0\})$.

Let $(x,0) \in \Omega \cap (\mathbb{R}^n \times \{0\})$. Without loss of generality we can assume that $x = 0$.

If $B(0,\varrho) \subset \Omega$ then $h(x,y) = \sum_{k=1}^{\infty} h_k(x,y)$, where $h_k(x,y)$ is a k-homogeneous harmonic polynomial on $\mathbb{R}^n \times \mathbb{R}^m$. This series converges almost uniformly on $B(0,\varrho)$. On the other hand, $h(x,y)$ is the sum of an almost uniformly convergent power series on $B(0,\varrho/\sqrt{2})$ (see [5]). This implies in particular that the terms of this power series can be permuted and regrouped without affecting the convergence of the series. The series $\sum_{k=0}^{\infty} h_k(x,y)$ can be regarded on $B(0,\varrho/\sqrt{2})$ as the permuted and regrouped power series of $h(x,y)$ at 0.

For every k, $h_k(x,y) = \sum_{i=0}^{k} w_{i,k}(y) v_{k-i,k}(x)$ where $w_{i,k}(y)$ is an i-homogeneous polynomial in y and $v_{k-i,k}(x)$ is a $(k-i)$-homogeneous polynomial in x. The polynomial $w_{i,k}(y)$ can be written as

$$w_{i,k}(y) = \sum_{j=0}^{i} \sum_{r=1}^{r(j)} P_{jr}(y) c_{jr} |y|^{2(\frac{i-j}{2})},$$

where $c_{jr} = 0$ if $(i-j)/2 \notin \mathbb{N}$.

As a result, the power series for $h(x,y)$ at zero can be written in the form

$$h(x,y) = \sum_{j=0}^{\infty} \sum_{r=1}^{r(j)} P_{jr}(y) \Big[\sum_{|\alpha|,\beta=0}^{\infty} c_{jr\alpha\beta} x^{\alpha} |y|^{2\beta} \Big].$$

The absolute convergence of the power series for $h(x,y)$ in $B(0,\varrho/\sqrt{2})$ implies that for each j,r the power series $\sum_{|\alpha|,\beta=0}^{\infty} c_{jr\alpha\beta} x^{\alpha} |t|^{2\beta}$ converges absolutely and almost uniformly to a real-analytic function $u_{jr}(x,t)$ on $B(0,\varrho/\sqrt{2}) \subset \widehat{\Omega}$. We have

$$\begin{aligned} 0 = \Delta h(x,y) &= \Delta\Big(\sum_{j=0}^{\infty} \sum_{r=1}^{r(j)} P_{jr}(y) u_{jr}(x,|y|) \Big) \\ &= \sum_{j=0}^{\infty} \sum_{r=1}^{r(j)} \Delta(P_{jr}(y) u_{jr}(x,|y|)) \\ &= \sum_{j=0}^{\infty} \sum_{r=1}^{r(j)} P_{jr}(y) \Big(\Delta u_{jr} + \frac{2j+m-1}{t} \frac{\partial u_{jr}}{\partial t} \Big)(x,|y|). \end{aligned}$$

(The fourth equality is valid because in the case of power series one can interchange differentiation and summation.) Hence each u_{jr} satisfies $(*)$ on $B(0,\varrho)$. Note that $u_{jr}(x,t) = u_{jr}(x,-t)$ on $B(0,\varrho)$ in $\widehat{\Omega}$ and $\frac{\partial u_{jr}}{\partial t}(x,0) = 0$ for $|x| < \varrho$.

Proof of Theorem 2. Fix j, r and $(x_0, 0) \in V$ and let $\varrho > 0$ be such that $B((x_0, 0), \varrho\sqrt{2}) \subset \widehat{\Omega}$. We already proved that

$$u_{jr}(x,t) = \sum_{|\alpha|,\beta=0}^{\infty} c_{jr\alpha\beta}(x-x_0)^{\alpha} t^{2\beta} \quad \text{on } B((x_0,0),\varrho).$$

Thus u_{jr} on $B((x_0,0),\varrho)$ can be written as $u_{jr}(x,t) = \sum_{\beta=0}^{\infty} g_\beta(x) t^{2\beta}$ where each g_β is analytic on $B(x_0,\varrho) \subset \mathbb{R}^n$. We have

$$\begin{aligned}
0 &= \Delta u_{jr} + \frac{2j+m-1}{t}\frac{\partial u_{jr}}{\partial t} \\
&= \sum_{\beta=0}^{\infty} \left[(\Delta_x g_\beta) t^{2\beta} + h_\beta (2\beta-1) 2\beta t^{2\beta-2} + \frac{2j+m-1}{t} 2\beta g_\beta t^{2\beta-1} \right] \\
&= \sum_{\beta=0}^{\infty} [(\Delta_x h_\beta) t^{2\beta} + (2\beta-1)(2\beta+2j+m-1) g_\beta t^{2\beta-1}] \\
&= \sum_{\beta=0}^{\infty} [\Delta_x g_\beta + (2\beta+1)(2\beta+2+2j+m-1) g_{\beta+1}] t^{2\beta}.
\end{aligned}$$

Hence for each $\beta \geq 0$ we have

$$g_{\beta+1} = -\frac{1}{(2\beta+1)(2\beta+2j+m+1)} \Delta g_\beta$$

and there exist constants $c_{jr\beta}$ such that $g_\beta = c_{jr\beta}\Delta^\beta g_0$ on $B(x_0,\varrho)$. Note that $g_0(x) = u_{jr}(x,0)$.

Since the same construction can be made for each $(x_0,0) \in V'$, we can put $f_{jr}(x) = u_{jr}(x,0)$ and get Theorem 2.

REMARK 1. Theorem 2 implies that if $u_{jr}(x,0) \equiv 0$ on some open subset of V' then $u_{jr}(x,t) \equiv 0$ on the whole $\widehat{\Omega}$. This is an interesting difference between our case and the case of harmonic functions on $\widehat{\Omega}$.

Proof of Theorem 3. A. Weinstein proved the following fact: If u satisfies on $\widehat{\Omega}$ the equation

$$\Delta u + \frac{k}{t}\frac{\partial u}{\partial t} = 0$$

then there exists v such that $u = \frac{1}{t}\frac{\partial v}{\partial t}$ and v satisfies the equation

$$\Delta v + \frac{k-2}{t}\frac{\partial v}{\partial t} = 0.$$

(For a generalized version of this result see [10].)

Let us outline the proof of Weinstein's result. Since $\widehat{\Omega}$ has connected vertical sections, there exists φ such that $\partial\varphi/\partial t = tu$. Hence

$$0 = \Delta\left(\frac{1}{t}\frac{\partial\varphi}{\partial t}\right) + \frac{k}{t}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial\varphi}{\partial t}\right) = \frac{1}{t}\frac{\partial}{\partial t}\left(\Delta\varphi + \frac{k-2}{t}\frac{\partial\varphi}{\partial t}\right).$$

The fact that $\widehat{\Omega}$ has connected vertical sections implies further that

$$\left(\Delta\varphi + \frac{k-2}{t}\frac{\partial\varphi}{\partial t}\right)(x,t) = f(x), \qquad x \in \Omega'.$$

Let $g(x)$ be any solution of the equation $\Delta g = f$. We have

$$\frac{1}{t}\frac{\partial}{\partial t}(\varphi - g) = u \quad \text{and} \quad \Delta(\varphi - g) + \frac{k-2}{t}\frac{\partial}{\partial t}(\varphi - g) = 0.$$

Thus, we can take $v = \varphi - g + h$, where h is a harmonic function on Ω'. In order to prove our Theorem 3 it suffices now to apply Weinstein's result $k + l$ times.

However, we must say that the functions h_{jr} are not uniquely determined by u_{jr}. They also depend on the choice of $j + l$ functions $h_1, \ldots, h_{j+l}$ on Ω' during the subsequent steps in the construction of h_{jr}.

3. Applications

3.1. *Conjugate harmonic functions.* We assume in this section that $n=1$. An $(m+1)$-tuple of harmonic functions $h_0, h_1, \ldots, h_m$ defined on a domain in $\mathbb{R} \times \mathbb{R}^m$ is called *conjugate harmonic functions* if the following equations are satisfied:

$$\frac{\partial h_0}{\partial y_j} = \frac{\partial h_j}{\partial x}, \qquad j = 1, \ldots, m,$$
$$\frac{\partial h_j}{\partial y_i} = \frac{\partial h_i}{\partial y_j}, \qquad i, j = 1, \ldots, m,$$

and

$$\frac{\partial h_0}{\partial x} + \sum_{j=1}^{m} \frac{\partial h_j}{\partial y_j} = 0.$$

The above equations are equivalent to the fact that every point of our domain has a neighborhood on which there exists a harmonic function H such that $h_0 = \partial H/\partial x$ and $h_j = \partial H/\partial y_j$ for $j = 1, \ldots, m$.

Assume now that Ω is a Hartogs domain in $\mathbb{R}$ which has the form either

$$\Omega = \{(x,y) : |y| < \varphi(x),\ x \in (a,b)\}$$

where φ is concave and there is $x_0 \in (a,b)$ such that $\varphi(x_0) = \sup_{s\in(a,b)} \varphi(s) = R$, or

$$\Omega = \{(x,y) : 0 \le \varrho(x) < |y| \le \varphi(x)\}$$

where ϱ is convex on (a,b), φ is concave on (a,b) and there is $x_0 \in (a,b)$ for which $\sup_{s\in(a,b)} \varphi(s) = \varphi(x_0) = R_2$, $\inf_{s\in(a,b)} \varrho(s) = \varrho(x_0) = R_1$ (an example of a domain of this last type is the filled-in torus in $\mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^3$).

THEOREM 4. *Let Ω be as above. If $h_0(x,y)$ is a harmonic function on Ω then there exists a harmonic function H on Ω such that $\partial H/\partial x = h_0$ on Ω. Hence $h_0, h_1, \ldots, h_m$ where $h_i = \partial H/\partial y_i$, $i = 1,\ldots,m$, form an $(m+1)$-tuple of conjugate harmonic functions on Ω. If*

$$h_0(x,y) = \sum_{j=0}^{\infty} \sum_{r=1}^{r(j)} P_{jr}(y) u_{jr}(x,|y|)$$

then

$$H(x,y) = g(y) + \sum_{jr} P_{jr}(y) \Big[\int_{x_0}^{x} u_{jr}(s,|y|)\, ds + \psi_{jr}(|y|) \Big]$$

where $\psi_{jr}(t)$ satisfies the equation

$$(***) \qquad \frac{\partial^2 \psi_{jr}}{\partial t^2} + \frac{2j+m-1}{t} \frac{\partial \psi_{jr}}{\partial t} = -\frac{\partial u_{jr}}{\partial x}(x_0,t)$$

and $g(y)$ is an arbitrary harmonic function defined either on $\{y \in \mathbb{R}^m : |y| < R\}$ or on $\{y \in \mathbb{R}^m : R_1 < |y| < R_2\}$. The equation $(***)$ is equivalent to the harmonicity of $P_{jr}(y)[\int_{x_0}^{x} u_{jr}(s,|y|)\, ds + \psi_{jr}(|y|)]$.

Proof. Define $f(x,y) = \int_{x_0}^{x} h_0(s,y)\, ds$. We have

$$\begin{aligned} \Delta f(x,y) &= \frac{\partial h_0}{\partial x}(x,y) + \int_{x_0}^{x} \Delta h_0(s,y)\, ds \\ &= \frac{\partial h_0}{\partial x}(x,y) + \int_{x_0}^{x} \Big(-\frac{\partial^2 h_0}{\partial s^2} \Big) ds = \frac{\partial h}{\partial x}(x_0,y). \end{aligned}$$

Let $g_0(y)$ be a solution of the equation $\Delta g_0 = \frac{\partial h}{\partial x}(x_0,y)$ on $\{y \in \mathbb{R}^m : |y| < R\}$ (or on $\{y \in \mathbb{R}^m : R_1 < |y| < R_2\}$).

Put $H(x,y) = f(x,y) - g_0(y)$. The function $H(x,y)$ is harmonic and $\partial H/\partial x = h_0$. By Theorem 1 we can write a Hartogs series for H,

$$H(x,y) = \sum_{j=0}^{\infty} \sum_{r=1}^{r(j)} P_{jr}(y) v_{jr}(x,|y|).$$

We have

$$
\begin{aligned}
- g_0(y) &= H(x,y) - f(x,y) \\
&= \sum_{j=0}^{\infty} \sum_{r=1}^{r(j)} \Big[v_{jr}(x,|y|) - \int_{x_0}^{x} u_{jr}(s,|y|)\,ds \Big] P_{jr}(y) \\
&= \sum_{j=0}^{\infty} \sum_{r=1}^{r(j)} \psi_{jr}(|y|) P_{jr}(y),
\end{aligned}
$$

where

$$
\psi_{jr}(|y|) = v_{jr}(x,|y|) - \int_{x_0}^{x} u_{jr}(s,|y|)\,ds
$$

depends only on $|y|$ since g_0 depends only on y. The fact that $\psi_{jr}(t)$ satisfies $(***)$ can be checked by simple calculation.

Remark 2. It follows from the proof of Theorem 4 that in contrast to the case of $m = 1$, the m-tuple $h_1, \ldots, h_m$ of harmonic functions which are conjugate to a given harmonic h_0 is not uniquely determined. If $h'_1, \ldots, h'_m$ is another such m-tuple and $g_i = h_i - h'_i$, $i = 1, \ldots, m$, then the g_i depend locally only on y and form locally an m-tuple of conjugate harmonic functions in $\mathbb{R}^m$.

Remark 3. Theorem 4 can be extended to a wider class of domains, namely those Ω which have connected horizontal sections. Let Ω'' denote the orthogonal projection of Ω onto $\mathbb{R}^m$. We assume that for each $y \in \Omega''$ the set $\{x \in \mathbb{R} : (x,y) \in \Omega\}$ is connected. Let $w(y)$ be a real-analytic function $\Omega'' \to \mathbb{R}^m$ whose graph is contained in Ω.

We can now define $f(x,y) = \int_{w(y)}^{x} h_0(s,y)\,ds$ (as in the proof of Theorem 4). The rest of the proof remains unchanged.

Remark 4. Conjugate harmonic functions on half-spaces of $\mathbb{R}^{m+1}$ were considered by E. Stein [8] in connection with Riesz transforms. Conjugate harmonic functions on domains in $\mathbb{R}^{m+1}$ were studied by R. Z. Yeh [11].

3.2. *The space of square integrable harmonic functions on Ω.* Let Ω be a bounded Hartogs domain in $\mathbb{R}^n \times \mathbb{R}^m$, $m \geq 2$. The following fact holds:

Proposition 1. *The space $L^2\,\mathrm{Harm}(\Omega)$ of square integrable harmonic functions can be expressed as the l^2-sum of spaces $L^2\,\mathrm{Harm}_{\alpha(j)}(\widehat{\Omega}, |t|^{2\alpha(j)+1})$, $\alpha(j) = (2j+m-1)/2$, with each $L^2\,\mathrm{Harm}_{\alpha(j)}(\widehat{\Omega}, |t|^{2\alpha(j)+1})$ repeated $r(j)$ times. Here $L^2\,\mathrm{Harm}_{\alpha(j)}(\widehat{\Omega}, |t|^{2\alpha(j)+1})$ denotes the space of functions on $\widehat{\Omega}$ which satisfy the equation $(*)$ and are square integrable with weight $|t|^{2\alpha(j)+1}$.*

P r o o f. It follows from Theorem 1 that for harmonic $h \in L^2(\Omega)$,

$$\|h\|^2_{L^2(\Omega)} = \sum_{j=0}^{\infty}\sum_{r=1}^{r(j)} \int_{\widehat{\Omega}} |u_{jr}(x,t)|^2 |t|^{2j+m-1}\, dV_x\, dt$$
$$= \sum_{j=0}^{\infty}\sum_{r=1}^{r(j)} \|u_{jr}\|_{L^2(\widehat{\Omega},|t|^{2\alpha(j)+1})}, \qquad \alpha(j) = \frac{2j+m-1}{2}.$$

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