Continuous linear extension operators on spaces of holomorphic functions on closed subgroups of a complex Lie group

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Abstract. We show that the restriction operator of the space of holomorphic functions on a complex Lie group to an analytic subset $V$ has a continuous linear right inverse if it is surjective and if $V$ is a finite branched cover over a connected closed subgroup $\Gamma$ of $G$. Moreover, we show that if $\Gamma$ and $G$ are complex Lie groups and $V \subset \Gamma \times G$ is an analytic set such that the canonical projection $\pi_1: V \to \Gamma$ is finite and proper, then $R_V: O(\Gamma \times G) \to \text{Im} \ R_V \subset O(V)$ has a right inverse.

Introduction. Let $M$ be a complex space. We denote by $O(M)$ the Fréchet space of analytic functions on $M$ equipped with the topology of uniform convergence on compacta. If $V$ is a closed subvariety of $M$ the question of whether one can find a continuous linear extension operator from $O(V)$ into $O(M)$ was studied by various authors (see [2], [10], [12]). For example if $V$ is a closed subvariety of $\mathbb{C}^n$ a continuous linear extension operator exists if $V$ is an algebraic variety of $\mathbb{C}^n$ [2]. Moreover, in [8] Vogt has given an important condition for existence of a right inverse of a continuous linear surjection between nuclear Fréchet spaces.

In this note we take up the question of existence of continuous extension operators from subvarieties of $\mathbb{C}^n$, in the category of analytic subsets in a complex Lie group, by using the splitting theorem of Vogt. Namely, we prove the following two theorems.

Theorem 1. Let $\Gamma$ be a connected closed subgroup of a complex Lie group $G$ and $\pi$ an analytic set in $G$ such that $\pi$ is a branched cover over $\Gamma$ and the restriction map $R_V: O(G) \to O(V)$ is surjective. Then $R_V$ has a right inverse.


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Theorem 2. Let $\Gamma$ and $G$ be complex Lie groups and $V \subset \Gamma \times G$ an analytic set such that the canonical projection $\pi_1 : V \to \Gamma$ is finite and proper. Then $R_V : O(\Gamma \times G) \to \operatorname{Im} R_V \subset O(V)$ has a right inverse.

We now recall some definitions and relevant properties. Let $E$ be a Fréchet space with a fundamental system $\{\|\cdot\|_k\}$ of seminorms. We say that $E$ has

• (DN) if there exists $p$ such that $\forall q, \exists k, \exists C > 0$:
  $$\|x\|_q^2 \leq C \|x\|_k \|x\|_p,$$

• (Ω) if $\forall p, \exists q, \forall k, \exists C, d > 0$:
  $$\|x\|_q^{1+d} \leq C \|x\|_k \|x\|_p^{1+d},$$

• (Ω) if $\exists d > 0, \forall p, \exists q, \forall k, \exists C > 0$:
  $$\|x\|_q^{1+d} \leq C \|x\|_k \|x\|_p^{1+d},$$

where for each $p$ we define $\|x^*\|_p^* = \sup\{x^*(x) : \|x\|_p \leq 1\}$ for $x^* \in E^*$, the dual space of $E$.

The properties (DN), (Ω), (Ω) and many other properties were introduced and investigated by Vogt. It is known [8] that a Fréchet space $F \in (\DN)$ (respectively $F \in (\Omega)$) if and only if $F$ is isomorphic to a subspace (respectively a quotient space) of the space $s$ of rapidly decreasing sequences of complex numbers. In [8], Vogt has proved that a continuous linear map $R$ from a nuclear Fréchet space $E$ onto a nuclear Fréchet space $F$ has a right inverse if $F \in (\DN)$ and $\ker R \in (\Omega)$.

By the above splitting theorem of Vogt, to prove Theorems 1 and 2, it suffices to show that

$$O(V), \operatorname{Im} R_V \in (\DN) \quad \text{and} \quad \ker R_V \in (\Omega).$$

The proofs of these relations are given in Sections 1 and 3 respectively.

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1. Proof of Theorem 1

Lemma 1.1. Let $\theta$ be a finite proper holomorphic map from a complex space $X$ onto a complex manifold $Y$. Then $O(X) \in (\DN)$ if and only if $O(Y) \in (\DN)$.

Proof. Since $O(Y)$ is a subspace of $O(X)$, the necessity is trivial.

Now, we prove the sufficiency. It is known [10] that a Fréchet space $F \in (\DN)$ if and only if every continuous linear map $T$ from $A_1(\alpha)$ into $F$ is bounded on a neighbourhood of zero in $A_1(\alpha)$, where $\alpha$ is any exponent.
sequence and

\[ A_1(\alpha) = \left\{ (\xi_j) \subset \mathbb{C}^n : \sum_{j=1}^{\infty} |\xi_j|^r < \infty \text{ for } 0 < r < 1 \right\}. \]

Assume that \( O(Y) \in (DN) \). We must prove \( O(X) \in (DN) \). By the above mentioned result of Vogt it suffices to show that every continuous linear map \( T \) from \( A_1(\alpha) \) into \( O(X) \) is bounded on a neighbourhood of zero in \( A_1(\alpha) \).

By the integrality lemma [3] it follows that there exists \( p \) such that

\[ f^p + a_{p-1}(f)f^{p-1} + \ldots + a_0(f) = 0 \]

for every \( f \in O(X) \), where \( a_{p-1}(f), \ldots, a_0(f) \in O(Y) \) are given by

\[ a_{p-1}(f) (y) = \sum_{\theta(x) = y} f(x), \]

\[ \ldots \]

\[ a_0(f) (y) = \prod_{\theta(x) = y} f(x). \]

Clearly \( a_{p-1}(f), \ldots, a_0(f) \) are continuous polynomials in \( f \) with values in \( O(Y) \). Hence \( a_{p-1}T, \ldots, a_0T \) are also continuous polynomials on \( A_1(\alpha) \). Since \( A_1(\alpha) \hat{\otimes} \ldots \hat{\otimes} A_1(\alpha), \ldots, A_1(\alpha) \in (\mathcal{F}) \), by the theorem of Vogt \( a_{p-1}T, \ldots, a_0T \) and hence \( T \) are bounded on a neighbourhood of zero in \( A_1(\alpha) \). □

**Lemma 1.2.** \( O(V) \in (DN) \).

**Proof.** As \( V \) is a branched cover over \( \Gamma \), by Lemma 1.1 it suffices to show that \( O(\Gamma) \in (DN) \).

Put \( \Gamma_\varepsilon = \{ z \in \Gamma : f(z) = f(e) \text{ for every } f \in O(\Gamma) \} \). It is well known [6] that \( \Gamma_\varepsilon \) is abelian and normal. Moreover \( \dim O(\Gamma_\varepsilon) = 1 \) and \( \Gamma/\Gamma_\varepsilon \) is Stein. This yields that \( O(\Gamma) \cong O(\Gamma/\Gamma_\varepsilon) \) and hence we may assume that \( \Gamma \) is Stein.

We now prove that \( O(\Gamma) \in (DN) \). By the theorem of Zaharyuta [12] it suffices to check that every plurisubharmonic function \( \varphi \) on \( \Gamma \) with \( \sup_{\Gamma} \varphi < \infty \) is constant.

Consider the exponential map \( \exp : T_\varepsilon \Gamma \to \Gamma \). Take a neighbourhood \( U \) of zero in \( T_\varepsilon \Gamma \) such that

\[ \exp : U \cong \exp U = V \quad \text{and} \quad V = V^{-1}. \]

Given \( b \in \Gamma \) and \( a \in V \), let \( z_0 \in U \) for which \( \exp z_0 = a^{-1} \) and \( \sigma(\lambda) = b(\exp \lambda z_0) a \) for every \( \lambda \in \mathbb{C} \). Since \( \varphi \sigma = \text{const.} \), we have

\[ \varphi(ba) = \varphi(\sigma(0)) = \varphi(\sigma(1)) = \varphi(b) \quad \text{for every } b \in \Gamma \text{ and every } a \in V. \]
Let $b$ be an arbitrary point in $\Gamma$. By the connectedness of $\Gamma$ we can find $a_1 = e, a_2, \ldots, a_n \in V$ such that $b = a_1a_2\ldots a_n$. By (*) we have

$$\varphi(b) = \varphi(a_1a_2\ldots a_n) = \varphi(a_1a_2\ldots a_{n-1}) = \ldots = \varphi(a_1) = \varphi(e).$$

Thus $\varphi = \text{const}$ and $O(\Gamma) \in (DN)$. 

**Lemma 1.3.** Let $X$ be a Stein space. Then $H^0(X, S) \in (\Omega)$ for every coherent sheaf $S$ on $X$.

**Proof.** Let $\{K_p\}$ be an increasing exhaustion sequence of compact sets in $X$. By the Cartan Theorem $\Lambda$, for each $x \in X$ there exist a neighbourhood $U_x$ of $x$ and $\sigma_1, \ldots, \sigma_n \in H^0(X, S)$ which generate $S_y$ for every $y \in U_x$.

By the compactness of $K_p$ there exists a sequence $\{\sigma_n\} \subset H^0(X, S)$ such that $\{\sigma_n\}$ generate $S_x$ for every $x \in X$.

Since $H^0(X, S)$ is Fréchet we may assume that $\{\sigma_n\}$ is bounded in $H^0(X, S)$. Consider the Banach coherent sheaf $O^\ell_X$ of germs of holomorphic functions on $X$ with values in $\ell^1$ and the morphism $\eta$ from $O^\ell_X$ into $S$ given by

$$\eta(f)(x) = \sum_{n \geq 1} \sigma_n(x)f_n(x) \quad \text{for } f = \{f_n\} \in O^\ell_X.$$

By the choice of $\sigma_n$ we infer that $\eta$ is surjective. By a theorem of Leiterer [5], $\text{Ker} \eta$ is a Banach coherent sheaf and hence $H^1(X, \text{Ker} \eta) = 0$ (see [5]). It follows that the map $\hat{\eta}: H^0(X, O^\ell_X) \cong O(X, \ell^1) \to H^0(X, S)$ is surjective.

On the other hand, since $O(X, \ell^1) \cong O(X) \hat{\otimes} \ell^1 \in (\Omega)$ when $O(X) \in (\Omega)$, it remains to check that $O(X) \in (\Omega)$. For each $n$, let $X_n$ denote the union of irreducible branches of $X$ of dimension $\leq n$. We have

$$O(X) \cong \lim \text{proj} O(X_n)$$

and the restriction maps $R_n: O(X) \to O(X_n)$ are surjective. Hence $O(X) \in (\Omega)$ if $O(X_n) \in (\Omega)$ for $n \geq 1$. For each $n \geq 1$, choose a proper injection $\theta: X_n \to \mathbb{C}^{2n+1}$. Since $O(\mathbb{C}^{2n+1}) \in (\Omega)$ we have

$$O(X_n) \cong H^0(\mathbb{C}^{2n+1}, (\theta_n) \ast O_{X_n}) \in (\Omega).$$

**Remark 1.4.** While this paper was in preparation, we were not aware of the results of D. Vogt [11] and A. Aytuna [1] who had proved Lemma 1.3 earlier. We thank the referee for pointing out these papers.

**Lemma 1.5.** $\text{Ker} R_V = J(V) = \{f \in O(G) : f|_V = 0\} \in (\Omega)$.

**Proof.** Let $\eta$ denote the canonical map from $G$ onto $G/G_e$ and let

$$\hat{V} = \{z \in G/G_e : f(z) = 0 \text{ for every } f \in J(V)\}. $$
Then \( J(V) = J(\hat{V}) \) and as \( G/G_e \) is Stein we have
\[
J(\hat{V}) = H^0(G/G_e, J_\hat{V})
\]
where \( J_\hat{V} \) denotes the coherent ideal sheaf defined by \( \hat{V} \). By Lemma 1.3, this yields that \( J(\hat{V}) \in (\Omega) \) and hence \( J(V) \in (\Omega) \).

Now Theorem 1 is deduced immediately from Lemmas 1.2 and 1.5.

2. It is known [7] that every non-compact connected complex Lie group \( G \) with \( \dim \Omega(G) = 1 \) contains a closed subgroup \( \Gamma \) for which \( R_\Gamma \) is not surjective.

Thus the following question arises naturally. When is the restriction map \( R_V \) in Theorem 1 surjective?

The following proposition gives an answer.

**Proposition 2.1.** Let \( \Gamma \) be a connected closed subgroup of a complex Lie group \( G \) such that \( G_e \subset \Gamma \). Then \( R_\Gamma : \Omega(G) \to \Omega(\Gamma) \) is surjective.

**Proof.** By [6] there exists a closed subgroup \( K \) of \( G \) such that for some \( n \) the groups \( G \) and \( K \times \mathbb{C}^n \) are isomorphic as complex Lie groups.

Moreover, there exists a closed Stein subgroup \( S_0 \) of \( K \) such that for the centre \( Z \) of \( K \), the map
\[
\varrho_0 : Z \times S_0 \to K, \quad (x, y) \mapsto xy,
\]
is a finite covering homomorphism.

By the result of [6], \( G_e \subset Z \) and \( Z \cong G_e \times \mathbb{C}^{\nu} \times \mathbb{C}^\mu \) for some non-negative integers \( \nu \) and \( \mu \).

Putting \( S = \mathbb{C}^{\nu} \times \mathbb{C}^\mu \times S_0 \times \mathbb{C}^n \) we get a finite covering homomorphism
\[
\varrho : G_e \times S \to G \text{ of degree } n, \text{ given by}
\]
\[
\varrho(x_0, x_1, x_2, x_3, x_4) = (\varrho_0((x_0, x_1, x_2), x_3), x_4).
\]

It is easy to see that
\[
\varrho^{-1}(\Gamma) = G_e \times (\Gamma \cap S).
\]

Since \( S \) is Stein, the restriction map \( \tilde{R} : \Omega(S) \to \Omega(\Gamma \cap S) \) and hence also the restriction map \( R : \Omega(G_e \times S) \to \Omega(G_e \times (\Gamma \cap S) \) is surjective.

Now given \( g \in \Omega(\Gamma) \), define \( f \in \Omega(G) \) by
\[
f(y) = \frac{1}{n} \sum_{\varrho(x,z) = y} \tilde{g}(x,z) \quad \text{with} \quad \tilde{g} \in \Omega(G_e \times S), \tilde{g}|_{G_e \times (\Gamma \cap S)} = g\varrho.
\]

Then \( f|_\Gamma = g \).

3. **Proof of Theorem 2.** Let \( SO(V) \) denote the spectrum of the Fréchet algebra \( \Omega(V) \) equipped with the weak topology. Since \( \pi_1 : V \to \Gamma \) is a
branched covering map and $SO(G) \cong SO(G/\Gamma_e) \cong \Gamma/\Gamma_e$ it follows that $\pi_1$ induces a branched covering map $\tilde{\pi}_1: SO(V) \to \Gamma/\Gamma_e$.

Then $SO(V)$ is a complex space and $O(V) \cong O(SO(V))$.

Now since $\Gamma/\Gamma_e \times G/G_e$ is Stein, there exists a commutative diagram of holomorphic maps

\[
\begin{array}{ccc}
V & \xrightarrow{\delta} & \Gamma \times G \\
\downarrow \beta & & \downarrow \eta \\
SO(V) & \xrightarrow{\tilde{\pi}_1} & \Gamma/\Gamma_e \times G/G_e \\
\downarrow \pi_1 & & \\
\Gamma/\Gamma_e & & 
\end{array}
\]

where $\delta$ and $\eta$ are canonical maps.

Then it is easy to see that $\beta$ is proper and hence $\text{Im} \beta$ is an analytic set in $\Gamma/\Gamma_e \times G/G_e$. Moreover, $O(\text{Im} \beta) \cong \text{Im} R_V$. By Lemma 1.1, $\text{Im} R_V \in (DN)$ and by Lemma 1.5, $\text{Ker} R_V \in (\Omega)$. Hence Vogt’s splitting theorem implies that $R_V: O(\Gamma \times G) \to \text{Im} R_V$ has a right inverse.

References


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