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Homogeneous extremal function for a ball in $\mathbb{R}^2$

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Abstract. We point out relations between Siciak's homogeneous extremal function Ψ_B and the Cauchy–Poisson transform in case B is a ball in $\mathbb{R}^2$. In particular, we find effective formulas for Ψ_B for an important class of balls. These formulas imply that, in general, Ψ_B is not a norm in $\mathbb{C}^2$.

0. Introduction. Let $\mathcal{P}(\mathbb{C}^n)$ and $\mathcal{H}(\mathbb{C}^n)$ denote the set of polynomials of n complex variables and the set of homogeneous polynomials of n variables, respectively. We denote by $\mathcal{L}(\mathbb{C}^n)$ the Lelong class of plurisubharmonic functions u in $\mathbb{C}^n$ with logarithmic growth: $u(z) \le \text{const} + \log(1 + \|z\|)$.

An important role in pluripotential theory and approximation theory of many variables is played by two extremal functions introduced by Siciak (see [Si1]–[Si5]) and called *Siciak's extremal function* (or *polynomial extremal function*) Φ_E and *Siciak's homogeneous extremal function* Ψ_E, respectively:

$$\Phi_E(z) = \sup\{|p(z)|^{1/\deg p} : p \in \mathcal{P}(\mathbb{C}^n),\ \deg p \ge 1,\ \|p\|_E \le 1\},\ z \in \mathbb{C}^n,$$
$$\Psi_E(z) = \sup\{|p(z)|^{1/\deg p} : p \in \mathcal{H}(\mathbb{C}^n),\ \deg p \ge 1,\ \|p\|_E \le 1\},\ z \in \mathbb{C}^n,$$

where E is a fixed compact subset of $\mathbb{C}^n$. It is well known (see [Si4], [Si5]) that

$$\log \Phi_E(z) = V_E(z) := \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^n),\ u|_E \le 0\}$$

and

$$\Psi_E(z) = \sup\{u(z) : u \text{ is homogeneous psh in } \mathbb{C}^n,\ u|_E \le 1\}.$$

If E is a circular set, there is a simple relation between Φ_E and Ψ_E (see [Si4]):

$$\Phi_E(z) = \max(1, \Psi_E(z)).$$

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In particular, if B is a closed unit ball with respect to a norm q in $\mathbb{C}^n$ then
$$\Psi_B(z) = q(z), \quad z \in \mathbb{C}^n$$
(see [Si4]).

The situation is much more complicated if B is a ball in $\mathbb{R}^n$ with respect to a norm q. Here we treat $\mathbb{R}^n$ as a subset of $\mathbb{C}^n$ such that $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$. It is known (see [Si1], [D]) that if B_n is the unit Euclidean ball in $\mathbb{R}^n$, then $\Psi_{B_n}(z)$ is equal to the Lie norm:
$$\Psi_{B_n}(z) = L_n(z) = \left(\frac{\|z\|^2 + |z^2|}{2}\right)^{1/2} + \left(\frac{\|z\|^2 - |z^2|}{2}\right)^{1/2},$$
where $z^2 = z_1^2 + \ldots + z_n^2$. The Lie norm is equal to the so-called projective crossnorm $\|z\|_\wedge$ for the projective tensor product $\mathbb{R}^n \widehat{\otimes}_{\mathbb{R}} \mathbb{C}$ (here $\mathbb{R}^n$ is understood to be the Euclidean space with its canonical inner product and norm). One can easily prove that in general we have the inequality
$$(*) \qquad \Psi_B(z) \ge \|z\|_\wedge, \quad z \in \mathbb{C}^n.$$
Here
$$\|z\|_\wedge = \inf\Big\{\sum_{j=1}^m |\alpha_j| q(x_j) : z = \sum_{j=1}^m \alpha_j x_j,\ \alpha_j \in \mathbb{C},\ x_j \in \mathbb{R}^n\Big\}$$
is a norm in $X \widehat{\otimes}_{\mathbb{R}} \mathbb{C}$, where $X = (\mathbb{R}^n, q)$ is a normed space such that $B = \{x \in \mathbb{R}^n : q(x) \le 1\}$. A few years ago Professor Siciak posed the question of whether in $(*)$ one has equality. In particular, is this true for the square $B = [-1,1] \times [-1,1]$?

In this paper, we show that, in general, equality in $(*)$ cannot hold for all $z \in \mathbb{C}^n$. This is a corollary to Theorem 2.3 where explicit formulas are given for Ψ_B for a wide family of norms in $\mathbb{R}^2$. The main goal of this paper is to show a relation between the extremal function Ψ_B, where B is a ball in $\mathbb{R}^2$ with respect to a norm q, and the Cauchy–Poisson transform which is an important tool in harmonic analysis (see [St], [SW])). Note that for $x \in \mathbb{R}^n$ one has
$$\Psi_B(x) = q(x).$$
In particular,
$$\log \Psi_B(1,t) = \log q(1,t)$$
if q is a norm in $\mathbb{R}^2$. Starting from the above fact, we show how to get an integral representation for Ψ_B. At the end of the paper we extend our result to a wider family of sets.

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1. Cauchy–Poisson transform. Let $\mathbb{H}_+$ and $\mathbb{H}_-$ be the upper and lower halfplanes, respectively. If q is a norm in $\mathbb{R}^2$, we put $u(t) = \log q(1,t)$. We denote by $\mathcal{P}u$ the Cauchy–Poisson transform of u in $\mathbb{H}_+$ (see e.g. [St]):

$$\mathcal{P}u(\zeta) = (\Im\zeta)\frac{1}{\pi}\int_{-\infty}^{\infty}|\zeta - t|^{-2}u(t)\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty}u(ty+x)\,\frac{dt}{1+t^2},$$

where $\zeta = x + iy \in \mathbb{H}_+$.

Lemma 1.1. *If $0 < \alpha < 1$ then there exists a constant $C = C(\alpha)$ such that for $x, x' \in \mathbb{R}$ and $y > 0$ we have*

$$|\mathcal{P}u(\zeta) - u(x')| \le C\{|x - x'| + y\}^\alpha, \qquad \zeta = x + iy.$$

Proof. Observe that for $t, \tau \in \mathbb{R}$ we have

$$\begin{aligned}|\log q(1,t) - \log q(1,\tau)| &\le M_\alpha[|q(1,t) - q(1,\tau)|(\min\{q(1,t), q(1,\tau)\})^{-1}]^\alpha\\ &\le M_\alpha[q(0,1)|t-\tau|(\min\{q(1,t), q(1,\tau)\})^{-1}]^\alpha\\ &\le M_\alpha\left[\frac{q(0,1)}{\inf_{t\in\mathbb{R}} q(1,t)}\right]^\alpha |t-\tau|^\alpha = M'_\alpha|t-\tau|^\alpha,\end{aligned}$$

where $M_\alpha = \sup_{x>0}(\log(1+x))/x^\alpha$. Now we have

$$\begin{aligned}|\mathcal{P}u(\zeta) - u(x')| &\le \frac{1}{\pi}\int_{-\infty}^{\infty}|u(ty+x) - u(x')|\,\frac{dt}{1+t^2}\\ &\le \frac{M'_\alpha}{\pi}\int_{-\infty}^{\infty}|ty + x - x'|^\alpha\,\frac{dt}{1+t^2}\\ &\le \frac{M'_\alpha}{\pi}\int_{-\infty}^{\infty}\frac{(1+|t|)^\alpha}{1+t^2}\,dt\,[|x-x'|+y]^\alpha = C(\alpha)[|x-x'|+y]^\alpha,\end{aligned}$$

which completes the proof.

Corollary 1.2. *The function $\mathcal{P}u$ extends to a continuous function in $\overline{\mathbb{H}}_+$ that is harmonic in $\mathbb{H}_+$. If we set*

$$\mathcal{P}u(\zeta) = \mathcal{P}u(\overline{\zeta}), \qquad \zeta \in \mathbb{H}_-,$$

we obtain a continuous function in $\mathbb{C}$, symmetric with respect to the real axis and harmonic in $\mathbb{H}_+ \cup \mathbb{H}_-$. Moreover, for $\zeta = x + iy$, we have

$$\mathcal{P}u(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty}u(t|y| + x)\,\frac{dt}{1+t^2}, \qquad \zeta \in \mathbb{C}.$$

Applying the maximum principle for subharmonic functions in $\mathbb{H}_+$ or $\mathbb{H}_-$, we easily obtain the following important

Corollary 1.3. *If* $B = \{x \in \mathbb{R}^2 : q(x) \le 1\}$ *then*

$$\log \Psi_B(1,\zeta) \le \mathcal{P}u(\zeta), \qquad \zeta \in \mathbb{C}.$$

Now we prove that $\mathcal{P}u \in \mathcal{SH}(\mathbb{C})$. To do this we need the following results which are interesting in themselves.

For a fixed $\alpha \in (-1,1)$, define

$$v(\alpha, y) := \tfrac{1}{2}\log(1 + 2\alpha y + y^2), \qquad y \in \mathbb{R},$$

and set $\beta = \sqrt{1-\alpha^2}$. Note that if $|y| < 1$ then

$$v(-\alpha, y) = -\sum_{k=1}^{\infty} \frac{1}{k} T_k(\alpha) y^k,$$

where $T_k(\alpha)$ denotes the kth Chebyshev polynomial $T_k(\alpha) = \cos(k \arccos \alpha)$ (see e.g. [SW]).

Lemma 1.4. *For all* $y \in \mathbb{R}$,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} v(\alpha, ty)\, \frac{dt}{1+t^2} = v(\beta, |y|).$$

P r o o f. Denote the left hand side of the above formula by $F_\alpha(y)$. Since $F_\alpha(y)$ and $v(\beta,|y|)$ are even functions that agree at 0, it suffices to show that $F'_\alpha(y) = v'(\beta, y)$ for $y > 0$. We can check this by applying the residue method. The calculation is rather simple but a little laborious so we omit it.

Lemma 1.5. *If* $\zeta = x + iy$ *then*

$$\mathcal{P}v(\alpha,\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} v(\alpha, t|y| + x)\, \frac{dt}{1+t^2} = \frac{1}{2}\log(1 + 2\alpha x + x^2 + 2\beta|y| + y^2).$$

P r o o f. We apply Lemma 1.4 with

$$\alpha' = \frac{\alpha + x}{\sqrt{1 + 2\alpha x + x^2}} \quad \text{and} \quad y' = \frac{|y|}{\sqrt{1+2\alpha x + x^2}}.$$

Lemma 1.6. $\mathcal{P}v(\alpha,\zeta) \in \mathcal{SH}(\mathbb{C})$.

P r o o f. We apply the Zaremba criterion (see [L, pp. 439–440]). Let $v \in \mathcal{C}(\Omega)$. Put

$$\Delta_h v(\zeta) = v(\zeta + h) + v(\zeta - h) + v(\zeta + ih) + v(\zeta - ih) - 4v(\zeta), \qquad h \in \mathbb{R}_*,$$

and define the Zaremba operator

$$\underline{\Delta} v(\zeta) := \limsup_{h \to 0} \frac{1}{h^2} \Delta_h v(\zeta).$$

Then $v \in \mathcal{SH}(\Omega)$ iff $\underline{\Delta} v \geq 0$ in Ω. We apply this criterion to $\mathcal{P}v(\alpha,\zeta)$. If $\zeta \in \mathbb{C}\setminus\mathbb{R}$ then $\underline{\Delta}\mathcal{P}v(\alpha,\zeta) = \Delta\mathcal{P}v(\alpha,\zeta) = 0$, since $\mathcal{P}v(\alpha,\zeta)$ is harmonic in $\mathbb{C}\setminus\mathbb{R}$. If $\zeta\in\mathbb{R}$, we easily calculate that $\underline{\Delta}\mathcal{P}v(\alpha,\zeta) = \infty$.

COROLLARY 1.7. *Let* $u(t) = \frac{1}{2}\log(at^2+bt+c)$ *and* $\alpha = b/(2\sqrt{ac})$, *where* $\Delta = b^2-4ac<0,\ c,a>0$. *Then* $\mathcal{P}u \in \mathcal{SH}(\mathbb{C})$.

Proof. We have

$$\mathcal{P}u(\zeta) = \mathcal{P}v\left(\alpha, \sqrt{\frac{a}{c}}\zeta\right) + \frac{1}{2}\log c,$$

whence we can apply Lemma 1.6.

Now we are in a position to prove the following

PROPOSITION 1.8. *If* q *is a norm in* $\mathbb{R}^2$ *and* $u(t) = \log q(1,t)$, *then* $\mathcal{P}u \in \mathcal{SH}(\mathbb{C})$. *This implies that* $\mathcal{P}u$ *belongs to the Lelong class* $\mathcal{L}(\mathbb{C})$.

Proof. Denote by q^* the dual norm: $q^*(x) = \sup\{x\cdot y : y \in B\}$. We can write (see [B3])

$$q(x) = \sup\{x\cdot y/q^*(y) : y\in S^1\} = \lim_{k\to\infty} q_k(x),$$

where $q_k(x) = [\int_{S^1}(x\cdot y/q^*(y))^{2k}\,d\sigma(y)]^{1/2k}$ is a (smooth) norm in $\mathbb{R}^2$ and q_k^{2k} is a polynomial of degree $2k$. Moreover (cf. [B3] again), the sequence q_k is increasing. Thus $q_k^{2k}(1,\zeta)$ is a polynomial of degree $2k$ with real coefficients and without any real zeros. Applying Corollary 1.7 we easily check that $\mathcal{P}u_k \in \mathcal{SH}(\mathbb{C})$, where $u_k(t) = \log q_k(1,t)$. Finally, we have

$$\begin{aligned}\mathcal{P}u(\zeta) &= \lim_{k\to\infty}\mathcal{P}u_k(\zeta) \leq \lim_{k\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{P}u_k(\zeta+re^{i\theta})\,d\theta\\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{P}u(\zeta+re^{i\theta})\,d\theta,\end{aligned}$$

which completes the proof.

2. Homogeneous extremal function for a ball in $\mathbb{R}^2$.

The main result of this paper is the following

THEOREM 2.1. *If* q *is a norm in* $\mathbb{R}^2$, $B = \{x\in\mathbb{R}^2 : q(x)\leq 1\}$ *and* $u(t) = \log q(1,t),\ t\in\mathbb{R}$, *then*

$$\Psi_B(1,\zeta) = \exp\mathcal{P}u(\zeta), \qquad \zeta\in\mathbb{C}.$$

Consequently,

$$\Psi_B(z_1,z_2) = |z_1|\exp\mathcal{P}u(z_2/z_1).$$

Proof. We know that $\log \Psi_B(1,\zeta) \le \mathcal{P}u(\zeta),\ \zeta \in \mathbb{C}$. To prove the opposite inequality, define

$$\phi(\zeta,z) = \begin{cases} |\zeta| \exp \mathcal{P}u(\zeta^{-1}z), & \zeta \in \mathbb{C}_*,\ z \in \mathbb{C}, \\ \limsup\limits_{\xi\to 0,\,\xi\neq 0} |\xi| \exp \mathcal{P}u(\xi^{-1}z), & \zeta = 0,\ z \in \mathbb{C} \end{cases}$$

(cf. [Kl, proof of Thm. 5.1.6]). Then $\phi \in \exp \mathcal{L}(\mathbb{C}^2)$ and $\phi(\zeta w) = |\zeta|\phi(w)$, $\phi_{|B} \le 1$. This means that

$$\phi(\zeta,z) \le \Psi_B(\zeta,z),$$

whence $\mathcal{P}u(\zeta) \le \log \Psi_B(1,\zeta)$. This completes the proof.

As an interesting application, we prove the following result on a harmonic foliation related to the extremal function $\log \Psi_B$. A similar foliation is related to the extremal function $V_B = \log \Phi_B$ (see [B1], [B2] for details).

Corollary 2.2. *Let $X = (\mathbb{R}^2, q)$, let $\check{X} = X \check{\otimes}_{\mathbb{R}} \mathbb{C}$ be the injective tensor product, and let $\check{S}$ be the unit sphere in $\check{X}$. Define*

$$\chi(\zeta,c) = \tfrac{1}{2}(\zeta c + \zeta^{-1}\overline{c}), \qquad \zeta \in \mathbb{D}^* = \mathbb{C} \setminus \overline{\mathbb{D}},\ c \in \check{S}.$$

Then $\log \Psi_B$ is harmonic on each leaf $\chi(\zeta,c)$, $c \in \check{S}$.

Proof. Let $\chi(\zeta,c) = (\chi_1(\zeta,c), \chi_2(\zeta,c))$, where $c = a + ib$. Then $\chi_j(\zeta,c) = g(\zeta)a_j + i\widehat{g}(\zeta)b_j$, $j = 1,2$, with $g(\zeta) = \frac{1}{2}(\zeta + \zeta^{-1})$ and $\widehat{g}(\zeta) = \frac{1}{2}(\zeta - \zeta^{-1})$. Without loss of generality we can assume that $c_1 \neq 0$ and $\det(a,b) = \det((a_1,a_2),(b_1,b_2)) \neq 0$. Then we can write

$$\log \Psi_B(\chi(\zeta,c)) = \log|\chi_1(\zeta,c)| + \mathcal{P}u(\chi_2(\zeta,c)/\chi_1(\zeta,c)).$$

Now observe that the mapping

$$\phi_c(\zeta) = \chi_2(\zeta,c)/\chi_1(\zeta,c) : \mathbb{D}^* \to \mathbb{C}$$

takes its values in $\mathbb{H}_+$ or in $\mathbb{H}_-$. Indeed, we have

$$\phi_c(\zeta) = |\chi_1(\zeta,c)|^{-2}\chi_2(\zeta,c)\overline{\chi_1(\zeta,c)}$$

and

$$\Im(\chi_2(\zeta,c)\overline{\chi_1(\zeta,c)}) = \tfrac{1}{4}\det(a,b)(|\zeta|^2 - |\zeta|^{-2}),$$

whence $\operatorname{sgn} \Im(\phi_c(\zeta))$ is constant in $\mathbb{D}^*$. Therefore $\mathcal{P}u(\phi_c(\zeta))$ is a harmonic function as a composition of a harmonic function with a holomorphic one.

Applying Lemmas 1.5, 1.6 and Theorem 2.1 we can explicitly calculate Ψ_B for an important class of norms.

Theorem 2.3. *If n is a fixed natural number, $q_n(x) = (x_1^{2n} + x_2^{2n})^{1/(2n)}$ and $S_n = \{x \in \mathbb{R}^2 : q_n(x) = 1\}$, then, for all $z \in \mathbb{C}^2$,*

$$\Psi_{S_n}(z) = \Big[\prod_{j=1}^{n}(|z_1|^2 - 2\alpha_j \Re(z_1\overline{z}_2) + |z_2|^2 + 2|\beta_j||\Im(z_1\overline{z}_2)|)^{1/2}\Big]^{1/n},$$

where $\zeta_j = \alpha_j + i\beta_j \in \sqrt[2n]{-1}$, $j = 1, \ldots, n$, *with* $\zeta_j \neq \overline{\zeta}_k$ *for* $j \neq k$.

Corollary 2.4. *If* $q_\infty(x) = \max(|x_1|, |x_2|)$ *and* $S_\infty = \{x \in \mathbb{R}^2 : q_\infty(x) = 1\}$, *then for all* $z \in \mathbb{C}^2$,

$$\Psi_{S_\infty}(z) = \exp\Bigg[\int_0^{2\pi} \log(|z_1|^2 - 2\cos\theta\,\Re(z_1\overline{z}_2) + |z_2|^2 + 2|\sin\theta\,\Im(z_1\overline{z}_2)|)^{1/2}\frac{d\theta}{2\pi}\Bigg].$$

Proof of Theorem 2.3. Fix an $n \in \mathbb{N}$. We have

$$(*)\qquad 1 + \zeta^{2n} = \prod_{j=1}^{n}(\zeta - \zeta_j)(\zeta - \overline{\zeta}_j) = \prod_{j=1}^{n}(1 - 2\alpha_j\zeta + \zeta^2).$$

Consider $u_n(t) = (2n)^{-1}\log(1 + t^{2n}) = \log f_n(t)$, where $f_n(t) = q_n(1, t)$. Applying Lemma 1.5 and $(*)$ we obtain

$$(**)\qquad \mathcal{P}u_n(\zeta) = \frac{1}{2n}\sum_{j=1}^{n}\log\left(1 - 2\alpha_j\Re\zeta + |\zeta|^2 + 2|\beta_j||\Im\zeta|\right).$$

By Theorem 2.1 we have $\Psi_{S_n}(1, \zeta) = \exp\mathcal{P}u_n(\zeta)$, whence, by homogeneity of Ψ,

$$\Psi_{S_n}(z_1, z_2) = |z_1|\exp\mathcal{P}u_n(z_2\overline{z}_1|z_1|^{-2}),$$

and applying $(**)$ we get the formula of Theorem 2.3.

Remark 2.5. If B is the unit ball and S is the unit sphere for a norm q in $\mathbb{R}^2$ then $\mathbb{T}B$ and $\mathbb{T}S$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$, are circular subsets of $\mathbb{C}^2$. Hence we obtain

$$\Phi_{\mathbb{T}S}(z) = \max(1, \Psi_{\mathbb{T}S}(z)) = \max(1, \Psi_B(z)), \qquad z \in \mathbb{C}^2.$$

Let $X = (\mathbb{R}^2, q)$, $\widehat{X} = X \mathbin{\widehat{\otimes}_{\mathbb{R}}} \mathbb{C}$ and let $\widehat{B}$ be the unit (closed) ball in $\widehat{X}$. It is well known that

$$\operatorname{extr}\widehat{B} = \{e^{i\theta}x : x \in \operatorname{extr}B,\ \theta \in [-\pi, \pi]\} = \mathbb{T}\operatorname{extr}B.$$

In particular, if X is a strictly convex space then

$$\operatorname{extr}\widehat{B} = \mathbb{T}S.$$

Hence we get the following

COROLLARY 2.6. *If* $(\mathbb{R}^2, q)$ *is a strictly convex space then*

$$\Phi_{\operatorname{extr}\widehat{B}}(z) = \max(1, \Psi_B(z)) = \max(1, |z_1| \exp \mathcal{P}u(z_2/z_1)), \qquad z \in \mathbb{C}^2,$$

where $u(t) = \log q(1,t)$.

COROLLARY 2.7. *If* q *is a norm in* $\mathbb{R}^2$, S *is its unit sphere and* $u(t) = \log q(1,t)$, *then*

$$\widehat{\mathbb{T}S} = \{z \in \mathbb{C}^2 : \log|z_1| + \mathcal{P}u(z_2/z_1) \le 0\},$$

where $\widehat{K}$ *denotes the polynomially convex hull of* K.

Note that the equality $\Psi_B(z) = \|z\|_\wedge$ is equivalent to

$$\widehat{\mathbb{T}S} = \operatorname{conv}(\mathbb{T}S).$$

In particular, if (X, q) is a strictly convex space then $\Psi_B(z) = \|z\|_\wedge$ iff $\widehat{(\operatorname{extr}\widehat{B})} = \widehat{B}$.

REMARK 2.8. Theorem 2.1 can be extended in the following way. Denote by Γ_0 the class of all continuous, nonnegative and absolutely homogeneous functions g on $\mathbb{R}^2$ (i.e. $g(tx) = |t|g(x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^2$) such that g has the form

$$g(x) = \max_{1 \le k \le n} Q_k(x)^{1/\deg Q_k},$$

where $Q_k \ge 0$ are homogeneous polynomials and $Q_1^{-1}(0) = \{0\}$. Denote by Γ the class of continuous, nonnegative and homogeneous functions g with $g^{-1}(0) = \{0\}$ which are generated by Γ_0 with respect to the operations: limit of monotonic sequences and $(g_1 \cdot \ldots \cdot g_N)^{1/N}$. We show that Theorem 2.1 extends to Γ. We need the following

LEMMA 2.9. *If* $g \in \Gamma_0$ *and* $u(t) = \log g(1,t)$ *then* $\mathcal{P}u$ *is a continuous function that belongs to* $\mathcal{L}(\mathbb{C})$.

P r o o f. The proof that $\mathcal{P}u$ is continuous is similar to that of Lemma 1.1. It is easily seen that all numbers $\deg Q_j$ are even. Put $N = \deg Q_1 \cdot \ldots \cdot \deg Q_n$ and define

$$q_k(x) = \frac{1}{n}(Q_1(x)^{2kN/\deg Q_1} + \ldots + Q_n(x)^{2kN/\deg Q_n}).$$

Then q_k is a sequence of homogeneous polynomials of degree $2kN$ and the sequence $g_k = q_k^{1/(2kN)}$ increases to g. Let $u_k(t) = \log g_k(1,t)$. Applying Corollary 1.7 we easily obtain $\mathcal{P}u_k \in \mathcal{SH}(\mathbb{C})$. Hence, similarly to the proof of Proposition 1.8, we show that $\mathcal{P}u \in \mathcal{SH}(\mathbb{C})$ and therefore $\mathcal{P}u \in \mathcal{L}(\mathbb{C})$.

COROLLARY 2.10. *Let* $g \in \Gamma$ *and let* $u(t) = \log g(1,t)$. *Then* $\mathcal{P}u \in \mathcal{L}(\mathbb{C})$.

Now one can easily obtain a generalization of Theorem 2.1; its proof is left to the reader.

THEOREM 2.11. *Let $g \in \Gamma$ and let $B = \{x \in \mathbb{R}^2 : g(x) \le 1\}$. Set $u(t) = \log g(1,t)$. Then*

$$\Psi_B(z_1, z_2) = |z_1| \exp \mathcal{P}u(z_2/z_1).$$

COROLLARY 2.12. *For $g_1, \ldots, g_n \in \Gamma$, put $B_j = \{x \in \mathbb{R}^2 : g_j(x) \le 1\}$. Define also $g(x) = (g_1 \cdot \ldots \cdot g_n)^{1/n}$ and $B = \{x \in \mathbb{R}^2 : g(x) \le 1\}$. Then*

$$\Psi_B = (\Psi_{B_1} \cdot \ldots \cdot \Psi_{B_n})^{1/n}.$$

COROLLARY 2.13. *Let $g \in \Gamma$ and let $S = \{x \in \mathbb{R}^2 : g(x) = 1\}$. If $u(t) = \log g(1,t)$ then*

$$\Phi_{\mathbb{T}S}(z_1, z_2) = \max(1, |z_1| \exp \mathcal{P}u(z_2/z_1))$$

and

$$\widehat{\mathbb{T}S} = \{z \in \mathbb{C}^2 : \log|z_1| + \mathcal{P}u(z_2/z_1) \le 0\}.$$

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