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Uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points

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Abstract. We prove a uniqueness theorem for meromorphic functions involving linear differential polynomials generated by them. As consequences of the main result we improve some previous results.

1. Introduction. Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for $a \in \mathbb{C} \cup \infty$, $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, we say that f and g *share the value a CM* (counting multiplicities), and if we do not consider the multiplicities, f and g are said to *share the value a IM* (ignoring multiplicities). It is assumed that the reader is familiar with the standard notations and definitions of value distribution theory (cf. [3]).

M. Ozawa [6] proved the following result:

Theorem A [6]. *If two nonconstant entire functions f, g share the value 1 CM with $\delta(0; f) > 0$ and 0 being lacunary for g then either $f \equiv g$ or $fg \equiv 1$.*

Improving the above result H. X. Yi [10] proved the following:

Theorem B [10]. *Let f and g be two nonconstant meromorphic functions satisfying $\delta(\infty; f) = \delta(\infty; g) = 1$. If f, g share the value 1 CM and $\delta(0; f) + \delta(0; g) > 1$ then either $f \equiv g$ or $fg \equiv 1$.*

In [9] C. C. Yang asked: What can be said if two nonconstant entire functions f and g share the value 0 CM and their first derivatives share the value 1 CM?

As an attempt to solve this question K. Shibazaki [7] proved the following:

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Theorem C [7]. *Let f and g be two entire functions of finite order. If f' and g' share the value 1 CM with $\delta(0;f) > 0$ and 0 being lacunary for g then either $f \equiv g$ or $f'g' \equiv 1$.*

Improving Theorem C, H. X. Yi [13] obtained the following result:

Theorem D [13]. *Let f and g be two entire functions such that $f^{(n)}$ and $g^{(n)}$ share the value 1 CM. If $\delta(0;f) + \delta(0;g) > 1$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Considering meromorphic functions H. X. Yi and and C. C. Yang [15] improved Theorem C as follows:

Theorem E [15]. *Let f and g be two meromorphic functions satisfying $\delta(\infty;f) = \delta(\infty;g) = 1$. If f' and g' share the value 1 CM with $\delta(0;f) + \delta(0;g) > 1$ then either $f \equiv g$ or $f'g' \equiv 1$.*

In [15] it is asked whether it is possible to replace the first derivatives f', g' in Theorem E by the nth derivatives $f^{(n)}$ and $g^{(n)}$.

In this direction the following two theorems can be noted.

Theorem F [13]. *Let f and g be two meromorphic functions sharing the value ∞ CM. If $f^{(n)}$ and $g^{(n)}$ share the value 1 CM with $\delta(0;f) + \delta(0;g) + (n+2)\Theta(\infty;f) > n+3$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

Theorem G [16]. *Let f and g be two meromorphic functions such that $\Theta(\infty;f) = \Theta(\infty;g) = 1$. If $f^{(n)}$ and $g^{(n)}$ share the value 1 CM and $\delta(0;f) + \delta(0;g) > 1$ then either $f \equiv g$ or $f^{(n)}g^{(n)} \equiv 1$.*

So it is not irrelevant to ask: What can be said if two linear differential polynomials generated by two meromorphic functions f and g share the value 1 CM?

In the paper we answer this question. Also as a consequence of the main theorem we prove a result which improves Theorem G and so some previous results.

2. Definitions and notations. In this section we present some necessary notations and definitions.

Notation 1. We denote by $\Psi(D)$ a linear differential operator with constant coefficients of the form $\Psi(D) = \sum_{i=1}^{p} \alpha_i D^i$, where $D \equiv d/dz$.

Definition 1. For a meromorphic function f and a positive integer k, $N_k(r,a;f)$ denotes the counting function of a-points of f where an a-point with multiplicity m is counted m times if $m \le k$ and k times if $m > k$.

Definition 2 (cf. [1]). For a meromorphic function f we put

$$T_0(r,f)=\int_1^r \frac{T(t,f)}{t}\,dt,$$

$$N_0(r,a;f)=\int_1^r \frac{N(t,a;f)}{t}\,dt, \qquad N_k^0(r,a;f)=\int_1^r \frac{N_k(t,a;f)}{t}\,dt,$$

$$m_0(r,f)=\int_1^r \frac{m(t,f)}{t}\,dt, \qquad S_0(r,f)=\int_1^r \frac{S(t,f)}{t}\,dt \qquad \text{etc.}$$

Definition 3. If f is a meromorphic function, then

$$\delta_k(a;f)=1-\limsup_{r\to\infty}\frac{N_k(r,a;f)}{T(r,f)}.$$

Clearly $0\le\delta(a;f)\le\delta_k(a;f)\le\delta_{k-1}(a;f)\le\ldots\le\delta_2(a;f)\le\delta_1(a;f)=\Theta(a;f)\le 1$.

Definition 4 (cf. [8]). For a meromorphic function f we put

$$\delta_0(a;f)=1-\limsup_{r\to\infty}\frac{N_0(r,a;f)}{T_0(r,f)}, \qquad \Theta_0(a;f)=1-\limsup_{r\to\infty}\frac{\overline{N}_0(r,a;f)}{T_0(r,f)},$$

$$\delta_k^0(a;f)=1-\limsup_{r\to\infty}\frac{N_k^0(r,a;f)}{T_0(r,f)} \qquad \text{where } a\in\mathbb{C}\cup\infty.$$

3. Lemmas. In this section we discuss some lemmas which will be required in the sequel.

Lemma 1 [1]. *For meromorphic f,*

$$\lim_{r\to\infty}\frac{S_0(r,f)}{T_0(r,f)}=0$$

through all values of r.

Lemma 2. *Let f be a meromorphic function and $a\in\mathbb{C}\cup\infty$. Then $\delta(a;f)\le\delta_0(a;f)$, $\Theta(a;f)\le\Theta_0(a;f)$ and $\delta_k(a;f)\le\delta_k^0(a;f)$.*

This lemma can be proved along the lines of [7, Proposition 6].

Lemma 3. *Let f_1, f_2 be nonconstant meromorphic functions such that $af_1+bf_2\equiv 1$, where a, b are nonzero constants. Then*

$$T_0(r,f_1)\le\overline{N}_0(r,0;f_1)+\overline{N}_0(r,0;f_2)+\overline{N}_0(r,\infty;f_1)+S_0(r,f_1).$$

Proof. By the second fundamental theorem we get

$$\begin{aligned}T(r,f_1)&\le\overline{N}(r,0;f_1)+\overline{N}(r,a^{-1};f_1)+\overline{N}(r,\infty;f_2)+S(r,f_1)\\&=\overline{N}(r,0;f_1)+\overline{N}(r,0,f_2)+\overline{N}(r,\infty;f_1)+S(r,f_1).\end{aligned}$$

From this inequality the lemma follows on integration.

Lemma 4 [4]. *For a meromorphic function f and any $a \in \mathbb{C}$,*

$$N(r,0;\Psi(D)f \mid f=a,\ \geq p) \geq N(r,0;f^{(p)} \mid f=a,\ \geq p) + S(r,f),$$

where $N(r,b;g \mid f=c,\ \geq k)$ is the counting function of those b-points of g, counted with proper multiplicities, which are the c-points of f with multiplicities not less than k.

Lemma 5. *Let f be a meromorphic function. Then*

(i) $$\liminf_{r\to\infty} \frac{T_0(r,\Psi(D)f)}{T_0(r,f)} \geq \sum_{a\neq\infty} \delta_p^0(a;f),$$

(ii) $$\delta_0(0;\Psi(D)f) \geq \frac{\sum_{a\neq\infty}\delta_0(a;f)}{1+p(1-\Theta_0(\infty;f))}.$$

Proof. For distinct finite complex numbers $a_1,\dots,a_n$ we put

$$A = \sum_{i=1}^{n} \frac{1}{f-a_i}.$$

Then by [3, inequality 2.1, p. 33] we get

$$\begin{aligned}\sum_{i=1}^{n} m(r,a_i;f) &\leq m(r,A)+O(1)\\ &\leq m(r,0;\Psi(D)f)+m(r,A\Psi(D)f)\\ &\leq m(r,0;\Psi(D)f)+\sum_{i=1}^{n} m\left(r,\frac{\Psi(D)f}{f-a_i}\right)\\ &= m(r,0;\Psi(D)f)+\sum_{i=1}^{n} m\left(r,\frac{\Psi(D)(f-a_i)}{f-a_i}\right)\\ &= m(r,0;\Psi(D)f)+S(r,f),\end{aligned}$$

by the Milloux theorem [3, p. 55], i.e.,

(1) $$\begin{aligned} nT(r,f) &\leq T(r,\Psi(D)f)+\sum_{i=1}^{n} N(r,a_i;f) - N(r,0;\Psi(D)f)+S(r,f)\\ &\leq T(r,\Psi(D)f)\\ &\quad+\sum_{i=1}^{n}\{N(r,a_i;f)-N(r,0;\Psi(D)f \mid f=a_i,\ \geq p)\}\\ &\quad+S(r,f).\end{aligned}$$

So by Lemma 4 we get

$$\begin{aligned} nT(r,f) &\le T(r,\Psi(D)f) + \sum_{i=1}^{n}\{N(r,a_i;f) - N(r,0;f^{(p)} \mid f = a_i,\ \ge p)\} \\ &\quad + S(r,f) \\ &\le T(r,\Psi(D)f) + \sum_{i=1}^{n} N_p(r,a_i;f) + S(r,f). \end{aligned}$$

This gives on integration

$$nT_0(r,f) \le T_0(r,\Psi(D)f) + \sum_{i=1}^{n} N_p^0(r,a_i;f) + S_0(r,f).$$

Hence by Lemma 1 we get

$$\liminf_{r\to\infty} \frac{T_0(r,\Psi(D)f)}{T_0(r,f)} \ge \sum_{i=1}^{n} \delta_p^0(a_i;f).$$

Since n is arbitrary, it follows that

$$\liminf_{r\to\infty} \frac{T_0(r,\Psi(D)f)}{T_0(r,f)} \ge \sum_{a\ne\infty} \delta_p^0(a;f).$$

Again by the Milloux theorem,

$$\begin{aligned} T(r,\Psi(D)f) &\le m\left(r,\frac{\Psi(D)f}{f}\right) + m(r,f) + N(r,f) \\ &\quad + p\overline{N}(r,f) + O(1) \\ &= T(r,f) + p\overline{N}(r,f) + S(r,f). \end{aligned}$$

This gives on integration

$$T_0(r,\Psi(D)f) \le T_0(r,f) + p\overline{N}_0(r,f) + S_0(r,f). \tag{2}$$

Also from (1) we get by integration

$$\begin{aligned} nT_0(r,f) &\le T_0(r,\Psi(D)f) + \sum_{i=1}^{n} N_0(r,a_i;f) \\ &\quad - N_0(r,0;\Psi(D)f) + S_0(r,f). \end{aligned}$$

So by (2) we obtain

$$\begin{aligned} n &\le \left(1 - \frac{N_0(r,0;\Psi(D)f)}{T_0(r,\Psi(D)f)}\right) \cdot \frac{T_0(r,f) + p\overline{N}_0(r,f) + S_0(r,f)}{T_0(r,f)} \\ &\quad + \sum_{i=1}^{n} \frac{N_0(r,a_i;f)}{T_0(r,f)} + \frac{S_0(r,f)}{T_0(r,f)}. \end{aligned}$$

In view of Lemma 1 this gives

$$\sum_{i=1}^{n} \delta_0(a_i; f) \le \delta_0(0; \Psi(D)f)\{1 - \Theta_0(\infty; f))\},$$

from which (ii) follows because n is arbitrary. This proves the lemma.

LEMMA 6 [11]. *Let f_1, f_2, f_3 be nonconstant meromorphic functions satisfying $f_1 + f_2 + f_3 \equiv 1$. If f_1, f_2, f_3 are linearly independent then $g_1 = -f_2/f_3$, $g_2 = 1/f_3$ and $g_3 = -f_1/f_3$ are also linearly independent.*

LEMMA 7. *Let f_1, f_2, f_3 be three linearly independent meromorphic functions such that $f_1 + f_2 + f_3 \equiv 1$. Then*

$$T_0(r, f_1) \le \sum_{j=1}^{3} N_2^0(r, 0; f_j) + \max_{1 \le i \ne j \le 3}\{N_2^0(r, \infty; f_i) + \overline{N}_0(r, \infty; f_j)\} + S_0(r),$$

where $S_0(r) = \sum_{j=1}^{3} S_0(r, f_j)$.

Proof. We prove under the hypotheses of the lemma the following inequality which on integration proves the lemma:

$$T(r, f_1) \le \sum_{j=1}^{3} N_2(r, 0; f_j) + \max_{1 \le i \ne j \le 3}\{N_2(r, \infty; f_i) + \overline{N}(r, \infty; f_j)\} + \sum_{j=1}^{3} S(r, f_j). \tag{3}$$

From the proof of a generalisation of Borel's theorem by Nevanlinna (cf. [2, p. 70]) we get

$$T(r, f_1) \le \sum_{j=1}^{3} N(r, 0; f_j) - N(r, 0; \Delta) + N(r, \Delta) - N(r, f_2) - N(r, f_3) + S(r), \tag{4}$$

where Δ is the wronskian determinant of f_1, f_2, f_3 and $S(r) = \sum_{j=1}^{3} S(r, f_j)$.

Now we need the following notations from [5]: for $z \in \mathbb{C}$ and $b \in \mathbb{C} \cup \{\infty\}$ we put

$$\mu_f^b(z) = \begin{cases} m & \text{if } z \text{ is a } b\text{-point of } f \text{ with multiplicity } m \ge 1, \\ 0 & \text{if } z \text{ is not a } b\text{-point of } f, \end{cases}$$

$$\overline{\mu}_f^b(z) = \begin{cases} 1 & \text{if } z \text{ is a } b\text{-point of } f \text{ with multiplicity } \ge 1, \\ 0 & \text{if } z \text{ is not a } b\text{-point of } f, \end{cases}$$

$$\nu_f^b(z) = \begin{cases} 2 & \text{if } z \text{ is a } b\text{-point of } f \text{ with multiplicity } m > 2, \\ m & \text{if } z \text{ is a } b\text{-point of } f \text{ with multiplicity } m \le 2. \end{cases}$$

Also we put

$$\mu(z) = \sum_{j=1}^{3} \mu^0_{f_j}(z) - \mu^0_{\Delta}(z) + \mu^{\infty}_{\Delta}(z) - \mu^{\infty}_{f_2}(z) - \mu^{\infty}_{f_3}(z)$$

and

$$\mu^*(z) = \sum_{j=1}^{3} \nu^0_{f_j}(z) + \max_{1\le i\ne j\le 3}\{\nu^{\infty}_{f_i}(z) + \overline{\mu}^{\infty}_{f_j}(z)\}.$$

Now (3) will follow from (4) if we can prove that for any $z \in \mathbb{C}$, $\mu(z) \le \mu^*(z)$.

We consider the following cases.

Case 1. Let z be not a pole of any f_i $(i = 1, 2, 3)$. Since any zero of f_i with multiplicity $m > 2$ is a zero of Δ with multiplicity at least $m - 2$, it follows that $\mu(z) \le \mu^*(z)$.

Case 2. Let z be a pole of at least one of f_i $(i = 1, 2, 3)$. So the following subcases come up for consideration.

Subcase 2.1. Let z be a zero of f_1 with multiplicity $m > 2$ and a pole of f_2, f_3 with multiplicity $k \ge 1$. Then z is a pole of Δ with multiplicity $k - m + 3$ provided $k - m + 3 > 0$ and otherwise z is a zero of Δ with multiplicity $m - k - 3$. Hence $\mu(z) = 3 - k$ and $\mu^*(z) \ge 3$. So $\mu(z) \le \mu^*(z)$.

Let z be a zero of f_1 with multiplicity $m \le 2$ and a pole of f_2, f_3 with multiplicity $k \ge 1$. Then z is a pole Δ with multiplicity not exceeding $k+2$. Hence $\mu(z) \le m + k + 2 - k - k \le 4 - k$ and $\mu^*(z) \ge 3$. So $\mu(z) \le \mu^*(z)$.

Subcase 2.2. Let z be a zero of f_2 with multiplicity $m > 2$ and a pole of f_1, f_3 with multiplicity $k \ge 1$. Then z is a pole of Δ with multiplicity $k - m + 3$ provided $k - m + 3 > 0$ and otherwise z is a zero of Δ with multiplicity $m - k - 3$. Hence $\mu(z) = 3$ and $\mu^*(z) \ge 3$. So $\mu(z) \le \mu^*(z)$.

Let z be a zero of f_2 with multiplicity $m \le 2$ and a pole of f_1, f_3 with multiplicity $k \ge 1$. Then z is a pole of Δ with multiplicity not exceeding $k+2$. Hence $\mu(z) \le m+k+2-k = m+2$ and $\mu^*(z) \ge m+2$. So $\mu(z) \le \mu^*(z)$.

Subcase 2.3. Let z be a zero of f_3 with multiplicity $m \ge 1$ and a pole of f_1, f_2 with multiplicity $k \ge 1$. Then as in Subcase 2.2 we can prove that $\mu(z) \le \mu^*(z)$.

Subcase 2.4. Let z be neither a zero nor a pole of f_1. Since $f_2 + f_3 = 1 - f_1$, it follows that z is not a pole of $f_2 + f_3$. Since z is a pole of at least one of f_i $(i = 1, 2, 3)$, it follows that z is a pole of f_2 and f_3 with the same multiplicity m, say (because the singularities of f_2 and f_3 at z cancel each other). Then z is a pole of Δ with multiplicity not exceeding $m + 2$. Hence $\mu(z) \le m + 2 - m - m \le 2$ and $\mu^*(z) \ge 2$. So $\mu(z) \le \mu^*(z)$.

Subcase 2.5. Let z be a pole of f_1, f_2 with multiplicity $m \ge 1$ and a pole of f_3 with multiplicity q $(1 \le q < m)$. Then z is a pole of Δ with multiplicity not exceeding $m+q+3$. Hence $\mu(z) \le m+q+3-m-q=3$ and $\mu^*(z)=2+1=3$. So $\mu(z)\le\mu^*(z)$.

SUBCASE 2.6. Let z be a pole of f_1, f_2, f_3 with multiplicity $m\ge 1$. Then there exist two functions ϕ,ψ analytic at z and $\phi(z)\ne 0, \psi(z)\ne 0$ such that in some neighbourhood of z, $f_2(\omega)=(\omega-z)^{-m}\phi(\omega)$ and $f_3(\omega) = (\omega-z)^{-m}\psi(\omega)$. Also $\Delta = f_2'f_3''-f_2''f_3'$ shows that z is a pole of Δ with multiplicity not exceeding $2m+3$ but by actual calculation we see that the coefficient of $(\omega-z)^{-(2m+3)}$ in Δ is $m^2(m+1)\phi\psi - m^2(m+1)\phi\psi \equiv 0$. So z is a pole of Δ with multiplicity not exceeding $2m+2$. Hence $\mu(z)\le 2m+2-m-m=2$ and $\mu^*(z)\ge 2$. So $\mu(z)\le\mu^*(z)$.

SUBCASE 2.7. Let z be a pole of f_1, f_2 with multiplicity $m\ge 1$ and neither a zero nor a pole of f_3. Then z is a pole of Δ with multiplicity not exceeding $m+2$. Hence $\mu(z)\le m+2-m=2$ and $\mu^*(z)\ge 2$. So $\mu(z)\le\mu^*(z)$.

SUBCASE 2.8. Let z be a pole of f_1 with multiplicity $m\ge 1$ and a pole of f_2 with multiplicity $m+q$ $(q\ge 1)$. Then z is also a pole of f_3 with multiplicity $m+q$ and the terms containing $(w-z)^{-(m+i)}$ $(i=1,\dots,q)$ of the Laurent expansions of f_2 and f_3 about z cancel each other because f_2+f_3 has a pole at z with multiplicity m. Also we see that Δ has a pole at z with multiplicity not exceeding $2m+q+3$. Hence $\mu(z)\le 2m+q+3-m-q-m-q=3-q$ and $\mu^*(z)=2+1=3$. So $\mu(z)\le\mu^*(z)$.

LEMMA 8. *If* $\sum_{a\ne\infty}\delta_p^0(a;f)>0$ *then*

$$\Theta_0(\infty;\Psi(D)f)\ge 1-\frac{1-\Theta_0(\infty;f)}{\sum_{a\ne\infty}\delta_p^0(a;f)}.$$

Proof. Since $\overline{N}_0(r,\Psi(D)f)=\overline{N}_0(r,f)$, the lemma follows from Lemma 5(i).

LEMMA 9 [14]. *Let F and G be two nonconstant meromorphic functions such that F and G share* 1 *CM. If*

$$\limsup_{r\to\infty,\, r\in I}\frac{N_2(r,0;F)+N_2(r,\infty;F)+N_2(r,0;G)+N_2(r,\infty;G)}{T(r)}<1,$$

where

$$T(r)=\max\{T(r,F),T(r,G)\}$$

and I is a set of r's $(0<r<\infty)$ *of infinite linear measure, then $F\equiv G$ or $FG\equiv 1$.*

4. Theorems. In this section we present the main results of the paper.

THEOREM 1. *Let f, g be two meromorphic functions such that*

(i) *$\Psi(D)f$, $\Psi(D)g$ are nonconstant and share* 1 *CM and*

(ii) $$\frac{\sum_{a\neq\infty}\delta(a;f)}{1+p(1-\Theta(\infty;f))}+\frac{\sum_{a\neq\infty}\delta(a;g)}{1+p(1-\Theta(\infty;g))} > 1+\frac{4(1-\Theta(\infty;f))}{\sum_{a\neq\infty}\delta_p(a;f)}+\frac{4(1-\Theta(\infty;g))}{\sum_{a\neq\infty}\delta_p(a;g)},$$

where $\sum_{a\neq\infty}\delta_p(a;f)>0$ *and* $\sum_{a\neq\infty}\delta_p(a;g)>0$. *Then either* $[\Psi(D)f]\cdot[\Psi(D)g]\equiv 1$ *or* $f-g\equiv s$, *where s is a solution of the differential equation* $\Psi(D)w=0$.

Theorem 2. *Let f, g be two meromorphic functions of finite order such that*

(i) *$\Psi(D)f$, $\Psi(D)g$ are nonconstant and share* 1 *CM and*

(ii) $$\frac{\sum_{a\neq\infty}\delta(a;f)}{1+p(1-\Theta(\infty;f))}+\frac{\sum_{a\neq\infty}\delta(a;g)}{1+p(1-\Theta(\infty;g))} > 1+\frac{2(1-\Theta(\infty;f))}{\sum_{a\neq\infty}\delta_p(a;f)}+\frac{2(1-\Theta(\infty;g))}{\sum_{a\neq\infty}\delta_p(a;g)},$$

where $\sum_{a\neq\infty}\delta_p(a;f)>0$ *and* $\sum_{a\neq\infty}\delta_p(a;g)>0$. *Then either* $[\Psi(D)f]\cdot[\Psi(D)g]\equiv 1$ *or* $f-g\equiv s$, *where s is a solution of the differential equation* $\Psi(D)w=0$.

The following example shows that the theorems are sharp.

Example 1. Let $f=\frac{1}{2}e^z(e^z-1)$, $g=\frac{1}{2}e^{-z}\left(\frac{1}{2}-\frac{1}{5}e^{-z}\right)$ and $\Psi(D)=D^2-3D$. Then $\sum_{a\neq\infty}\delta(a;f)=\sum_{a\neq\infty}\delta(a;g)=1/2$, $\Theta(\infty;f)=\Theta(\infty;g)=1$ and $\sum_{a\neq\infty}\delta_2(a;f)>0$, $\sum_{a\neq\infty}\delta_2(a;g)>0$. Also $\Psi(D)f=e^z(1-e^z)$ and $\Psi(D)g=e^{-z}(1-e^{-z})$ share 1 CM but neither $[\Psi(D)f]\cdot[\Psi(D)g]\equiv 1$ nor $f-g\equiv c_1+c_2e^{3z}$ for any constants c_1 and c_2.

Proof of Theorem 1. Let $F=\Psi(D)f$ and $G=\Psi(D)g$. Then in view of Lemmas 2, 5 and 8 the condition (ii) implies

$$\delta_0(0;F)+\delta_0(0;G)+4\Theta_0(\infty;F)+4\Theta_0(\infty;G)>9. \tag{5}$$

We put

$$H=\frac{F-1}{G-1}. \tag{6}$$

Since F, G share 1 CM, the poles and zeros of H occur only at the poles of F and G respectively. Also $\overline{N}_0(r,\infty;H)\leq\overline{N}_0(r,\infty;F)$ and $\overline{N}_0(r,0;H)\leq\overline{N}_0(r,\infty;G)$.

Let $F_1=F$, $F_2=-GH$ and $F_3=H$. Then from (6) it follows that

$$F_1+F_2+F_3\equiv 1. \tag{7}$$

First we suppose that $F_3 = H \equiv k$, a constant. Then from (7) we get $F - kG = 1 - k$. If $k \neq 1$, we see that

$$\frac{1}{1-k}F - \frac{k}{1-k}G \equiv 1.$$

Since $k \neq 0$, from Lemma 3 it follows that

$$T_0(r,F) \leq N_0(r,0;F) + N_0(r,0;G) + \overline{N}_0(r,\infty;F) + S_0(r,F),$$
$$T_0(r,G) \leq N_0(r,0;F) + N_0(r,0;G) + \overline{N}_0(r,\infty;G) + S_0(r,G).$$

So

$$\max\{T_0(r,F),T_0(r,G)\} \leq N_0(r,0;F) + N_0(r,0;G) + \overline{N}_0(r,\infty;F) + \overline{N}_0(r,\infty;G) + o(\max\{T_0(r,F),T_0(r,G)\}).$$

This gives $\delta_0(0;F) + \delta_0(0;G) + \Theta_0(\infty;F) + \Theta_0(\infty;G) \leq 3$ and so from (5) we see that $9 < 3\Theta_0(\infty;F) + 3\Theta_0(\infty;G) + 3 \leq 9$, a contradiction. So $k = 1$ and hence $F \equiv G$. Therefore $\Psi(D)(f-g) \equiv 0$ and so $f - g \equiv s$, where $s = s(z)$ is a solution of $\Psi(D)w = 0$.

Similarly if $F_2 \equiv k$, a constant, we can show that $[\Psi(D)f] \cdot [\Psi(D)g] \equiv 1$.

Now we suppose that F_1, F_2 and F_3 are nonconstant. If possible, let F_1, F_2, F_3 be linearly independent. Then from Lemma 7 we get

$$(8) \quad \begin{aligned} T_0(r,F) &\leq N_2^0(r,0;F) + N_2^0(r,0;G) + 2N_2^0(r,0;H) \\ &\qquad + \max_{1\leq i\neq j\leq 3}\{N_2^0(r,\infty;F_i) + \overline{N}_0(r,\infty;F_j)\} + \sum_{j=1}^{3} S_0(r,F_j) \\ &\leq N_0(r,0;F) + N_0(r,0;G) + 4\overline{N}_0(r,\infty;G) \\ &\qquad + \max_{1\leq 1\neq j\leq 3}\{N_2^0(r,\infty;F_i) + \overline{N}_0(r,\infty;F_j)\} + \sum_{j=1}^{3} S_0(r,F_j). \end{aligned}$$

Now in view of (6) we see that

$$\sum_{j=1}^{3} S_0(r,F_j) = o(\max\{T_0(r,F),T_0(r,G)\})$$

and

$$\begin{aligned} N_2^0(r,\infty;F_1) + \overline{N}_0(r,\infty;F_2) &= N_2^0(r,\infty;F) + \overline{N}_0(r,\infty;H(G-1)) \\ &= N_2^0(r,\infty;F) + \overline{N}_0(r,\infty;F) \leq 3\overline{N}_0(r,\infty;F), \\ N_2^0(r,\infty;F_2) + \overline{N}_0(r,\infty;F_3) &= N_2^0(r,\infty;H(G-1)) + N_0(r,\infty;H) \\ &\leq N_2^0(r,\infty;F) + \overline{N}_0(r,\infty;F) \leq 3\overline{N}_0(r,\infty;F), \\ N_2^0(r,\infty;F_3) + \overline{N}_0(r,\infty;F_1) &= N_2^0(r,\infty;H) + \overline{N}_2^0(r,\infty;F) \\ &\leq 2\overline{N}_0(r,\infty;H) + \overline{N}_0(r,\infty;F) \leq 3\overline{N}_0(r,\infty;F) \end{aligned}$$

and similarly for the other three terms. So from (8) we get

$$\text{(9)}\qquad T_0(r,F) \le N_0(r,0;F) + N_0(r,0;G) + 3\overline{N}_0(r,\infty;F) + 4\overline{N}_0(r,\infty;G) + o(\max\{T_0(r,F), T_0(r,G)\}).$$

Now we put $G_1 = -F_2/F_3 = G$, $G_2 = 1/F_3 = 1/H$ and $G_3 = -F_1/F_3 = -F/H$. Then by Lemma 6, G_1, G_2, G_3 are linearly independent and so proceeding as above we get

$$\text{(10)}\qquad T_0(r,G) \le N_0(r,0;F) + N_0(r,0;G) + 3\overline{N}_0(r,\infty;G) + 4\overline{N}_0(r,\infty;F) + o(\max\{T_0(r,F), T_0(r,G)\}).$$

From (9) and (10) we get

$$\begin{aligned}\max\{T_0(r,F), T_0(r,G)\} &\le (10 - \delta_0(0;F) - \delta_0(0;G) - 4\Theta_0(\infty;F) \\ &\quad - 4\Theta_0(\infty;G) + o(1))\max\{T_0(r,F), T_0(r,G)\} \\ &< (1 - \varepsilon + o(1))\max\{T_0(r,F), T_0(r,G)\},\end{aligned}$$

which is a contradiction, where by (5) we choose

$$0 < \varepsilon < \delta_0(0;F) + \delta_0(0;G) + 4\Theta_0(\infty;F) + 4\Theta_0(\infty;G) - 9.$$

Hence there exist constants c_1, c_2, c_3, not all zero, such that

$$\text{(11)}\qquad c_1F_1 + c_2F_2 + c_3F_3 \equiv 0.$$

Clearly $c_1 \ne 0$. For, otherwise from (11) we get $H(c_3 - c_2G) \equiv 0$, which is impossible because F and G are nonconstant.

Now eliminating F_1 from (7) and (11) we get

$$\text{(12)}\qquad cF_2 + dF_3 \equiv 1,$$

where $c = 1 - c_2/c_1$ and $d = 1 - c_3/c_1$.

If possible let $cd \ne 0$. Then from (12) we get $(c/d)(G) + 1/(dH) \equiv 1$. So by Lemma 3 we get

$$T_0(r,G) \le N_0(r,0;G) + \overline{N}_0(r,\infty;H) + \overline{N}_0(r,\infty;G) + S_0(r,G),$$

i.e.

$$\text{(13)}\qquad T_0(r,G) \le N_0(r,0;G) + \overline{N}_0(r,\infty;F) + \overline{N}_0(r,\infty;G) + S_0(r,G).$$

By the second fundamental theorem we get on integration

$$\begin{aligned}T_0(r,F) &\le N_0(r,0;F) + N_0(r,1;F) + \overline{N}_0(r,\infty;F) + S_0(r;F) \\ &= N_0(r,0;F) + N_0(r,1;G) + \overline{N}_0(r,\infty;F) + S_0(r,F) \\ &\le N_0(r,0;F) + T_0(r,G) + \overline{N}_0(r,\infty;F) + S_0(r,F).\end{aligned}$$

So by (13) we obtain

$$\text{(14)}\qquad T_0(r,F) \le N_0(r,0;F) + N_0(r,0;G) + 2\overline{N}_0(r,\infty;F) + \overline{N}_0(r,\infty;G) + S_0(r,F) + S_0(r,G).$$

From (13) and (14) we get

$$\max\{T_0(r,F),T_0(r,G)\} \le N_0(r,0;F)+N_0(r,0;G)+2\overline{N}_0(r,\infty;F) + \overline{N}_0(r,\infty;G)+o(\max\{T_0(r,F),T_0(r,G)\})$$

and so $\delta_0(0;F)+\delta_0(0;G)+2\Theta_0(\infty;F)+\Theta_0(\infty;G)\le 4$.

Now by (5) we see that

$$9<\delta_0(0;F)+\delta_0(0;G)+4\Theta_0(\infty;F)+4\Theta_0(\infty;G) \le 4+2\Theta_0(\infty;F)+3\Theta_0(\infty;G)\le 9,$$

which is a contradiction. Therefore $cd=0$. From (12) we see that c and d are not simultaneously zero. So we consider the following cases.

Case I. Let $d=0$. Then from (12) we get $-cF+1/G\equiv 1-c$. If $c\ne 1$, we obtain $(-c/(1-c))F+1/((1-c)G)\equiv 1$. So by Lemma 3 it follows that

$$T_0(r,F)\le N_0(r,0;F)+\overline{N}_0(r,\infty;G)+\overline{N}_0(r,\infty;F)+S_0(r,F)$$

and

$$T_0(r,G)=T_0(r,1/G)+S_0(r,G) \le N_0(r,0;F)+N_0(r,0;G)+\overline{N}_0(r,\infty;G)+S_0(r,G).$$

Hence

$$\max\{T_0(r,F),T_0(r,G)\}\le N_0(r,0;F)+N_0(r,0;G)+\overline{N}_0(r,\infty;F) + \overline{N}_0(r,\infty;G)+o(\max\{T_0(r,F),T_0(r,G)\}),$$

and so $\delta_0(0;F)+\delta_0(0;G)+\Theta_0(\infty;F)+\Theta_0(\infty;G)\le 3$.

From (5) we see that

$$9<\delta_0(0;F)+\delta_0(0;G)+4\Theta_0(\infty;F)+4\Theta_0(\infty;G) \le 3+3\Theta_0(\infty;F)+3\Theta_0(\infty;G)\le 9,$$

which is a contradiction. Therefore $c=1$ and so $FG\equiv 1$, i.e., $[\Psi(D)f]\cdot[\Psi(D)g]\equiv 1$.

Case II. Let $c=0$. Then from (12) we get $dF-G\equiv d-1$. If $d\ne 1$ it follows that $(d/(d-1))F-(1/(d-1))G\equiv 1$. Now by Lemma 3 we obtain

$$T_0(r,F)\le N_0(r,0;F)+N_0(r,0;G)+\overline{N}_0(r,\infty;F)+S_0(r,F),$$

$$T_0(r,G)\le N_0(r,0;F)+N_0(r,0;G)+\overline{N}_0(r,\infty;G)+S_0(r,G).$$

So we get

$$\max\{T_0(r,F),T_0(r,G)\}\le N_0(r,0;F)+N_0(r,0;G)+\overline{N}_0(r,\infty;F) + \overline{N}_0(r,\infty;G)+o(\max\{T_0(r,F),T_0(r,G)\})$$

and as in Case I this leads to a contradiction. So $d=1$ and hence $F\equiv G$, i.e., $\Psi(D)(f-g)\equiv 0$. Therefore $f-g\equiv s$ where $s=s(z)$ is a solution of $\Psi(D)w=0$. This proves the theorem.

Proof of Theorem 2. If f and g are of finite order, we can prove along the lines of Lemmas 5 and 8 that

$$\delta(0;\Psi(D)f)\geq \frac{\sum_{a\neq\infty}\delta(a;f)}{1+p(1-\Theta(\infty;f))},\qquad \Theta(\infty;\Psi(D)f)\geq 1-\frac{1-\Theta(\infty;f)}{\sum_{a\neq\infty}\delta_p(a;f)},$$

and the corresponding results for g. Let $F=\Psi(D)f$ and $G=\Psi(D)g$. Then by the condition (ii) of the theorem we get

$$\delta(0;F)+\delta(0;G)+2\Theta(\infty;F)+2\Theta(\infty;G)>5.$$

This implies

$$\limsup_{r\to\infty}\frac{N(r,0;F)}{T(r,F)}+\limsup_{r\to\infty}\frac{N(r,0;G)}{T(r,G)} +2\limsup_{r\to\infty}\frac{\overline{N}(r,\infty;F)}{T(r,F)}+2\limsup_{r\to\infty}\frac{\overline{N}(r,\infty;G)}{T(r,G)}<1,$$

i.e.,

$$\limsup_{r\to\infty}\frac{N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)}{\max\{T(r,F),T(r,G)\}}<1$$

and so by Lemma 9 the theorem follows.

Considering $f=-2^{-n}e^{2z}+e^{z}$, $g(z)=-(-1)^n2^{-n}e^{-2z}+(-1)^ne^{-z}$ where n is a positive integer, Yi and Yang [16] claimed that for $n\geq 1$ the condition $\delta(0;f)+\delta(0;g)>1$ of Theorem G is necessary. In the following example we see that this claim is not justified.

EXAMPLE 2. Let $f=e^z-1$ and $g=1+(-1)^ne^{-z}$. Then $\delta(0;f)=\delta(0;g)=0$ and $f^{(n)}$, $g^{(n)}$ share 1 CM. Also $f^{(n)}g^{(n)}\equiv 1$.

In the first corollary we improve Theorem G for $n\geq 1$.

COROLLARY 1. *Let f, g be two meromorphic function with $\Theta(\infty;f)=\Theta(\infty;g)=1$. If for $n\geq 1$ the derivatives $f^{(n)}$ and $g^{(n)}$ are nonconstant and share* 1 *CM with*

(i) $\sum_{a\neq\infty}\delta(a;f)+\sum_{a\neq\infty}\delta(a;g)>1$,
(ii) $\Theta(0;f)+\Theta(0;g)>1$,

then either (a) $f^{(n)}g^{(n)}\equiv 1$ *or* (b) $f\equiv g$.

Proof. Choosing $\Psi(D)=D^n$, from Theorem 1 it follows that either $f^{(n)}g^{(n)}\equiv 1$ or $f-g\equiv Q$, where Q is a polynomial of degree at most $n-1$. If possible, let $Q\not\equiv 0$. Then from [3, Theorem 2.5, p. 47] it follows that

$$\begin{aligned}T(r,f)&\leq \overline{N}(r,0;f)+\overline{N}(r,Q;f)+\overline{N}(r,\infty;f)+S(r,f)\\&=\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

Since $f-g\equiv Q$, it follows that $T(r,f)=T(r,g)+O(\log r)$. So we get $\Theta(0;f)+\Theta(0;g)\le 1$, which is a contradiction. Therefore $Q\equiv 0$ and so $f\equiv g$.

The following examples show that the condition $\Theta(0;f)+\Theta(0;g)>1$ is necessary for the validity of case (b).

EXAMPLE 3. Let $f=e^z+1$ and $g=e^z$. Then $\Theta(0;f)=0$, $\Theta(0;g)=1$, $\sum_{a\neq\infty}\delta(a;f)+\sum_{a\neq\infty}\delta(a;g)=2$, $\Theta(\infty;f)=\Theta(\infty;g)=1$ and $f^{(n)}$, $g^{(n)}$ share 1 CM but $f-g\equiv 1$.

EXAMPLE 4. Let $f=e^z+1$ and $g=(-1)^n e^{-z}$. Then $\Theta(0;f)=0$, $\Theta(0;g)=1$, $\sum_{a\neq\infty}\delta(a;f)+\sum_{a\neq\infty}\delta(a;g)=2$, $\Theta(\infty;f)=\Theta(\infty;g)=1$ and $f^{(n)}$, $g^{(n)}$ share 1 CM but $f^{(n)}g^{(n)}\equiv 1$.

Answering the question of C. C. Yang [9], mentioned in the introduction, H. X. Yi [12] proved the following theorem.

THEOREM H [12]. *Let f and g be two nonconstant entire functions. Assume that f, g share 0 CM and $f^{(n)}$, $g^{(n)}$ share 1 CM, where n is a nonnegative integer. If $\delta(0;f)>1/2$ then either $f\equiv g$ or $f^{(n)}g^{(n)}\equiv 1$.*

Considering $f=-2^{-n}e^{2z}+(-1)^{n+1}2^{-n}e^z$ and $g=(-1)^{n+1}2^{-n}e^{-2z}-2^{-n}e^{-z}$, Yi [12] claimed that the condition $\delta(0;f)>1/2$ is necessary. The following example shows that for $n\ge 1$ this is not always the case.

EXAMPLE 5. Let $f=e^z-1$ nad $g=(-1)^{n+1}+(-1)^n e^{-z}$. Then f, g share 0 CM and $f^{(n)}$, $g^{(n)}$ $(n\ge 1)$ share 1 CM, $\delta(0;f)=0$ but $f^{(n)}g^{(n)}\equiv 1$.

In the following corollary we provide an answer to a question of Yang [9].

COROLLARY 2. *Let f and g be two meromorphic functions with $\Theta(\infty;f)=\Theta(\infty;g)=1$. Suppose that $f^{(n)}$, $g^{(n)}$ $(n\ge 1)$ share 1 CM and f,g share a value b $(\neq\infty)$ IM. If $\sum_{a\neq\infty}\delta(a;f)+\sum_{a\neq\infty}\delta(a;g)>1$ then either $f\equiv g$ or $f^{(n)}g^{(n)}\equiv 1$.*

Proof. The condition $\sum_{a\neq\infty}\delta(a;f)+\sum_{a\neq\infty}\delta(a;g)>1$ implies that f and g are transcendental so that $f^{(n)}$, $g^{(n)}$ are nonconstant. Choosing $\Psi(D)=D^n$ we see from Theorem 1 that either $f^{(n)}g^{(n)}\equiv 1$ or $f-g\equiv Q$, where Q is a polynomial. Now we consider the case $f-g\equiv Q$. If possible, let $Q\not\equiv 0$. Also we suppose that f has at most a finite number of b-points and so g also has a finite number of b-points. Now by [3, Theorem 2.5, p. 47] it follows that

$$\begin{aligned}T(r,f)&\le \overline{N}(r,b;f)+\overline{N}(r,b+Q;f)+\overline{N}(r,\infty;f)+S(r,f)\\&=\overline{N}(r,b;f)+\overline{N}(r,b;g)+S(r,f)=O(\log r)+S(r,f),\end{aligned}$$

which is a contradiction. Therefore f has infinitely many b-points and so $f-g$ has infinitely many zeros. This again implies a contradiction because $f-g\equiv Q$ and $Q\not\equiv 0$. So $Q\equiv 0$ and hence $f\equiv g$.

Considering $f = -2^{-n}e^{2z} + (-1)^{n+1}2^{-n}e^{z}$ and $g = (-1)^{n+1}2^{-n}e^{-2z} - 2^{-n}e^{-z}$ we can verify that the condition $\sum_{a\neq\infty}\delta(a;f) + \sum_{a\neq\infty}\delta(a;g) > 1$ of Corollary 2 is necessary.

COROLLARY 3. *Let* $\Psi(D) = D(D-\lambda_1)(D-\lambda_2)\dots(D-\lambda_{p-1})$ *where* λ_i*'s are nonzero pairwise distinct complex numbers. Also suppose that* f *and* g *are two meromorphic functions with the following properties*:

(i) $\Psi(D)f$, $\Psi(D)g$ *are nonconstant and share* 1 *CM*,

(ii) $\sum_{a\neq\infty}\delta(a;f) + \sum_{a\neq\infty}\delta(a;g) > 1$ *and* $\Theta(\infty;f) = \Theta(\infty;g) = 1$,

(iii) f *and* g *have* b*-points* ($b \neq \infty$) *with multiplicities not less than* $p+1$ *at the origin.*

Then $f \equiv g$.

Proof. From the theorem we get either $[\Psi(D)f]\cdot[\Psi(D)g] \equiv 1$ or $f-g \equiv c_0 + c_1e^{\lambda_1 z} + c_2e^{\lambda_2 z} + \dots + c_{p-1}e^{\lambda_{p-1}z}$ where c_i's are constants. Since f has a b-point with multiplicity at least $p+1$ at the origin, it follows that $\Psi(D)f$ has at least a simple zero at the origin. Similarly $\Psi(D)g$ has at least a simple zero at the origin. So the case $[\Psi(D)f]\cdot[\Psi(D)g] \equiv 1$ does not occur. If possible, let $f \not\equiv g$. Then the constants $c_0, c_1, \dots, c_{p-1}$ are not all zero. Also by condition (iii) it follows that $f-g$ has a zero at the origin with multiplicity at least $p+1$. This implies that

$$\sum_{i=0}^{p-1} c_i = 0, \quad \sum_{i=0}^{p-1}\lambda_i c_i = 0, \quad \sum_{i=0}^{p-1}\lambda_i^2 c_i = 0, \ \dots, \ \sum_{i=0}^{p-1}\lambda_i^p c_i = 0.$$

This system of equations gives $c_0 = c_1 = c_2 = \dots = c_{p-1} = 0$, which is a contradiction. Therefore $f \equiv g$. This proves the corollary.

The following examples show that condition (iii) of Corollary 3 is necessary.

EXAMPLE 6. Let $f = e^{z^3}$, $g = e^{z^3} + 1$ and $\Psi(D) = D(D-1)$. Then $\Psi(D)f$, $\Psi(D)g$ share 1 CM, $\sum_{a\neq\infty}\delta(a;f) + \sum_{a\neq\infty}\delta(a;g) = 2$, $\Theta(\infty;f) = \Theta(\infty;g) = 1$ and $f-1$, $g-2$ have zeros with multiplicity three at the origin, but $f \not\equiv g$.

EXAMPLE 7. Let $f = e^z - 1$, $g = 1 - e^{-z}$ and $\Psi(D) = D$. Then $\Psi(D)f$, $\Psi(D)g$ share 1 CM, $\sum_{a\neq\infty}\delta(a;f) + \sum_{a\neq\infty}\delta(a;g) = 2$, $\Theta(\infty;f) = \Theta(\infty;g) = 1$ and f, g have simple zeros at the origin, but $f \not\equiv g$.

Let us conclude the paper with the following question: What can be said if two nonlinear differential polynomials generated by two meromorphic functions share 1 CM?

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