Uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points

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Abstract. We prove a uniqueness theorem for meromorphic functions involving linear differential polynomials generated by them. As consequences of the main result we improve some previous results.

1. Introduction. Let \( f \) and \( g \) be two nonconstant meromorphic functions defined in the open complex plane \( \mathbb{C} \). If for \( a \in \mathbb{C} \cup \infty \), \( f - a \) and \( g - a \) have the same set of zeros with the same multiplicities, we say that \( f \) and \( g \) share the value \( a \) \( CM \) (counting multiplicities), and if we do not consider the multiplicities, \( f \) and \( g \) are said to share the value \( a \) \( IM \) (ignoring multiplicities). It is assumed that the reader is familiar with the standard notations and definitions of value distribution theory (cf. [3]).

M. Ozawa [6] proved the following result:

Theorem A [6]. If two nonconstant entire functions \( f, g \) share the value 1 \( CM \) with \( \delta(0; f) > 0 \) and 0 being lacunary for \( g \) then either \( f \equiv g \) or \( fg \equiv 1 \).

Improving the above result H. X. Yi [10] proved the following:

Theorem B [10]. Let \( f \) and \( g \) be two nonconstant meromorphic functions satisfying \( \delta(\infty; f) = \delta(\infty; g) = 1 \). If \( f, g \) share the value 1 \( CM \) and \( \delta(0; f) + \delta(0; g) > 1 \) then either \( f \equiv g \) or \( fg \equiv 1 \).

In [9] C. C. Yang asked: What can be said if two nonconstant entire functions \( f \) and \( g \) share the value 0 \( CM \) and their first derivatives share the value 1 \( CM \)?

As an attempt to solve this question K. Shibazaki [7] proved the following:

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Theorem C [7]. Let \( f \) and \( g \) be two entire functions of finite order. If \( f' \) and \( g' \) share the value \( 1 \) CM with \( \delta(0; f) > 0 \) and \( 0 \) being lacunary for \( g \) then either \( f \equiv g \) or \( f'g' \equiv 1 \).

Improving Theorem C, H. X. Yi [13] obtained the following result:

Theorem D [13]. Let \( f \) and \( g \) be two entire functions such that \( f^{(n)} \) and \( g^{(n)} \) share the value \( 1 \) CM. If \( \delta(0; f) + \delta(0; g) > 1 \) then either \( f \equiv g \) or \( f^{(n)}g^{(n)} \equiv 1 \).

Considering meromorphic functions H. X. Yi and C. C. Yang [15] improved Theorem C as follows:

Theorem E [15]. Let \( f \) and \( g \) be two meromorphic functions satisfying \( \delta(\infty; f) = \delta(\infty; g) = 1 \). If \( f' \) and \( g' \) share the value \( 1 \) CM with \( \delta(0; f) + \delta(0; g) > 1 \) then either \( f \equiv g \) or \( f'g' \equiv 1 \).

In [15] it is asked whether it is possible to replace the first derivatives \( f', g' \) in Theorem E by the \( n \)th derivatives \( f^{(n)} \) and \( g^{(n)} \).

In this direction the following two theorems can be noted.

Theorem F [13]. Let \( f \) and \( g \) be two meromorphic functions sharing the value \( \infty \) CM. If \( f^{(n)} \) and \( g^{(n)} \) share the value \( 1 \) CM with \( \delta(0; f) + \delta(0; g) + (n + 2)\Theta(\infty; f) > n + 3 \) then either \( f \equiv g \) or \( f^{(n)}g^{(n)} \equiv 1 \).

Theorem G [16]. Let \( f \) and \( g \) be two meromorphic functions such that \( \Theta(\infty; f) = \Theta(\infty; g) = 1 \). If \( f^{(n)} \) and \( g^{(n)} \) share the value \( 1 \) CM and \( \delta(0; f) + \delta(0; g) > 1 \) then either \( f \equiv g \) or \( f^{(n)}g^{(n)} \equiv 1 \).

So it is not irrelevant to ask: What can be said if two linear differential polynomials generated by two meromorphic functions \( f \) and \( g \) share the value \( 1 \) CM?

In the paper we answer this question. Also as a consequence of the main theorem we prove a result which improves Theorem G and so some previous results.

2. Definitions and notations. In this section we present some necessary notations and definitions.

Notation 1. We denote by \( \Psi(D) \) a linear differential operator with constant coefficients of the form \( \Psi(D) = \sum_{i=1}^{p} \alpha_i D^i \), where \( D \equiv d/dz \).

Definition 1. For a meromorphic function \( f \) and a positive integer \( k \), \( N_k(r, a; f) \) denotes the counting function of \( a \)-points of \( f \) where an \( a \)-point with multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k \) times if \( m > k \).
Definition 2 (cf. [1]). For a meromorphic function \( f \) we put
\[
T_0(r, f) = \frac{1}{r} \int_1^r \frac{T(t, f)}{t} \, dt,
\]
\[
N_0(r, a; f) = \frac{1}{r} \int_1^r \frac{N(t, a; f)}{t} \, dt, \quad N_k^0(r, a; f) = \frac{1}{r} \int_1^r \frac{N_k(t, a; f)}{t} \, dt,
\]
\[
m_0(r, f) = \frac{1}{r} \int_1^r \frac{m(t, f)}{t} \, dt, \quad S_0(r, f) = \frac{1}{r} \int_1^r \frac{S(t, f)}{t} \, dt \quad \text{etc.}
\]

Definition 3. If \( f \) is a meromorphic function, then
\[
\delta_k(a; f) = 1 - \limsup_{r \to \infty} \frac{N_k(r, a; f)}{T(r, f)}.
\]
Clearly \( 0 \leq \delta(a; f) \leq \delta_k(a; f) \leq \ldots \leq \delta_2(a; f) \leq \delta_1(a; f) = \Theta(a; f) \leq 1. \)

Definition 4 (cf. [8]). For a meromorphic function \( f \) we put
\[
\delta_0(a; f) = 1 - \limsup_{r \to \infty} \frac{N_0(r, a; f)}{T_0(r, f)}, \quad \Theta_0(a; f) = 1 - \limsup_{r \to \infty} \frac{N_0(r, a; f)}{T_0(r, f)},
\]
\[
\delta_0^k(a; f) = 1 - \limsup_{r \to \infty} \frac{N_k^0(r, a; f)}{T_0(r, f)} \quad \text{where } a \in \mathbb{C} \cup \infty.
\]

3. Lemmas. In this section we discuss some lemmas which will be required in the sequel.

Lemma 1 [1]. For meromorphic \( f \),
\[
\lim_{r \to \infty} \frac{S_0(r, f)}{T_0(r, f)} = 0
\]
through all values of \( r \).

Lemma 2. Let \( f \) be a meromorphic function and \( a \in \mathbb{C} \cup \infty \). Then
\[
\delta(a; f) \leq \delta_0(a; f), \quad \Theta(a; f) \leq \Theta_0(a; f) \quad \text{and} \quad \delta_k(a; f) \leq \delta_0^k(a; f).
\]

This lemma can be proved along the lines of [7, Proposition 6].

Lemma 3. Let \( f_1, f_2 \) be nonconstant meromorphic functions such that \( af_1 + bf_2 \equiv 1 \), where \( a, b \) are nonzero constants. Then
\[
T_0(r, f_1) \leq N_0(r, 0; f_1) + N_0(r, 0; f_2) + N_0(r, \infty; f_1) + S_0(r, f_1).
\]

Proof. By the second fundamental theorem we get
\[
T(r, f_1) \leq N(r, 0; f_1) + N(r, a^{-1}; f_1) + N(r, \infty; f_2) + S(r, f_1)
\]
\[
= N(r, 0; f_1) + N(r, 0, f_2) + N(r, \infty; f_1) + S(r, f_1).
\]
From this inequality the lemma follows on integration.
Lemma 4 [4]. For a meromorphic function $f$ and any $a \in \mathbb{C}$,

$$N(r, 0; \Psi(D)f \mid f = a, \geq p) \geq N(r, 0; f^{(p)} \mid f = a, \geq p) + S(r, f),$$

where $N(r, b; g \mid f = c, \geq k)$ is the counting function of those $b$-points of $g$, counted with proper multiplicities, which are the $c$-points of $f$ with multiplicities not less than $k$.

Lemma 5. Let $f$ be a meromorphic function. Then

(i) \[ \liminf_{r \to \infty} \frac{T_0(r, \Psi(D)f)}{T_0(r, f)} \geq \sum_{a \neq \infty} \delta^0_p(a; f), \]

(ii) \[ \delta_0(0; \Psi(D)f) \geq \frac{\sum_{a \neq \infty} \delta_0(a; f)}{1 + p(1 - \Theta_0(\infty; f))}. \]

Proof. For distinct finite complex numbers $a_1, \ldots, a_n$ we put

$$A = \sum_{i=1}^{n} \frac{1}{f - a_i}.$$

Then by [3, inequality 2.1, p. 33] we get

$$\sum_{i=1}^{n} m(r, a_i; f) \leq m(r, A) + O(1)$$

$$\leq m(r, 0; \Psi(D)f) + m(r, A\Psi(D)f)$$

$$\leq m(r, 0; \Psi(D)f) + \sum_{i=1}^{n} m\left(r, \frac{\Psi(D)f}{f - a_i}\right)$$

$$= m(r, 0; \Psi(D)f) + \sum_{i=1}^{n} m\left(r, \frac{\Psi(D)(f - a_i)}{f - a_i}\right)$$

$$= m(r, 0; \Psi(D)f) + S(r, f),$$

by the Milloux theorem [3, p. 55], i.e.,

(1) \[ nT(r, f) \leq T(r, \Psi(D)f) + \sum_{i=1}^{n} N(r, a_i; f) - N(r, 0; \Psi(D)f) + S(r, f) \]

$$\leq T(r, \Psi(D)f)$$

$$+ \sum_{i=1}^{n} (N(r, a_i; f) - N(r, 0; \Psi(D)f \mid f = a_i, \geq p))$$

$$+ S(r, f).$$
So by Lemma 4 we get
\[
nT(r, f) \leq T(r, \Psi(D)f) + \sum_{i=1}^{n} \{N(r, a_i; f) - N(r, 0; f^{(p)} | f = a_i \geq p)\} + S(r, f)
\]
\[
\leq T(r, \Psi(D)f) + \sum_{i=1}^{n} N_p(r, a_i; f) + S(r, f).
\]
This gives on integration
\[
nT_0(r, f) \leq T_0(r, \Psi(D)f) + \sum_{i=1}^{n} N^0_p(r, a_i; f) + S_0(r, f).
\]
Hence by Lemma 1 we get
\[
\liminf_{r \to \infty} \frac{T_0(r, \Psi(D)f)}{T_0(r, f)} \geq \sum_{i=1}^{n} \delta^0_p(a_i; f).
\]
Since \(n\) is arbitrary, it follows that
\[
\liminf_{r \to \infty} \frac{T_0(r, \Psi(D)f)}{T_0(r, f)} \geq \sum_{a \neq \infty} \delta^0_p(a; f).
\]
Again by the Milloux theorem,
\[
T(r, \Psi(D)f) \leq m \left( r, \frac{\Psi(D)f}{f} \right) + m(r, f) + N(r, f) + pN(r, f) + O(1)
\]
\[
= T(r, f) + pN_0(r, f) + S(r, f).
\]
This gives on integration
\[
(2) \quad T_0(r, \Psi(D)f) \leq T_0(r, f) + pN_0(r, f) + S_0(r, f).
\]
Also from (1) we get by integration
\[
nT_0(r, f) \leq T_0(r, \Psi(D)f) + \sum_{i=1}^{n} N_0(r, a_i; f)
\]
\[
- N_0(r, 0; \Psi(D)f) + S_0(r, f).
\]
So by (2) we obtain
\[
n \leq \left( 1 - \frac{N_0(r, 0; \Psi(D)f)}{T_0(r, \Psi(D)f)} \right) \cdot \frac{T_0(r, f) + pN_0(r, f) + S_0(r, f)}{T_0(r, f)}
\]
\[
+ \sum_{i=1}^{n} \frac{N_0(r, a_i; f)}{T_0(r, f)} + \frac{S_0(r, f)}{T_0(r, f)}.
\]
In view of Lemma 1 this gives
\[\sum_{i=1}^{n} \delta_0(a_i; f) \leq \delta_0(0; \Psi(D)f)\{1 - \Theta_0(\infty; f))\},\]
from which (ii) follows because \(n\) is arbitrary. This proves the lemma.

**Lemma 6** [11]. Let \(f_1, f_2, f_3\) be nonconstant meromorphic functions satisfying \(f_1 + f_2 + f_3 \equiv 1\). If \(f_1, f_2, f_3\) are linearly independent then \(g_1 = -f_2/f_3, g_2 = 1/f_3\) and \(g_3 = -f_1/f_3\) are also linearly independent.

**Lemma 7.** Let \(f_1, f_2, f_3\) be three linearly independent meromorphic functions such that \(f_1 + f_2 + f_3 \equiv 1\). Then
\[T_0(r, f_1) \leq \sum_{j=1}^{3} N_2^0(r, 0; f_j) + \max_{1 \leq i \neq j \leq 3} \{N_2^0(r, \infty; f_i) + \overline{N}_0(r, \infty; f_j)\}\]
\[+ S_0(r),\]
where \(S_0(r) = \sum_{j=1}^{3} S_0(r, f_j)\).

**Proof.** We prove under the hypotheses of the lemma the following inequality which on integration proves the lemma:

\[(3) \quad T(r, f_1) \leq \sum_{j=1}^{3} N_2(r, 0; f_j) + \max_{1 \leq i \neq j \leq 3} \{N_2(r, \infty; f_i) + \overline{N}(r, \infty; f_j)\}\]
\[+ \sum_{j=1}^{3} S(r, f_j).\]

From the proof of a generalisation of Borel’s theorem by Nevanlinna (cf. [2, p. 70]) we get
\[(4) \quad T(r, f_1) \leq \sum_{j=1}^{3} N(r, 0; f_j) - N(r, 0; \Delta) + N(r, \Delta)\]
\[- N(r, f_2) - N(r, f_3) + S(r),\]
where \(\Delta\) is the wronskian determinant of \(f_1, f_2, f_3\) and \(S(r) = \sum_{j=1}^{3} S(r, f_j)\).

Now we need the following notations from [5]: for \(z \in \mathbb{C}\) and \(b \in \mathbb{C} \cup \{\infty\}\) we put
\[\mu^b_f(z) = \begin{cases} m & \text{if } z \text{ is a } b \text{-point of } f \text{ with multiplicity } m \geq 1, \\ 0 & \text{if } z \text{ is not a } b \text{-point of } f, \end{cases}\]
\[\overline{\mu}^b_f(z) = \begin{cases} 1 & \text{if } z \text{ is a } b \text{-point of } f \text{ with multiplicity } \geq 1, \\ 0 & \text{if } z \text{ is not a } b \text{-point of } f, \end{cases}\]
\[\nu^b_f(z) = \begin{cases} 2 & \text{if } z \text{ is a } b \text{-point of } f \text{ with multiplicity } m > 2, \\ m & \text{if } z \text{ is a } b \text{-point of } f \text{ with multiplicity } m \leq 2. \end{cases}\]
Also we put
\[\mu(z) = \sum_{j=1}^{3} \mu_{f_j}^0(z) - \mu_{f_j}^{\infty}(z) + \mu_{f_k}^\infty(z) - \mu_{f_3}^\infty(z)\]
and
\[\mu^*(z) = \sum_{j=1}^{3} \nu_{f_j}^0(z) + \max_{1 \leq i \neq j \leq 3} \{\nu_{f_i}^\infty(z) + \overline{\nu}_{f_j}^\infty(z)\}.
\]
Now (3) will follow from (4) if we can prove that for any \(z \in \mathbb{C}\), \(\mu(z) \leq \mu^*(z)\).

We consider the following cases.

**Case 1.** Let \(z\) be not a pole of any \(f_i\) \((i = 1, 2, 3)\). Since any zero of \(f_i\) with multiplicity \(m > 2\) is a zero of \(\Delta\) with multiplicity at least \(m - 2\), it follows that \(\mu(z) \leq \mu^*(z)\).

**Case 2.** Let \(z\) be a pole of at least one of \(f_i\) \((i = 1, 2, 3)\). So the following subcases come up for consideration.

**Subcase 2.1.** Let \(z\) be a zero of \(f_1\) with multiplicity \(m > 2\) and a pole of \(f_2, f_3\) with multiplicity \(k \geq 1\). Then \(z\) is a pole of \(\Delta\) with multiplicity \(k - m + 3\) provided \(k - m + 3 > 0\) and otherwise \(z\) is a zero of \(\Delta\) with multiplicity \(m - k - 3\). Hence \(\mu(z) = 3 - k\) and \(\mu^*(z) \geq 3\). So \(\mu(z) \leq \mu^*(z)\).

Let \(z\) be a zero of \(f_1\) with multiplicity \(m \leq 2\) and a pole of \(f_2, f_3\) with multiplicity \(k \geq 1\). Then \(z\) is a pole of \(\Delta\) with multiplicity not exceeding \(k + 2\). Hence \(\mu(z) \leq m + k + 2 - k \leq 4 - k\) and \(\mu^*(z) \geq 3\). So \(\mu(z) \leq \mu^*(z)\).

**Subcase 2.2.** Let \(z\) be a zero of \(f_2\) with multiplicity \(m > 2\) and a pole of \(f_1, f_3\) with multiplicity \(k \geq 1\). Then \(z\) is a pole of \(\Delta\) with multiplicity \(k - m + 3\) provided \(k - m + 3 > 0\) and otherwise \(z\) is a zero of \(\Delta\) with multiplicity \(m - k - 3\). Hence \(\mu(z) = 3\) and \(\mu^*(z) \geq 3\). So \(\mu(z) \leq \mu^*(z)\).

Let \(z\) be a zero of \(f_2\) with multiplicity \(m \leq 2\) and a pole of \(f_1, f_3\) with multiplicity \(k \geq 1\). Then \(z\) is a pole of \(\Delta\) with multiplicity not exceeding \(k + 2\). Hence \(\mu(z) \leq m + k + 2 - k = m + 2\) and \(\mu^*(z) \geq m + 2\). So \(\mu(z) \leq \mu^*(z)\).

**Subcase 2.3.** Let \(z\) be a zero of \(f_3\) with multiplicity \(m \geq 1\) and a pole of \(f_1, f_2\) with multiplicity \(k \geq 1\). Then as in Subcase 2.2 we can prove that \(\mu(z) \leq \mu^*(z)\).

**Subcase 2.4.** Let \(z\) be neither a zero nor a pole of \(f_1\). Since \(f_2 + f_3 = 1 - f_1\), it follows that \(z\) is not a pole of \(f_2 + f_3\). Since \(z\) is a pole of at least one of \(f_i\) \((i = 1, 2, 3)\), it follows that \(z\) is a pole of \(f_2\) and \(f_3\) with the same multiplicity \(m\), say (because the singularities of \(f_2\) and \(f_3\) at \(z\) cancel each other). Then \(z\) is a pole of \(\Delta\) with multiplicity not exceeding \(m + 2\). Hence \(\mu(z) \leq m + 2 - m = m \leq 2\) and \(\mu^*(z) \geq 2\). So \(\mu(z) \leq \mu^*(z)\).

**Subcase 2.5.** Let \(z\) be a pole of \(f_1, f_2\) with multiplicity \(m \geq 1\) and a pole of \(f_3\) with multiplicity \(q \leq m\). Then \(z\) is a pole of \(\Delta\) with
multiplicity not exceeding \( m + q + 3 \). Hence \( \mu(z) \leq m + q + 3 - m - q = 3 \) and \( \mu^*(z) = 2 + 1 = 3 \). So \( \mu(z) \leq \mu^*(z) \).

**Subcase 2.6.** Let \( z \) be a pole of \( f_1, f_2, f_3 \) with multiplicity \( m \geq 1 \). Then there exist two functions \( \phi, \psi \) analytic at \( z \) and \( \phi(z) \neq 0, \psi(z) \neq 0 \) such that in some neighbourhood of \( z \), \( f_2(\omega) = (\omega - z)^{-m}\phi(\omega) \) and \( f_3(\omega) = (\omega - z)^{-m}\psi(\omega) \). Also \( \Delta = f''_2f'_3 - f''_3f'_2 \) shows that \( z \) is a pole of \( \Delta \) with multiplicity not exceeding \( 2m + 3 \) but by actual calculation we see that the coefficient of \( (\omega - z)^{-(2m+3)} \) in \( \Delta \) is \( m^2(m+1)\phi\psi - m^2(m+1)\phi\psi \equiv 0 \). So \( z \) is a pole of \( \Delta \) with multiplicity not exceeding \( 2m + 2 \). Hence \( \mu(z) \leq 2m + 2 - m - m = 2 \) and \( \mu^*(z) \geq 2 \). So \( \mu(z) \leq \mu^*(z) \).

**Subcase 2.7.** Let \( z \) be a pole of \( f_1, f_2 \) with multiplicity \( m \geq 1 \) and neither a zero nor a pole of \( f_3 \). Then \( z \) is a pole of \( \Delta \) with multiplicity not exceeding \( m + 2 \). Hence \( \mu(z) \leq m + 2 - m = 2 \) and \( \mu^*(z) \geq 2 \). So \( \mu(z) \leq \mu^*(z) \).

**Subcase 2.8.** Let \( z \) be a pole of \( f_1 \) with multiplicity \( m \geq 1 \) and a pole of \( f_2 \) with multiplicity \( m + q \) (\( q \geq 1 \)). Then \( z \) is also a pole of \( f_3 \) with multiplicity \( m + q \) and the terms containing \( (w - z)^{-(m+i)} \) \((i = 1, \ldots, q)\) of the Laurent expansions of \( f_2 \) and \( f_3 \) about \( z \) cancel each other because \( f_2 + f_3 \) has a pole at \( z \) with multiplicity \( m \). Also we see that \( \Delta \) has a pole at \( z \) with multiplicity not exceeding \( 2m + q + 3 \). Hence \( \mu(z) \leq 2m + q + 3 - m - q - m = 3 - q \) and \( \mu^*(z) = 2 + 1 = 3 \). So \( \mu(z) \leq \mu^*(z) \).

**Lemma 8.** If \( \sum_{a \neq \infty} \delta^0_p(a; f) > 0 \) then

\[ \Theta_0(\infty; \Psi(D)f) \geq 1 - \frac{1 - \Theta_0(\infty; f)}{\sum_{a \neq \infty} \delta^0_p(a; f)} \]

**Proof.** Since \( N_0(r, \Psi(D)f) = N_0(r, f) \), the lemma follows from Lemma 5(i).

**Lemma 9 [14].** Let \( F \) and \( G \) be two nonconstant meromorphic functions such that \( F \) and \( G \) share \( 1 \) CM. If

\[ \lim_{r \to \infty, r \in I} \frac{N_2(r, 0; F) + N_2(r, \infty; F) + N_2(r, 0; G) + N_2(r, \infty; G)}{T(r)} < 1, \]

where \( T(r) = \max\{T(r, F), T(r, G)\} \)

and \( I \) is a set of \( r \)'s \((0 < r < \infty)\) of infinite linear measure, then \( F \equiv G \) or \( FG \equiv 1 \).

**4. Theorems.** In this section we present the main results of the paper.

**Theorem 1.** Let \( f, g \) be two meromorphic functions such that
(i) $\Psi(D)f, \Psi(D)g$ are nonconstant and share 1 CM and

$$\frac{\sum_{a \neq \infty} \delta(a; f)}{1 + p(1 - \Theta(\infty; f))} + \frac{\sum_{a \neq \infty} \delta(a; g)}{1 + p(1 - \Theta(\infty; g))} > 1 + \frac{4(1 - \Theta(\infty; f))}{\sum_{a \neq \infty} \delta_p(a; f)} + \frac{4(1 - \Theta(\infty; g))}{\sum_{a \neq \infty} \delta_p(a; g)},$$

where $\sum_{a \neq \infty} \delta_p(a; f) > 0$ and $\sum_{a \neq \infty} \delta_p(a; g) > 0$. Then either $[\Psi(D)f] \cdot [\Psi(D)g] \equiv 1$ or $f - g \equiv s$, where $s$ is a solution of the differential equation $\Psi(D)w = 0$.

**Theorem 2.** Let $f, g$ be two meromorphic functions of finite order such that

(i) $\Psi(D)f, \Psi(D)g$ are nonconstant and share 1 CM and

(ii) $\frac{\sum_{a \neq \infty} \delta(a; f)}{1 + p(1 - \Theta(\infty; f))} + \frac{\sum_{a \neq \infty} \delta(a; g)}{1 + p(1 - \Theta(\infty; g))} > 1 + \frac{2(1 - \Theta(\infty; f))}{\sum_{a \neq \infty} \delta_p(a; f)} + \frac{2(1 - \Theta(\infty; g))}{\sum_{a \neq \infty} \delta_p(a; g)},$

where $\sum_{a \neq \infty} \delta_p(a; f) > 0$ and $\sum_{a \neq \infty} \delta_p(a; g) > 0$. Then either $[\Psi(D)f] \cdot [\Psi(D)g] \equiv 1$ or $f - g \equiv s$, where $s$ is a solution of the differential equation $\Psi(D)w = 0$.

The following example shows that the theorems are sharp.

**Example 1.** Let $f = \frac{1}{2}e^{z}(e^{z} - 1)g = \frac{1}{2}e^{-z}(\frac{1}{2} - \frac{1}{3}e^{-z})$ and $\Psi(D) = D^{2} - 3D$. Then $\sum_{a \neq \infty} \delta(a; f) = \sum_{a \neq \infty} \delta(a; g) = 1/2$, $\Theta(\infty; f) = \Theta(\infty; g) = 1$ and $\sum_{a \neq \infty} \delta_{2}(a; f) > 0$, $\sum_{a \neq \infty} \delta_{2}(a; g) > 0$. Also $\Psi(D)f = e^{z}(1 - e^{z})$ and $\Psi(D)g = e^{-z}(1 - e^{-z})$ share 1 CM but neither $[\Psi(D)f] \cdot [\Psi(D)g] \equiv 1$ nor $f - g \equiv c_{1} + c_{2}e^{3z}$ for any constants $c_{1}$ and $c_{2}$.

**Proof of Theorem 1.** Let $F = \Psi(D)f$ and $G = \Psi(D)g$. Then in view of Lemmas 2, 5 and 8 the condition (ii) implies

$$\delta_{0}(0; F) + \delta_{0}(0; G) + 4\Theta_{0}(\infty; F) + 4\Theta_{0}(\infty; G) > 9.$$  

We put

$$H = \frac{F - 1}{G - 1}.$$  

Since $F, G$ share 1 CM, the poles and zeros of $H$ occur only at the poles of $F$ and $G$ respectively. Also $N_{0}(r, \infty; H) \leq N_{0}(r, \infty; F)$ and $N_{0}(r, 0; H) \leq N_{0}(r, \infty; G)$.

Let $F_{1} = F, F_{2} = -GH$ and $F_{3} = H$. Then from (6) it follows that

$$F_{1} + F_{2} + F_{3} \equiv 1.$$
First we suppose that $F_3 = H \equiv k$, a constant. Then from (7) we get $F - kG = 1 - k$. If $k \neq 1$, we see that
$$
\frac{1}{1 - k}F - \frac{k}{1 - k}G \equiv 1.
$$
Since $k \neq 0$, from Lemma 3 it follows that
$$
T_0(r, F) \leq N_0(r, 0; F) + N_0(r, 0; G) + \overline{N}_0(r, \infty; F) + \overline{S}_0(r, F),
$$
$$
T_0(r, G) \leq N_0(r, 0; F) + N_0(r, 0; G) + \overline{N}_0(r, \infty; G) + \overline{S}_0(r, G).
$$
So
$$
\max\{T_0(r, F), T_0(r, G)\} \leq N_0(r, 0; F) + N_0(r, 0; G) + \overline{N}_0(r, \infty; F) + \overline{N}_0(r, \infty; G) + o(\max\{T_0(r, F), T_0(r, G)\}).
$$
This gives $\delta_0(0; F) + \delta_0(0; G) + \Theta_0(\infty; F) + \Theta_0(\infty; G) \leq 3$ and so from (5) we see that $9 < 3\Theta_0(\infty; F) + 3\Theta_0(\infty; G) + 3 \leq 9$, a contradiction. So $k = 1$ and hence $F \equiv G$. Therefore $\Psi(D)(f - g) \equiv 0$ and so $f - g \equiv s$, where $s = s(z)$ is a solution of $\Psi(D)w = 0$.

Similarly if $F_2 \equiv k$, a constant, we can show that $|\Psi(D)f| \cdot |\Psi(D)g| \equiv 1$.

Now we suppose that $F_1$, $F_2$ and $F_3$ are nonconstant. If possible, let $F_1$, $F_2$, $F_3$ be linearly independent. Then from Lemma 7 we get
$$
(8) \quad T_0(r, F) \leq N_2^0(r, 0; F) + N_2^0(r, 0; G) + 2N_2^0(r, 0; H)
$$
$$
+ \max_{1 \leq i \neq j \leq 3}\{N_2^0(r, \infty; F_i) + \overline{N}_0(r, \infty; F_j)\} + \sum_{j=1}^{3} S_0(r, F_j)
$$
$$
\leq N_0(r, 0; F) + N_0(r, 0; G) + 4\overline{N}_0(r, \infty; G)
$$
$$
+ \max_{1 \leq i \neq j \leq 3}\{N_2^0(r, \infty; F_i) + \overline{N}_0(r, \infty; F_j)\} + \sum_{j=1}^{3} S_0(r, F_j).
$$
Now in view of (6) we see that
$$
\sum_{j=1}^{3} S_0(r, F_j) = o(\max\{T_0(r, F), T_0(r, G)\})
$$
and
$$
N_2^0(r, \infty; F_1) + \overline{N}_0(r, \infty; F_2) = N_2^0(r, \infty; F) + \overline{N}_0(r, \infty; H(G - 1))
$$
$$
= N_2^0(r, \infty; F) + \overline{N}_0(r, \infty; F) \leq 3\overline{N}_0(r, \infty; F),
$$
$$
N_2^0(r, \infty; F_2) + \overline{N}_0(r, \infty; F_3) = N_2^0(r, \infty; H(G - 1)) + N_0(r, \infty; H)
$$
$$\leq N_2^0(r, \infty; F) + \overline{N}_0(r, \infty; F) \leq 3\overline{N}_0(r, \infty; F),
$$
$$
N_2^0(r, \infty; F_3) + \overline{N}_0(r, \infty; F_1) = N_2^0(r, \infty; H) + \overline{N}_0(r, \infty; F)
$$
$$\leq 2\overline{N}_0(r, \infty; H) + \overline{N}_0(r, \infty; F) \leq 3\overline{N}_0(r, \infty; F)$$
and similarly for the other three terms. So from (8) we get
\[ T_0(r, F) \leq N_0(r, 0; F) + N_0(r, 0; G) + 3\overline{N}_0(r, \infty; F) + 4\overline{N}_0(r, \infty; G) + o(\max\{T_0(r, F), T_0(r, G)\}). \]

From (9) and (10) we get
\[ \max\{T_0(r, F), T_0(r, G)\} \leq (10 - \delta_0(0; F) - \delta_0(0; G) - 4\Theta_0(\infty; F) - 4\Theta_0(\infty; G) + o(1) \max\{T_0(r, F), T_0(r, G)\} < (1 - \varepsilon + o(1)) \max\{T_0(r, F), T_0(r, G)\}, \]
which is a contradiction, where by (5) we choose
\[ 0 < \varepsilon < \delta_0(0; F) + \delta_0(0; G) + 4\Theta_0(\infty; F) + 4\Theta_0(\infty; G) - 9. \]

Hence there exist constants \( c_1, c_2, c_3 \), not all zero, such that
\[ c_1F_1 + c_2F_2 + c_3F_3 \equiv 0. \]
Clearly \( c_1 \neq 0 \). For, otherwise from (11) we get \( H(c_3 - c_2G) \equiv 0 \), which is impossible because \( F \) and \( G \) are nonconstant.

Now eliminating \( F_1 \) from (7) and (11) we get
\[ cF_2 + dF_3 \equiv 1, \]
where \( c = 1 - c_2/c_1 \) and \( d = 1 - c_3/c_1 \).

If possible let \( cd \neq 0 \). Then from (12) we get \( (c/d)(G) + 1/(dH) \equiv 1 \). So by Lemma 3 we get
\[ T_0(r, G) \leq N_0(r, 0; G) + \overline{N}_0(r, \infty; H) + \overline{N}_0(r, \infty; G) + S_0(r, G), \]
\[ i.e. \]
\[ T_0(r, G) \leq N_0(r, 0; G) + \overline{N}_0(r, \infty; F) + \overline{N}_0(r, \infty; G) + S_0(r, G). \]
By the second fundamental theorem we get on integration
\[ T_0(r, F) \leq N_0(r, 0; F) + N_0(r, 1; F) + \overline{N}_0(r, \infty; F) + S_0(r; F) = N_0(r, 0; F) + N_0(r, 1; G) + \overline{N}_0(r, \infty; F) + S_0(r, F) \leq N_0(r, 0; F) + T_0(r, G) + \overline{N}_0(r, \infty; F) + S_0(r, F). \]
So by (13) we obtain
\[ T_0(r, F) \leq N_0(r, 0; F) + N_0(r, 0; G) + 2\overline{N}_0(r, \infty; F) + \overline{N}_0(r, \infty; G) + S_0(r, F) + S_0(r, G). \]
From (13) and (14) we get
\[
\max \{ T_0(r, F), T_0(r, G) \} \leq N_0(r, 0; F) + N_0(r, 0; G) + 2N_0(r, \infty; F) + \nabla_0(r, \infty; G) + o(\max \{ T_0(r, F), T_0(r, G) \})
\]
and so \( \delta_0(0; F) + \delta_0(0; G) + 2\Theta_0(\infty; F) + \Theta_0(\infty; G) \leq 4. \)

Now by (5) we see that
\[
9 < \delta_0(0; F) + \delta_0(0; G) + 4\Theta_0(\infty; F) + 4\Theta_0(\infty; G) \leq 9,
\]
which is a contradiction. Therefore \( cd = 0. \) From (12) we see that \( c \) and \( d \) are not simultaneously zero. So we consider the following cases.

**Case I.** Let \( d = 0. \) Then from (12) we get \(-cF + 1/G = 1 - c. \) If \( c \neq 1, \) we obtain \((-c/(1-c))F + 1/((1-c)G) = 1. \) So by Lemma 3 it follows that
\[
T_0(r, F) \leq N_0(r, 0; F) + \nabla_0(r, \infty; G) + N_0(r, \infty; F) + S_0(r, F)
\]
and
\[
T_0(r, G) = T_0(r, 1/G) + S_0(r, G)
\]
\[
\leq N_0(r, 0; F) + N_0(r, 0; G) + \nabla_0(r, \infty; G) + S_0(r, G).
\]
Hence
\[
\max \{ T_0(r, F), T_0(r, G) \} \leq N_0(r, 0; F) + N_0(r, 0; G) + \nabla_0(r, \infty; F) + \nabla_0(r, \infty; G) + o(\max \{ T_0(r, F), T_0(r, G) \}),
\]
and so \( \delta_0(0; F) + \delta_0(0; G) + \Theta_0(\infty; F) + \Theta_0(\infty; G) \leq 3. \)

From (5) we see that
\[
9 < \delta_0(0; F) + \delta_0(0; G) + 4\Theta_0(\infty; F) + 4\Theta_0(\infty; G) \leq 9,
\]
which is a contradiction. Therefore \( c = 1 \) and so \( FG = 1, \) i.e., \( [\Psi(D)f] \cdot [\Psi(D)g] = 0. \)

**Case II.** Let \( c = 0. \) Then from (12) we get \( dF - G = d - 1. \) If \( d \neq 1 \) it follows that \((d/(d-1))F - (1/(d-1))G = 1. \) Now by Lemma 3 we obtain
\[
T_0(r, F) \leq N_0(r, 0; F) + N_0(r, 0; G) + \nabla_0(r, \infty; F) + S_0(r, F),
\]
\[
T_0(r, G) \leq N_0(r, 0; F) + N_0(r, 0; G) + \nabla_0(r, \infty; G) + S_0(r, G).
\]
So we get
\[
\max \{ T_0(r, F), T_0(r, G) \} \leq N_0(r, 0; F) + N_0(r, 0; G) + \nabla_0(r, \infty; F) + \nabla_0(r, \infty; G) + o(\max \{ T_0(r, F), T_0(r, G) \})
\]
and as in Case I this leads to a contradiction. So \( d = 1 \) and hence \( F \equiv G, \) i.e., \( \Psi(D)(f - g) \equiv 0. \) Therefore \( f - g \equiv s \) where \( s = s(z) \) is a solution of \( \Psi(D)w = 0. \) This proves the theorem.
Proof of Theorem 2. If \( f \) and \( g \) are of finite order, we can prove along the lines of Lemmas 5 and 8 that
\[
\delta(0; \Psi(D)f) \geq \frac{\sum_{a \neq \infty} \delta(a; f)}{1 + p(1 - \Theta(\infty; \Psi(D)f))}, \quad \Theta(\infty; \Psi(D)f) \geq 1 - \frac{1 - \Theta(\infty; f)}{\sum_{a \neq \infty} \delta_p(a; f)},
\]
and the corresponding results for \( g \). Let \( F = \Psi(D)f \) and \( G = \Psi(D)g \). Then by the condition (ii) of the theorem we get
\[
\delta(0; F) + \delta(0; G) + 2\Theta(\infty; F) + 2\Theta(\infty; G) > 5.
\]
This implies
\[
\lim_{r \to \infty} \frac{N(r, 0; F) + N(r, 0; G)}{T(r, F)} + \lim_{r \to \infty} \frac{N(r, \infty; F) + N(r, \infty; G)}{T(r, G)} < 1,
\]
and so by Lemma 9 the theorem follows.

Considering \( f = -2^{-n}e^{2z} + e^z \), \( g(z) = -(-1)^n2^{-n}e^{-2z} + (-1)^ne^{-z} \) where \( n \) is a positive integer, Yi and Yang [16] claimed that for \( n \geq 1 \) the condition \( \delta(0; f) + \delta(0; g) > 1 \) of Theorem G is necessary. In the following example we see that this claim is not justified.

Example 2. Let \( f = e^z - 1 \) and \( g = 1 + (-1)^ne^{-z} \). Then \( \delta(0; f) = \delta(0; g) = 0 \) and \( f^{(n)}, g^{(n)} \) share 1 CM. Also \( f^{(n)}g^{(n)} \equiv 1 \).

In the first corollary we improve Theorem G for \( n \geq 1 \).

Corollary 1. Let \( f, g \) be two meromorphic function with \( \Theta(\infty; f) = \Theta(\infty; g) = 1 \). If for \( n \geq 1 \) the derivatives \( f^{(n)} \) and \( g^{(n)} \) are nonconstant and share 1 CM with
\[
(i) \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) > 1,
(ii) \Theta(0; f) + \Theta(0; g) > 1,
\]
then either (a) \( f^{(n)}g^{(n)} \equiv 1 \) or (b) \( f \equiv g \).

Proof. Choosing \( \Psi(D) = D^n \), from Theorem 1 it follows that either \( f^{(n)}g^{(n)} \equiv 1 \) or \( f - g \equiv Q \), where \( Q \) is a polynomial of degree at most \( n - 1 \). If possible, let \( Q \neq 0 \). Then from [3, Theorem 2.5, p. 47] it follows that
\[
T(r, f) \leq N(r, 0; f) + N(r, Q; f) + N(r, \infty; f) + S(r, f) = N(r, 0; f) + N(r, 0; g) + N(r, \infty; f) + S(r, f).
\]
Since \( f - g \equiv Q \), it follows that \( T(r, f) = T(r, g) + O(\log r) \). So we get 
\[ \Theta(0; f) + \Theta(0; g) \leq 1, \]
which is a contradiction. Therefore \( Q \equiv 0 \) and so \( f \equiv g \).

The following examples show that the condition \( \Theta(0; f) + \Theta(0; g) > 1 \) is necessary for the validity of case (b).

**Example 3.** Let \( f = e^z + 1 \) and \( g = e^z \). Then \( \Theta(0; f) = 0, \Theta(0; g) = 1 \), 
\[ \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) = 2, \Theta(\infty; f) = \Theta(\infty; g) = 1 \text{ and } f^{(n)}, g^{(n)} \]
share 1 CM but \( f - g \equiv 1 \).

**Example 4.** Let \( f = e^z + 1 \) and \( g = (-1)^n e^z \). Then \( \Theta(0; f) = 0, \Theta(0; g) = 1 \), 
\[ \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) = 2, \Theta(\infty; f) = \Theta(\infty; g) = 1 \text{ and } f^{(n)}, g^{(n)} \]
share 1 CM but \( f^{(n)} g^{(n)} \equiv 1 \).

Answering the question of C. C. Yang [9], mentioned in the introduction, H. X. Yi [12] proved the following theorem.

**Theorem H [12].** Let \( f \) and \( g \) be two nonconstant entire functions. Assume that \( f \) and \( g \) share 0 CM and \( f^{(n)}, g^{(n)} \) share 1 CM, where \( n \) is a non-negative integer. If \( \delta(0; f) > 1/2 \) then either \( f \equiv g \) or \( f^{(n)} g^{(n)} \equiv 1 \).

Considering \( f = -2^{-n} e^{2z} + (-1)^{n+1} e^z \) and \( g = (-1)^{n+1} 2^{-n} e^{-2z} - 2^{-n} e^{-z} \), Yi [12] claimed that the condition \( \delta(0; f) > 1/2 \) is necessary. The following example shows that for \( n \geq 1 \) this is not always the case.

**Example 5.** Let \( f = e^z - 1 \) and \( g = (-1)^{n+1} + (-1)^n e^z \). Then \( f \) and \( g \) share 0 CM and \( f^{(n)}, g^{(n)} \) (\( n \geq 1 \)) share 1 CM, \( \delta(0; f) = 0 \) but \( f^{(n)} g^{(n)} \equiv 1 \).

In the following corollary we provide an answer to a question of Yang [9].

**Corollary 2.** Let \( f \) and \( g \) be two meromorphic functions with \( \Theta(\infty; f) = \Theta(\infty; g) = 1 \). Suppose that \( f^{(n)}, g^{(n)} \) (\( n \geq 1 \)) share 1 CM and \( f \) and \( g \) share a value \( b \) (\( \neq \infty \)) IM. If \( \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) > 1 \) then either \( f \equiv g \) or \( f^{(n)} g^{(n)} \equiv 1 \).

**Proof.** The condition \( \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) > 1 \) implies that \( f \) and \( g \) are transcendental so that \( f^{(n)}, g^{(n)} \) are nonconstant. Choosing \( \Psi(D) = D^n \) we see from Theorem 1 that either \( f^{(n)} g^{(n)} \equiv 1 \) or \( f - g \equiv Q \), where \( Q \) is a polynomial. Now we consider the case \( f - g \equiv Q \). If possible, let \( Q \equiv 0 \). Also we suppose that \( f \) has at most a finite number of \( b \)-points and so \( g \) also has a finite number of \( b \)-points. Now by [3, Theorem 2.5, p. 47] it follows that
\[
T(r, f) = \overline{N}(r, b; f) + \overline{N}(r, b; Q; f) + \overline{N}(r, \infty; f) + N(r, f) + S(r, f) \leq \overline{N}(r, b; f) + \overline{N}(r, b; g) + N(r, f) + O(\log r) + S(r, f),
\]
which is a contradiction. Therefore \( f \) has infinitely many \( b \)-points and so \( f - g \) has infinitely many zeros. This again implies a contradiction because \( f - g \equiv Q \) and \( Q \equiv 0 \). So \( Q \equiv 0 \) and hence \( f \equiv g \).
Considering \( f = -2^{-n}e^{2z} + (-1)^{n+1}2^{-n}e^{-2z} \) and \( g = (-1)^{n+1}2^{-n}e^{2z} - 2^{-n}e^{-2z} \) we can verify that the condition \( \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) > 1 \) of Corollary 2 is necessary.

**Corollary 3.** Let \( \Psi(D) = D(D - \lambda_1)(D - \lambda_2) \ldots (D - \lambda_{p-1}) \) where \( \lambda_i \)'s are nonzero pairwise distinct complex numbers. Also suppose that \( f \) and \( g \) are two meromorphic functions with the following properties:

(i) \( \Psi(D)f, \Psi(D)g \) are nonconstant and share 1 CM,
(ii) \( \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) > 1 \) and \( \Theta(\infty; f) = \Theta(\infty; g) = 1 \),
(iii) \( f \) and \( g \) have b-points (\( b \neq \infty \)) with multiplicities not less than \( p + 1 \) at the origin.

Then \( f \equiv g \).

**Proof.** From the theorem we get either \( |\Psi(D)f|, |\Psi(D)g| \equiv 1 \) or \( f - g \equiv c_0 + c_1e^{-1}z + c_2e^{-2}z + \ldots + c_{p-1}e^{-p-1}z \) where \( c_i \)'s are constants. Since \( f \) has a b-point with multiplicity at least \( p + 1 \) at the origin, it follows that \( \Psi(D)f \) has at least a simple zero at the origin. Similarly \( \Psi(D)g \) has at least a simple zero at the origin. So the case \( |\Psi(D)f|, |\Psi(D)g| \equiv 1 \) does not occur. If possible, let \( f \neq g \). Then the constants \( c_0, c_1, \ldots, c_{p-1} \) are not all zero. Also by condition (iii) it follows that \( f - g \) has a zero at the origin with multiplicity at least \( p + 1 \). This implies that

\[
\sum_{i=0}^{p-1} c_i = 0, \quad \sum_{i=0}^{p-1} \lambda_i c_i = 0, \quad \sum_{i=0}^{p-1} \lambda_i^2 c_i = 0, \ldots, \quad \sum_{i=0}^{p-1} \lambda_i^p c_i = 0.
\]

This system of equations gives \( c_0 = c_1 = c_2 = \ldots = c_{p-1} = 0 \), which is a contradiction. Therefore \( f \equiv g \). This proves the corollary.

The following examples show that condition (iii) of Corollary 3 is necessary.

**Example 6.** Let \( f = e^{z^3}, g = e^{z^3} + 1 \) and \( \Psi(D) = D(D - 1) \). Then \( \Psi(D)f, \Psi(D)g \) share 1 CM, \( \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) = 2 \), \( \Theta(\infty; f) = \Theta(\infty; g) = 1 \) and \( f - 1, g - 2 \) have zeros with multiplicity three at the origin, but \( f \neq g \).

**Example 7.** Let \( f = e^{z^2} - 1, g = 1 - e^{-z} \) and \( \Psi(D) = D \). Then \( \Psi(D)f, \Psi(D)g \) share 1 CM, \( \sum_{a \neq \infty} \delta(a; f) + \sum_{a \neq \infty} \delta(a; g) = 2 \), \( \Theta(\infty; f) = \Theta(\infty; g) = 1 \) and \( f, g \) have simple zeros at the origin, but \( f \neq g \).

Let us conclude the paper with the following question: What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM?

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